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NEW TYPE DEGENERATE SIMSEK NUMBERS AND RELATED ASPECTS

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In this paper, we introduce a new type degenerate Simsek numbers and their generating function, which are different from degenerate Simsek number studied so far. We establish the explicit formula, recurrence relation and other identities for these numbers. We also derive several interesting expressions and relations between these numbers and certain other special numbers in the literature. In addition, several numerical examples and graphical illustrations are provided to illustrate the behavior of the introduced numbers.

1. INTRODUCTION AND PRELIMINARIES

Special numbers and polynomials play crucial rule to solve explicitly several problems in many scientific areas, such as in mathematical physics, biology, quantum mechanics, discrete mathematics, probability and statistics. The very well known special numbers are Stirling numbers of the first and second kind introduced by James Stirling in 1730.

Stirling numbers of the second kind $S_1(n,k)$ count the number of partitions of a set of size n into k disjoint, non-empty subsets. They appear as coefficients in the following expansion:

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

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where

$$(x)_k = \begin{cases} 1 & \text{if } k = 0, \\ x(x-1)(x-2)\cdots(x-k+1) & \text{if } k \geq 1, \end{cases}$$

denotes the falling factorial (cf. [**3, 25**]). The generating function of $S_2(n,k)$ is given by

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{1}$$

Stirling numbers of the first kind, denoted by $S_1(n,k)$, are given by the following:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \tag{2}$$

with its generating function

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!},$$

(cf. [**3, 25**]).

The modifications and degenerate versions of these numbers were studied by several researchers, see for instance [**4, 5, 7, 9, 14, 19, 20, 22, 23, 24, 26, 27, 29**] and the references therein.

For a nonzero $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function $e_\alpha^x(t)$ defined as follows

$$e_\alpha^x(t) = (1+\alpha t)^{\frac{x}{\alpha}} \text{ and } e_\alpha(t) = (1+\alpha t)^{\frac{1}{\alpha}}, \quad \text{(see, [2, 10, 11, 15, 16])}. \tag{3}$$

Using the Taylor expansion, the degenerate exponential function can be expressed as follows:

$$e_\alpha^x(t) = \sum_{n=0}^{\infty}(x)_{n,\alpha}\frac{t^n}{n!},$$

where

$$(x)_{n,\alpha} = \begin{cases} 1 & \text{if } n = 0, \\ x(x-\alpha)(x-2\alpha)\cdots(x-(n-1)\alpha) & \text{if } n \geq 1, \end{cases}$$

is called the degenerate falling factorial. It follows that

$$\lim_{\alpha\to 0} e_\alpha^x(t) = \sum \frac{x^n t^n}{n!} = e^{xt}.$$

The degenerate Stirling numbers of the second kind $S_{2,\alpha}(n,l)$ are given as follows [**8, 10, 13, 15**]:

$$(x)_{n,\alpha} = \sum_{l=0}^{n} S_{2,\alpha}(n,l)(x)_l. \tag{4}$$

Moreover, as an inversion formula of (4), the degenerate Stirling numbers of the fist kind $S_{1,\alpha}(n,l)$ are defined as follows [**8, 10, 15**]:

$$(x)_n = \sum_{l=0}^{n} S_{1,\alpha}(n,l)(x)_{l.\alpha}.$$

Note that

$$\lim_{\alpha\to 0} S_{1,\alpha}(n,l) = S_1(n,l) \text{ and } \lim_{\alpha\to 0} S_{2,\alpha}(n,l) = S_2(n,l).$$

In addition, a new type degenerate Stirling numbers of the second kind $S_2^*(n,k|\alpha)$ were introduced in [**7**] as

$$\frac{1}{k!}(e^t-1)_{k,\alpha} = \sum_{n=k}^{\infty} S_2^*(n,k|\alpha)\frac{t^n}{n!} \quad \text{and} \quad S_2^*(n,0|\alpha) = 0, \quad (n \geq 0).$$

They reduced, for $\alpha = 0$, to Stirling numbers of the second kind (1).

In 2018, Simsek [**28**] introduced new families of special numbers $y_1(n,k;\lambda)$, for computing negative order Euler numbers and related numbers and polynomials. Simsek numbers $y_1(n,k;\lambda)$ are defined by the generating function

$$\frac{\left(\lambda e^t + 1\right)^k}{k!} = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}. \tag{5}$$

They can be expressed explicitly by the following identity

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n \lambda^j. \tag{6}$$

Simsek numbers $y_1(n,k;\lambda)$ are related to the many well-known special numbers in the literature, among them, Bernoulli numbers, Fibonacci numbers, Stirling numbers of the second kind, Lucas numbers and the central numbers.

Recently, the author introduced in [**24**] a degenerate version of Simsek numbers $y_1(n,k;\lambda|\alpha)$ by means of the generating function

$$\frac{\left(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + 1\right)^k}{k!} = \sum_{n=0}^{\infty} y_1(n,k;\lambda|\alpha)\frac{t^n}{n!}, \tag{7}$$

with the explicit formulas

$$y_1(n,k;\lambda|\alpha)=\frac{1}{k!}\sum_{m=0}^{n}\sum_{j=0}^{k}\binom{k}{j}j^m\lambda^j\alpha^{n-m}S_1(n,m)$$

and

$$y_1(n,k;\lambda|\alpha)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j\alpha^n\left(\frac{j}{\alpha}\right)_n.$$

For more details about the numbers $y_1(n,k;\lambda|\alpha)$ and related aspects, we refer the readers to [**24**].

Kucukoglu [**17, 18**] investigated the q-combinatorial Simsek numbers $y_{1,q}(n,k;\lambda)$ and polynomials $y_{1,q}(x;n,k;\lambda)$ of the first kind, respectively, as follows

$$y_{1,q}(n,k;\lambda)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q[j]_q^n\lambda^j$$

and

$$y_{1,q}(x;n,k;\lambda)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q[x+j]_q^n\lambda^j.$$

In [**30**], Kucukoglu et al. defined the higher order expansion of the Simsek numbers and polynomials. Moreover, Kilar [**6**] introduced degenerate Simsek-type numbers and polynomials of higher order.

In this paper, we establish a new type degenerate Simsek numbers, which are different from degenerate Simsek numbers (7).

2. NEW TYPE DEGENERATE SIMSEK NUMBERS AND RELATED FORMULAS

In this section, we define a new type degenerate Simsek numbers and we derive some related formulas including generating function, derivative formulas, recurrence formulas, and integral formulas. Moreover, we establish also a relation between these numbers and certain special numbers.

We define a new type degenerate Simsek numbers $y^*_{1,\alpha}(n,k;\lambda)$ by means of the generating function:

$$F_k(t;\alpha,\lambda):=\frac{(\lambda e^t+1)_{k,\alpha}}{k!}=\sum_{n=0}^{\infty}y^*_{1,\alpha}(n,k;\lambda)\frac{t^n}{n!}. \tag{8}$$

It follows that for $\alpha=0$, we obtain $y^*_{1,\alpha}(n,k;\lambda)=y_1(n,k;\lambda)$ Simsek numbers (5). Moreover, the function $F_k(t;\alpha,k)$ can be expressed as follows

$$F_k(t;\alpha,\lambda)=\frac{\alpha^k}{k!}\left(\frac{\lambda e^t+1}{\alpha}\right)_k=\frac{\alpha^k}{k!}B_k^{(k+1)}\left(\frac{\lambda e^t+1}{\alpha}+1\right), \tag{9}$$

where $B_k^{(n)}(x)$ denotes the Bernoulli polynomials of order n, given by

$$\frac{t^n e^{xt}}{(e^t-1)^n} = \sum_{k=0}^{\infty} B_k^{(n)}(x) \frac{t^k}{k!}$$

and satisfies the recurrence relation

$$B_k^{(n+1)}(x) = \left(1 - \frac{k}{n}\right) B_k^{(n)}(x) + k\left(\frac{x}{n} - 1\right) B_{k-1}^{(n)}(x), \tag{10}$$

(cf. [**21**]).

By using (8), we derive the following functional equation

$$\alpha^k \left(\frac{\lambda e^t}{\alpha}\right)_k = (\lambda e^t)_{k,\alpha} = \sum_{\ell=0}^{k} (-1)_{k-\ell,\alpha} \binom{k}{\ell} \ell! F_\ell(t;\alpha,\lambda).$$

Combining (8) and (2) with the above equation, yields

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{k} \alpha^{k-j} S_1(k,j) \lambda^j j^n\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{k} (-1)_{k-\ell,\alpha} \ell! \binom{k}{\ell} y_{1,\alpha}^*(n,\ell;\lambda)\right) \frac{t^n}{n!}. \tag{11}$$

Comparing the coefficients of $\frac{t^n}{n1}$ on both sides of (11), we obtain the following relation between the numbers $y_{1,\alpha}^*(n,k;\lambda)$ and Stirling numbers of the first kind $S_1(n,k)$:

Theorem 2.1.

$$\sum_{j=0}^{k} \alpha^{k-j} S_1(k,j) \lambda^j j^n = \sum_{\ell=0}^{k} (-1)_{k-\ell,\alpha} \ell! \binom{k}{\ell} y_{1,\alpha}^*(n,\ell;\lambda).$$

Remark 2.2. *It follows that for $\alpha = 0$, we obtain*

$$\lambda^k k^n = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell! y_1(n,\ell;\lambda)$$

*the result given in [**28**].*

Theorem 2.3. *The numbers $y_{1,\alpha}^*(n,k;\lambda)$ can be expressed explicitly as follows:*

$$y_{1,\alpha}^*(n,k;\lambda) = \frac{1}{k!} \sum_{\ell=0}^{k} \sum_{j=0}^{\ell} \binom{\ell}{j} \alpha^{k-\ell} S_1(k,\ell) \lambda^j j^n \tag{12}$$

or

$$y_{1,\alpha}^*(n,k;\lambda) = \frac{1}{k!} \sum_{\ell=0}^{k} \sum_{j=0}^{\ell} \binom{k}{\ell} \alpha^{\ell-j} \lambda^j j^n (1)_{k-\ell,\alpha} S_1(\ell,j). \tag{13}$$

Proof. Using (2), we have

$$\begin{aligned}\frac{(\lambda e^t+1)_{k,\alpha}}{k!} &= \frac{\alpha^k}{k!}\Big(\frac{\lambda e^t+1}{\alpha}\Big)_k\\ &= \frac{1}{k!}\sum_{\ell=0}^{k}\alpha^{k-\ell}S_1(k,\ell)(\lambda e^t+1)^\ell\\ &= \frac{1}{k!}\sum_{\ell=0}^{k}\alpha^{k-\ell}S_1(k,\ell)\sum_{j=0}^{\ell}\binom{\ell}{j}\lambda^j e^{jt}\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{\ell=0}^{k}\sum_{j=0}^{\ell}\alpha^{k-\ell}S_1(k,\ell)\binom{\ell}{j}\lambda^j j^n\right)\frac{t^n}{n!}.\end{aligned}$$

Hence, by (8), we obtain (12).

To prove Eq. (13), we use the fact that $(x+y)_{k,\alpha}=\sum\limits_{j=0}^{k}\binom{k}{j}(x)_{j,\alpha}(y)_{k-j,\alpha}$. Then

$$\begin{aligned}\frac{\big(\lambda e^t+1\big)_{k,\alpha}}{k!} &= \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}(\lambda e^t)_{\ell,\alpha}(1)_{k-\ell,\alpha}\\ &= \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}\alpha^\ell\Big(\frac{\lambda e^t}{\alpha}\Big)_\ell(1)_{k-\ell,\alpha}\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{\ell=0}^{k}\sum_{j=0}^{\ell}\binom{k}{\ell}\alpha^{\ell-j}\lambda^j j^n(1)_{k-\ell,\alpha}S_1(\ell,j)\right)\frac{t^n}{n!},\end{aligned}$$

and the result follows. □

On the other hand, using the fact that $F_k(t;\alpha,\lambda)=\frac{\alpha^k}{k!}\left(\frac{\lambda e^t+1}{\alpha}\right)_k$ and the identity

$$(x)_k=\sum_{j=0}^{k}\binom{k}{j}\frac{j}{k}x^jB_{k-j}^{(k)},$$

where $B_n^{(k)}$ denotes the Bernoulli numbers of order k, defined by

$$\Big(\frac{t}{e^t-1}\Big)^k=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}$$

(cf. [**21**]), we obtain the following proposition:

Proposition 2.4. *The numbers* $y_{1,\alpha}^*(n,k;\lambda)$ *can be expressed in terms of the Bernoulli numbers* $B_n^{(k)}$ *of order* k *as follows:*

$$y_{1,\alpha}^*(n,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\alpha^{k-j}\binom{k}{j}\frac{j}{k}B_{k-j}^{(k)}\sum_{\ell=0}^{j}\binom{j}{\ell}\lambda^\ell\ell^n. \tag{14}$$

Remark 2.5. *Substituting* $\alpha = 0$ *in Equations* (12), (13) *and* (14), *we obtain*

$$y^*_{1,0}(n,k;\lambda) = y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n \lambda^j$$

Simsek numbers (6).

Example 2.6. *Some special cases of the numbers* $y^*_{1,\alpha}(n,k;\lambda)$ *are given as follows:*

- $y^*_{1,\alpha}(n,0;\lambda) = \delta_{n0}$,
- $y^*_{1,\alpha}(n,1;\lambda) = \delta_{n0} + \lambda$,
- $y^*_{1,\alpha}(0,k;\lambda) = \frac{\alpha^k}{k!}\sum\limits_{\ell=0}^{k} S_1(k,\ell)\left(\frac{\lambda+1}{\alpha}\right)^{\ell}$.

In the following theorem, we give a relation between the numbers $y^*_{1,\alpha}(n,k;\lambda)$, Simsek numbers $y_1(n,k;\lambda)$, Stirling numbers of the first kind $S_1(n,k)$ and the degenerate Stirling numbers of the second kind $S_{2.\alpha}(n,k)$.

Theorem 2.7. *The numbers* $y^*_{1,\alpha}(n,k;\lambda)$ *can be expressed as follows:*

$$y^*_{1,\alpha}(n,k;\lambda) = \frac{1}{k!}\sum_{\ell=0}^{k}\sum_{j=0}^{\ell} S_{2,\alpha}(k,\ell) S_1(\ell,j) j! y_1(n,j;\lambda).$$

Proof. Using (2), (4), (5) and (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty} y^*_{1,\alpha}(n,k;\lambda)\frac{t^n}{n!} &= \frac{1}{k!}\sum_{\ell=0}^{k} S_{2,\alpha}(k,\ell)(\lambda e^t+1)_\ell \\ &= \frac{1}{k!}\sum_{\ell=0}^{k} S_{2,\alpha}(k,\ell)\sum_{j=0}^{\ell} S_1(\ell,j)(\lambda e^t+1)^j \\ &= \frac{1}{k!}\sum_{\ell=0}^{k} S_{2,\alpha}(k,\ell)\sum_{j=0}^{\ell} S_1(\ell,j) j! \sum_{n=0}^{\infty} y_1(n,j;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above identity, we obtain the desired result. □

On the other hand, the numbers $y^*_{1,\alpha}(n,k;\lambda)$ are related to the degenerate Stirling numbers of the second kind $S^*_2(n,k|\alpha)$ as follows:

Theorem 2.8.

$$y^*_{1,\alpha}(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j! \lambda^j (\lambda+1)_{k-j,\alpha} S^*_2\left(n,k|\frac{\alpha}{\lambda}\right).$$

Proof. We have

$$\begin{aligned}\frac{(\lambda e^t+1)_{k,\alpha}}{k!} &= \frac{1}{k!}\left(\lambda(e^t-1)+\lambda+1\right)_{k,\alpha}\\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\left(\lambda(e^t-1)\right)_{j,\alpha}(\lambda+1)_{k-j,\alpha}\\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j(e^t-1)_{j,\frac{\alpha}{\lambda}}(\lambda+1)_{k-j,\alpha}\\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j j!\lambda^j(\lambda+1)_{k-j,\alpha}\sum_{n\geq 0}S_2^*\left(n,j|\frac{\alpha}{\lambda}\right)\frac{t^n}{n!}\end{aligned}$$

Using (8) and equating the coefficients in both sides of the above identity finishes the proof. □

Theorem 2.9. *The numbers* $y_{1,\alpha}^*(n,k;\lambda)$ *satisfy the following recurrence formula with respect to* k

$$y_{1,\alpha}^*(n,k+1;\lambda)=\frac{1}{k+1}\left(\lambda\sum_{\ell=0}^{n}\binom{n}{\ell}y_{1,\alpha}^*(\ell,k;\lambda)+(1-k\alpha)y_{1,\alpha}^*(n,k;\lambda)\right). \tag{15}$$

Proof. From (8), we have

$$F_k(t;\alpha,\lambda)=\left(\frac{\lambda e^t+\alpha-k\alpha+1}{k}\right)F_{k-1}(t;\alpha,\lambda). \tag{16}$$

Hence,

$$\begin{aligned}\sum_{n=0}^{\infty}y_{1,\alpha}^*(n,k+1;\lambda)\frac{t^n}{n!} &= \frac{\lambda}{k+1}\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}y_{1,\alpha}^*(n,k;\lambda)\frac{t^n}{n!}\\ &\quad+\frac{1-k\alpha}{k+1}\sum_{n=0}^{\infty}y_{1,\alpha}^*(n,k;\lambda)\frac{t^n}{n!}\\ &= \frac{1}{k+1}\sum_{n=0}^{\infty}\left(\lambda\sum_{\ell=0}^{n}y_{1,\alpha}^*(\ell,k;\lambda)\binom{n}{\ell}\right.\\ &\quad\left.+(1-k\alpha)y_{1,\alpha}^*(n,k;\lambda)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides yields (15). □

Differentiating both sides of (16) with respect to t, we obtain

$$F_k'(t;\alpha,\lambda)=\frac{\lambda e^t}{k}\left(F_{k-1}(t;\alpha,\lambda)+F_{k-1}'(t;\alpha.\lambda)\right)+\left(\frac{1+\alpha-k\alpha}{k}\right)F_{k-1}'(t;\alpha.\lambda).$$

Then, using (8), the above equation becomes

$$\sum_{n=0}^{\infty} y^*_{1,\alpha}(n+1,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\frac{\lambda}{k}\sum_{j=0}^{n}\binom{n}{j}\left(y^*_{1,\alpha}(j,k-1;\lambda)+y^*_{1,\alpha}(j+1,k-1;\lambda)\right)\right.$$
$$\left. +\left(\frac{1+\alpha-k\alpha}{k}\right)y^*_{1,\alpha}(n+1,k-1;\lambda)\right]\frac{t^n}{n!}.$$

Therefore, equating the coefficients of $\frac{t^n}{n!}$, yields another recursive formula for the numbers $y^*_{1,\alpha}(n,k;\lambda)$, given in the following theorem.

Theorem 2.10. *The numbers $y^*_{1,\alpha}(n,k;\lambda)$ satisfy the following recurrence formula*

$$y^*_{1,\alpha}(n+1,k;\lambda) = \frac{\lambda}{k}\sum_{j=0}^{n}\binom{n}{j}\Big(y^*_{1,\alpha}(j,k-1;\lambda)+y^*_{1,\alpha}(j+1,k-1;\lambda)\Big)$$
$$+\left(\frac{1+\alpha-k\alpha}{k}\right)y^*_{1,\alpha}(n+1,k-1;\lambda), \qquad \text{for } k\geq 1$$

and

$$y^*_{1,\alpha}(n,0;\lambda)=\delta_{n,0}.$$

Remark 2.11. *Note that, Theorem 2.10 can also be proved using* (9), (10), *and the derivative of the Bernoulli polynomials of order k:*

$$\frac{d}{dt}B_n^{(k)}(t) = nB_{n-1}^{(k)}(t),$$

*(cf. [**21**]).*

We define the generating function $\phi^*_n(x|\alpha,\lambda)$ of the numbers $y^*_{1,\alpha}(n,k;\lambda)$ as follows:

$$\phi^*_n(x|\alpha,\lambda) := \sum_{k=0}^{\infty} y^*_{1,\alpha}(n,k;\lambda)x^k.$$

Furthermore, we derive formulas related to the generating function $\phi^*_n(x|\alpha,\lambda)$, including the exponential generating function, a recurrence formula, an integral formula, a derivative formula and a relation with some well-know numbers.

Theorem 2.12. *For a non-negative integer n, we have the following exponential generating function of $\phi^*_n(x|\alpha,\lambda)$:*

$$\sum_{n=0}^{\infty}\phi^*_n(x|\alpha,\lambda)\frac{t^n}{n!} = e_{\alpha}^{\lambda e^t+1}(x) \tag{17}$$

Proof. we have

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n^*(x|\alpha,\lambda)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty}y_{1,\alpha}^*(n,k;\lambda)x^k\Big)\frac{t^n}{n!}\\&=\sum_{k=0}^{\infty}\Big(\sum_{n=0}^{\infty}y_{1,\alpha}^*(n,k;\lambda)\frac{t^n}{n!}\Big)x^k\\&=\sum_{k=0}^{\infty}(\lambda e^t+1)_{k,\alpha}\frac{x^k}{k!}\\&=e_{\alpha}^{\lambda e^t+1}(x),\end{aligned}$$

where in the third equation we used (8) and in the last equation we used (3). □

The following result, describe a relation between the generating functions $\phi_n^*(x|\alpha,\lambda)$ and $\phi_n^*(x|0,\lambda)$ of the numbers $y_{1,\alpha}^*(n,k;\lambda)$ and Simsek numbers $y_1(n,k;\lambda)$, respectively.

Theorem 2.13. *The generating function $\phi_n^*(x|\alpha,\lambda)$ can be expressed as follows:*

$$\phi_n^*(x|\alpha,\lambda)=\sum_{k=0}^{\infty}\left(\frac{\log(1+\alpha x)}{\alpha}\right)^k y_1(n,k;\lambda)=\phi_n^*\left(\frac{\log(1+\alpha x)}{\alpha}|0,\lambda\right),$$

where $\phi_n^*(x|0,\lambda)$ is the generating function of Simsek numbers $y_1(n,k;\lambda)$.

Proof. According to Theorem 2.12, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n^*(x|\alpha,\lambda)\frac{t^n}{n!}&=e_{\alpha}^{\lambda e^t+1}(x)\\&=(1+\alpha x)^{\frac{\lambda e^t+1}{\alpha}}\\&=e^{\frac{\lambda e^t+1}{\alpha}\log(1+\alpha x)}\\&=\sum_{k=0}^{\infty}\frac{1}{\alpha^k}\big(\log(1+\alpha x)\big)^k\frac{(\lambda e^t+1)^k}{k!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\Big(\frac{\log(1+\alpha x)}{\alpha}\Big)^k y_1(n,k;\lambda)\frac{t^n}{n!}.\end{aligned}\tag{18}$$

Hence, the result follows by comparing the coefficients on both sides of (18). □

Theorem 2.14. *For $n\geq 0$, the generating function $\phi_n^*(x,\alpha|\lambda)$ satisfies the following recursive formula:*

$$\phi_{n+1}^*(x|\alpha,\lambda)=\frac{\lambda}{\alpha}\log(1+\alpha x)\sum_{\ell=0}^{n}\binom{n}{\ell}\phi_\ell^*(x|\alpha,\lambda).$$

Proof. Taking the derivative with respect to t in both sides of (17), we obtain

$$\frac{d}{dt}\sum_{n=0}^{\infty}\phi_n^*(x|\alpha,\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\phi_{n+1}^*(x|\alpha,\lambda)\frac{t^n}{n!}, \tag{19}$$

and

$$\begin{aligned}\frac{d}{dt}\left(e_\alpha^{\lambda e^t+1}(x)\right)&=\frac{\lambda}{\alpha}e^t\log(1+\alpha x)e_\alpha^{\lambda e^t+1}(x)\\&=\frac{\lambda}{\alpha}\log(1+\alpha x)\sum_{j=0}^{\infty}\frac{t^j}{j!}\sum_{\ell=0}^{\infty}\phi_\ell^*(x|\alpha,\lambda)\frac{t^\ell}{\ell!}\\&=\frac{\lambda}{\alpha}\log(1+\alpha x)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\binom{n}{\ell}\phi_\ell^*(x|\alpha,\lambda)\frac{t^n}{n!}. \qquad (20)\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$ in (20) and (19) gives the desired result. □

We express the generating function $\phi_n^*(x|\alpha,\lambda)$ in terms of the degenerate first kind Apostol-Euler numbers $E_n^{(k)}(\lambda|\alpha)$ of order k, defined in [**24**], as follows:

$$\left(\frac{2}{\lambda e^{\frac{\log(1+\alpha t)}{\alpha}}+1}\right)^k=\sum_{n=0}^{\infty}E_n^{(k)}(\lambda|\alpha)\frac{t^n}{n!}. \tag{21}$$

Theorem 2.15. *For a non-negative integer n, we have*

$$\sum_{m=0}^{n}\binom{n}{m}E_{n-m}^{(1)}(\lambda|0)\frac{d}{dx}\phi_m^*(x|\alpha,\lambda)=\frac{2}{1+\alpha x}\phi_n^*(x|\alpha,\lambda).$$

Proof. Multiplying both sides of (17) by $\frac{2}{1+\lambda e^t}$, we obtain

$$\frac{2}{1+\lambda e^t}\sum_{n=0}^{\infty}\frac{d}{dx}\phi_n^*(x|\alpha,\lambda)\frac{t^n}{n!}=\frac{2}{1+\alpha x}e_\alpha^{\lambda e^t+1}(x). \tag{22}$$

The right-hand side of (22) can be written as

$$\frac{2}{1+\alpha x}\sum_{n=0}^{\infty}\phi_n^*(x|\alpha,\lambda)\frac{t^n}{n!}. \tag{23}$$

Using (21), the left-hand side of (22) can be expressed as

$$\begin{aligned}\frac{2}{1+\lambda e^t}\sum_{n=0}^{\infty}\frac{d}{dx}\phi_n^*(x|\alpha,\lambda)\frac{t^n}{n!}&=\sum_{j=0}^{\infty}E_j^{(1)}(\lambda|0)\frac{t^j}{j!}\sum_{m=0}^{\infty}\frac{d}{dx}\phi_m^*(x|\alpha,\lambda)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}E_{n-m}^{(1)}(\lambda|0)\frac{d}{dx}\phi_m^*(x|\alpha,\lambda)\frac{t^n}{n!}, \qquad (24)\end{aligned}$$

and the result follows from (23) and (24). □

Using Theorem 2.12 and the fact that

$$\frac{d}{dx}\big(e_{\alpha}^{\lambda e^t+1}(x)\big) = \frac{\lambda e^t+1}{1+\alpha x}\, e_{\alpha}^{\lambda e^t+1}(x),$$

we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}\phi_n^*(x|\alpha,\lambda)\frac{t^n}{n!} &= \frac{1}{1+\alpha x}\Big((1+\lambda e^t)e_{\alpha}^{\lambda e^t+1}(x)\Big)\\ &= \frac{1}{1+\alpha x}\Big((1+\lambda e^t)\sum_{\ell=0}^{\infty}\phi_\ell^*(x|\alpha,\lambda)\frac{t^\ell}{\ell!}\Big)\\ &= \frac{1}{1+\alpha x}\sum_{n=0}^{\infty}\left(\phi_n^*(x|\alpha,\lambda)+\lambda\sum_{\ell=0}^{n}\binom{n}{\ell}\phi_\ell^*(x|\alpha,\lambda)\right)\frac{t^n}{n!},\end{aligned}$$

which, by equating the coefficients of $\frac{t^n}{n!}$ on both sides, gives the following derivative formula for the generating function $\phi_n^*(x|\alpha,\lambda)$.

Theorem 2.16. *For $n \geq 0$, we have*

$$\frac{d}{dx}\phi_n^*(x|\alpha,\lambda) = \frac{1}{1+\alpha x}\left(\phi_n^*(x|\alpha,\lambda)+\lambda\sum_{\ell=0}^{n}\binom{n}{\ell}\phi_\ell^*(x|\alpha,\lambda)\right).$$

By virtue of Theorem 2.12, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\int_0^x \phi_n^*(y|\alpha,\lambda)dy\frac{t^n}{n!} &= \int_0^x e_{\alpha}^{\lambda e^t+1}(y)dy\\ &= \left[\frac{1+\alpha y}{\lambda e^t+1}e_{\alpha}^{\lambda e^t+1}(y)\right]_0^x\\ &= \frac{1}{\lambda e^t+1}\Big((1+\alpha x)e_{\alpha}^{\lambda e^t+1}(x)-1\Big)\\ &= \frac{1+\alpha x}{2}\sum_{n=0}^{\infty}E_n^{(1)}(\lambda|0)\frac{t^n}{n!}\sum_{l=0}^{\infty}\phi_l^*(x,\alpha|\lambda)\frac{t^l}{l!}\\ &\quad -\frac{1}{2}\sum_{n=0}^{\infty}E_n^{(1)}(\lambda|0)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\frac{1+\alpha x}{2}\sum_{l=0}^{n}\binom{n}{l}E_{n-l}^{(1)}(\lambda|0)\phi_l^*(x,\alpha|\lambda)\right.\\ &\quad \left.-\frac{1}{2}E_n^{(1)}(\lambda|0)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficient on both sides yields the following integral formula for the generating function $\phi_n^*(x|\alpha,\lambda)$:

Theorem 2.17. *For $n \geq 1$, one has*

$$\int_0^x \phi_n^*(y|\alpha,\lambda)dy = \frac{1+\alpha x}{2}\sum_{l=0}^{n}\binom{n}{l}E_{n-l}^{(1)}(\lambda|0)\phi_l^*(x|\alpha,\lambda) - \frac{1}{2}E_n^{(1)}(\lambda|0).$$

The following result from [**1, 12**] will be used in the proof of the next theorem. Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{k=0}^{\infty} b_k x^k$ be power series convergent on some open disk centered at the origin. Then

$$\sum_{k=0}^{\infty}\frac{g^{(k)}(0)}{k!}f(k)x^k = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\sum_{k=0}^{n}S_2(n,k)g^{(k)}(x)x^k. \tag{25}$$

Theorem 2.18. *For $n \geq 0$, one has*

$$\begin{aligned}\sum_{m=0}^{\infty} y_{1,\alpha}^*(n,m;\lambda)f(m)x^m = \sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{\infty}\sum_{k=0}^{m}S_2(m,k)\Big(\frac{x}{1+\alpha x}\Big)^k k!\frac{f^{(m)}(0)}{m!}\\ \times y_{1,\alpha}^*(j,k;\lambda)\phi_{n-j}^*(x|\alpha,\lambda).\end{aligned}$$

Proof. Consider the function $g(x)=e_\alpha^{\lambda e^t+1}(x)$. Then

$$g^{(k)}(x) = \Big(\frac{d}{dx}\Big)^k e_\alpha^{\lambda e^t+1}(x) = \frac{(\lambda e^t+1)_{k,\alpha}}{(1+\alpha x)_{k,\alpha}}e_\alpha^{\lambda e^t+1}(x). \tag{26}$$

Using (8), (25) and (26), we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(\lambda e^t+1)_{m,\alpha}}{m!}f(m)x^m &= \sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}\sum_{k=0}^{m}S_2(m,k)x^k\frac{(\lambda e^t+1)_{k,\alpha}}{(1+\alpha x)^k}e_\alpha^{\lambda e^t+1}(x)\\ &= e_\alpha^{\lambda e^t+1}(x)\sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}\sum_{k=0}^{m}S_2(m,k)\Big(\frac{x}{1+\alpha x}\Big)^k k!\sum_{j=0}^{\infty}y_{1,\alpha}^*(j,k;\lambda)\frac{t^j}{j!}\\ &= \sum_{l=0}^{\infty}\phi_l^*(x|\alpha,\lambda)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}\sum_{k=0}^{m}S_2(m,k)\Big(\frac{x}{1+\alpha x}\Big)^k k!\sum_{j=0}^{\infty}y_{1,\alpha}^*(j,k;\lambda)\frac{t^j}{j!}\\ &= \sum_{n=0}^{\infty}\Bigg(\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{\infty}\sum_{k=0}^{m}S_2(m,k)\Big(\frac{x}{1+\alpha x}\Big)^k k!\,\frac{f^{(m)}(0)}{m!}\\ &\quad\times y_{1,\alpha}^*(j,k;\lambda)\phi_{n-j}^*(x|\alpha,\lambda)\Bigg)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides yields the desired result. □

2.1 Numerical Results and Graphical Illustrations

In this subsection, similarly to the recent computational implementations given by Kucukoglu [**18**] for q-combinatorial Simsek numbers and polynomials, we present numerical illustrations and graphical plots to highlight the behavior of the generalized numbers $y^*_{1,\alpha}(n,k;\lambda)$ with respect to the parameters n, k, α, and λ. All computations are based on the explicit formula (12), adopting the convention $0^0 = 1$. We first present explicit numerical values of $y^*_{1,\alpha}(n,k;\lambda)$ in tabular form for small values of n and k. Table 1 lists the values for fixed $\lambda = 1$ and free α, while Table 2 provides the corresponding values for fixed $\alpha = 1$ and free λ, in both cases for $0 \le n,k \le 4$.

Table 1: Values of $y^*_{1,\alpha}(n,k;1)$ for $0 \le n,k \le 4$.

$n \setminus k$	0	1	2	3	4
0	1	2	$\alpha+2$	$\frac{2\alpha^2+6\alpha+4}{3}$	$\frac{3\alpha^3+11\alpha^2+12\alpha+4}{6}$
1	0	1	$\frac{\alpha+4}{2}$	$\frac{\alpha^2+6\alpha+6}{3}$	$\frac{3\alpha^3+22\alpha^2+36\alpha+16}{12}$
2	0	1	$\frac{\alpha+6}{2}$	$\frac{\alpha^2+9\alpha+12}{3}$	$\frac{3\alpha^3+33\alpha^2+72\alpha+40}{12}$
3	0	1	$\frac{\alpha+10}{2}$	$\frac{\alpha^2+15\alpha+27}{3}$	$\frac{3\alpha^3+55\alpha^2+162\alpha+112}{12}$
4	0	1	$\frac{\alpha+18}{2}$	$\frac{\alpha^2+27\alpha+66}{3}$	$\frac{3\alpha^3+99\alpha^2+396\alpha+340}{12}$

Table 2: Values of $y^*_{1,1}(n,k;\lambda)$ for $0 \le n,k \le 4$.

$n \setminus k$	0	1	2	3	4
0	1	$\lambda+1$	$\frac{\lambda^2+3\lambda+2}{2}$	$\frac{\lambda^3+6\lambda^2+11\lambda+6}{6}$	$\frac{\lambda^4+10\lambda^3+35\lambda^2+50\lambda+24}{24}$
1	0	λ	$\frac{2\lambda^2+3\lambda}{2}$	$\frac{3\lambda^3+12\lambda^2+11\lambda}{6}$	$\frac{2\lambda^4+15\lambda^3+35\lambda^2+25\lambda}{12}$
2	0	λ	$\frac{4\lambda^2+3\lambda}{2}$	$\frac{9\lambda^3+24\lambda^2+11\lambda}{6}$	$\frac{8\lambda^4+45\lambda^3+70\lambda^2+25\lambda}{12}$
3	0	λ	$\frac{8\lambda^2+3\lambda}{2}$	$\frac{27\lambda^3+48\lambda^2+11\lambda}{6}$	$\frac{32\lambda^4+135\lambda^3+140\lambda^2+25\lambda}{12}$
4	0	λ	$\frac{16\lambda^2+3\lambda}{2}$	$\frac{81\lambda^3+96\lambda^2+11\lambda}{6}$	$\frac{128\lambda^4+405\lambda^3+280\lambda^2+25\lambda}{12}$

These tables clearly illustrate the polynomial dependence of $y^*_{1,\alpha}(n,k;\lambda)$ on the free parameter. For fixed $\lambda = 1$, Table 1 shows that the entries are polynomials in α whose degrees increase with k, reflecting the role of the degeneracy parameter in deforming the classical structure. Similarly, Table 2 demonstrates that, for fixed $\alpha = 1$, the values are polynomial expressions in λ with coefficients depending on n and k.

Moreover, the boundary cases $k = 0$ and $k = 1$ are consistent with the theoretical properties of the numbers, namely $y^*_{1,\alpha}(n,0;\lambda) = \delta_{n0}$ and $y^*_{1,\alpha}(n,1;\lambda) = \delta_{n0} + \lambda$.

For completeness, we also include in Table 3 the numerical values of the numbers $y^*_{1,\alpha}(n,k;\lambda)$ for the representative choice $(\alpha,\lambda)=(1,1)$ and $0\leq n,k\leq 4$. This table provides a direct numerical verification of the polynomial expressions given in Tables 1 and 2, and confirms the values displayed in the corresponding graphical plots in Figures 1 and 2 .

Table 3: Numerical values of $y^*_{1,1}(n,k;1)$ for $0\leq n,k\leq 4$.

$n\backslash k$	0	1	2	3	4
0	1	2	3	4	5
1	0	1	$\frac{5}{2}$	$\frac{13}{3}$	$\frac{77}{12}$
2	0	1	$\frac{7}{2}$	$\frac{22}{3}$	$\frac{37}{3}$
3	0	1	$\frac{11}{2}$	$\frac{43}{3}$	$\frac{83}{3}$
4	0	1	$\frac{19}{2}$	$\frac{94}{3}$	$\frac{419}{6}$

In addition to the numerical illustrations, We also examine the variation of $y^*_{1,\alpha}(n,k;\lambda)$ as a function of the order n for a fixed combinatorial parameter k.

Figure 1 illustrates the case $k=3$. Panels (a) and (c) show the values for fixed $\alpha=1$ and $\alpha=2$, respectively, with $\lambda\in\{1,2,3,4\}$, while panels (b) and (d) illustrate the influence of the degeneracy parameter $\alpha\in\{\frac{1}{2},1,2,3\}$ for fixed $\lambda=1$ and $\lambda=2$, respectively. In both cases, the values increase rapidly with n, and larger values of λ or α lead to a pronounced growth rate, indicating a strong dependence on these parameters.

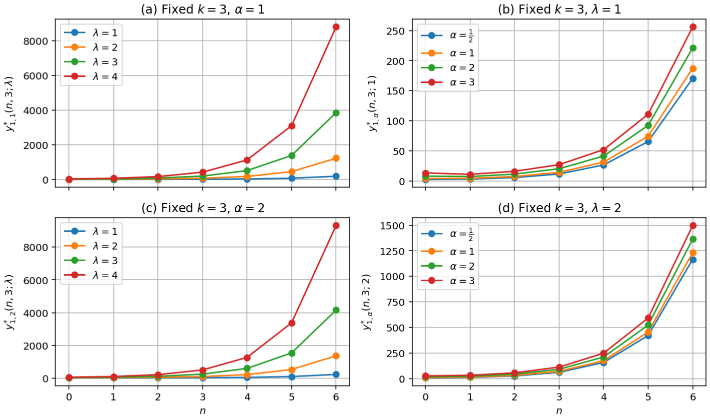

Figure 1: Numerical comparison of the degenerate Simsek numbers $y^*_{1,\alpha}(n,3;\lambda)$. Panels (a) and (c) illustrate the dependence on λ for fixed α, while panels (b) and (d) show the effect of varying α for fixed λ.

To complement this analysis, we next fix the order $n = 3$ and study the dependence on the parameter k.

Figure 2 presents a four-panel comparison. Panels (a) and (c) correspond to fixed $\alpha = 1$ and $\alpha = 2$, respectively, with $\lambda \in \{1, 2, 3, 4\}$. Panels (b) and (d) show the effect of varying α over $\{\frac{1}{2}, 1, 2, 3\}$ for fixed $\lambda = 1$ and $\lambda = 2$, respectively. In contrast to the growth in n, the dependence on k is more moderate and exhibits a structured polynomial-type behavior. The influence of the degeneracy parameter α becomes more visible as k increases, especially for larger values of λ.

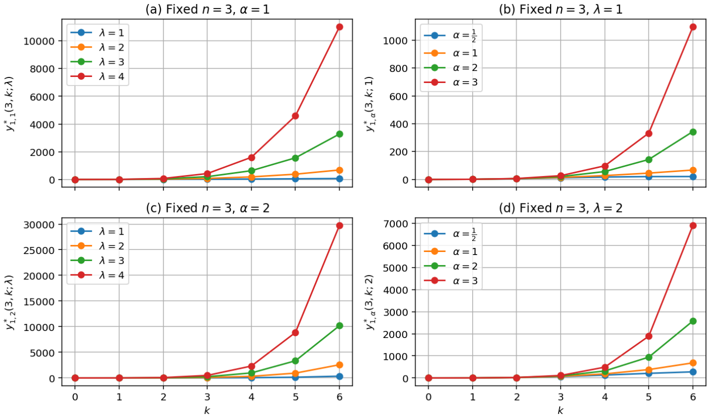

Figure 2: Numerical comparison of the degenerate Simsek numbers $y^*_{1,\alpha}(3, k; \lambda)$. Panels (a) and (c) illustrate the dependence on λ for fixed α, while panels (b) and (d) show the effect of varying α for fixed λ.

Overall, these numerical and graphical experiments confirm the analytical structure of $y^*_{1,\alpha}(n, k; \lambda)$ and clearly illustrate the distinct roles played by the parameters n, k, α, and λ. In particular, the parameter λ primarily controls the growth rate, whereas α governs the deformation effect, which becomes more pronounced for larger values of k and n.

3. CONCLUSION

In the present work, we have introduced a new type of degenerate Simsek numbers $y^*_{1,\alpha}(n, k; \lambda)$ along with their generating function, defined using the degenerate falling function. These numbers are distinct from the degenerate Simsek number studied previously. We analyzed several of their properties, including derivative formula, recurrence relation, and integral formula. In addition, we described the relationships between these numbers and certain well-known special numbers, such as the Stirling numbers of the first and second kind, the degenerate first kind Apostol-Euler numbers, the degenerate Stirling numbers of the second kind, the Bernoulli numbers, and the classical Simsek numbers. To complement the theoretical developments, we included numerical results and graphical illustrations to demonstrate the behavior of the introduced numbers for various parameter choices. Consequently, the results of this paper have potential applications in different areas of mathematics, particularly in topics related to discrete mathematics.

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