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AFFINE-PERIODIC SOLUTIONS OF DISCRETE DYNAMICAL SYSTEMS: MASSERA'S CRITERION, AFFINE-PERIODIC FLOQUET DECOMPOSITION AND EXISTENCE RESULTS

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In this manuscript, we focus on discrete dynamical systems and propose some important results according to their affine periodic solutions. We reconsider two landmark results, namely Massera's theorem and Floquet's theorem, for linear discrete dynamical systems with respect to affine-periodicity. We adapt affine periodicity in Floquet's theory in order to obtain an affine-periodic Floquet's decomposition. We also examine the existence of (Q, T)-affine periodic solutions for certain kinds of difference systems.

1. INTRODUCTION

In the qualitative theory of dynamical systems, periodic structures, and analysis of periodic systems has been a fruitful research direction for scholars due to their remarkable potential for application in different fields, such as as biology, physics, economics, engineering sciences, etc. Analysis of discrete-time dynamical systems have taken great interest as much as the analysis of continuous-time dynamical systems since difference equations are the best outlet to study discrete phenomena in real-life models. Investigating periodic solutions of discrete dynamical

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systems have been a popular task of applied mathematics in the last three decades, and undoubtably, there is a vast literature based on the existence and uniqueness of the periodic solutions of difference equations and difference systems, so-called discrete dynamical systems. It is worth noting that the classical periodicity perception is a relaxable and generalizable concept for function classes. As a relaxation of conventional periodicity, we refer to almost periodic and almost automorphic functions, which are introduced at the beginning of century. On the other hand, we refer to affine periodicity, which is commonly used for vector-valued functions, as a generalization of the classical periodicity. Indeed, affine periodicity can be regarded as a special symmetry. property for functions, and it covers several periodicity notions involving anti-periodicity, rotating-periodicity, or quasi-periodicity in particular cases. This special periodicity (symmetry) perception, namely affine-periodicity, has received noticeable attention in the existing literature, and affine periodic solutions of continuous, discrete, and hybrid time dynamical systems have been extensively studied (see [**7**], [**8**], [**12**], [**16**], [**24**], [**29**], [**30**], [**31**], [**33**], [**36**]). Motivated by this hot topic, we put the discrete affine-periodicity notion at the center of this research. Affine periodicity arises naturally in several applied contexts where strict periodicity is too restrictive but recurrent patterns still appear up to a linear transformation. In population dynamics and ecology, seasonal variations often cause species or infection levels to repeat with alternating signs or rotated profiles depending on environmental conditions, leading to anti-periodic or rotation-type affine periodic behaviors. In engineering and control systems, repetitive motions or switching processes frequently return to states that are transformed versions of earlier ones, for example in robotics where a manipulator repeats tasks rotated by a fixed angle. In signal processing, alternating current waveforms provide a prototypical case of anti-periodicity, where the signal flips sign each half-period and thus satisfies $x(t + T) = -x(t)$. Even in economics and finance, business or market cycles may exhibit recurrent patterns that resemble previous cycles up to scaling or reflection, which can be effectively modeled within the affine periodicity framework. These examples illustrate that affine periodicity extends the notion of classical periodicity in a natural way and provides a flexible mathematical tool for describing recurrent phenomena in diverse real-world applications.

This manuscript is a multi-task research project that aims to fill the existing gaps. regarding affine-periodic solutions of discrete dynamical systems. Our motivation and the highlights of this work are summarized as follows:

i. Establishing a linkage between the existence of periodic solutions and the existence of bounded solutions has been always an interesting objective in the qualitative theory of dynamical systems. Based on the conventional periodicity, J. L. Massera gave the original result in his famous paper [**23**] which claims that a periodic equation has a bounded solution if and only if it has a periodic solution. This result is known as Massera's theorem in the applied mathematics. Besides, similar results were constructed for almost periodic and almost automorphic functions, and they are acknowledged as Favard's theorem and Bohr-Neugebauer theorem, respectively. By a quick literature review, one may easily find numerous papers providing Massera type theorems for almost periodic, almost automorphic, anti-periodic, and quasi-periodic solutions of dynamical equations and systems (see [**5**], [**6**], [**13**], [**21**], [**25**], [**27**]). It should be pointed out that authors of the recent paper [**17**] provide a Massera type theorem for affine-periodic, continuous-time dynamical systems by proposing an alternative boundedness concept for the solutions. As the first aim of this research, we adapt affine-periodicity to linear discrete dynamical systems, and propose a discrete counterpart of Massera's result with respect to the notion of affine-periodicity.

ii. Floquet theory is a landmark century-old theory which is introduced by the French mathematician G. Floquet in 1883 (see [**11**]). Basically, Floquet theory is one of the well-known tools in classical analysis to analyze solutions of linear dynamical systems with periodic coefficients. This theory has been reestablished by several researchers on continuous, discrete, and hybrid time domains for various kind of equations, such as summation, integral, integro-differential, and partial differential equations. In addition to conventional periodicity, this important theory has been constructed for almost periodic and quasi-periodic systems (see [**14**], [**19**], [**26**], [**35**]). To the best of our knowledge, there is no research which gathers affine-periodicity and Floquet theory. Motivated by this gap, we aim to initiate an affine-periodic Floquet theory by providing a discrete Floquet decomposition theorem for affine-periodic linear systems. We shall underline that Floquet theory has tremendous applications in quantum physics, classical physics, chemistry, and electronics. In our mathematical point of view, the introduced generalized Floquet decomposition has an important application potential in various disciplines since it covers already existing Floquet theorems on discrete-time domains in special cases.

iii. Dichotomies are very important tools for the stability theory, and they are commonly used for analyzing asymptotic behaviors of nonautonomous dynamical systems. Besides, dichotomy notion is utilized in various researches to focus on the existence of periodic solutions since the solutions of nonautonomous dynamical systems can be expressed in terms of Green's function on the complete line when the system possesses a dichotomy. It is possible to find a wide range of studies which handle the existence of periodic, almost periodic, almost automorphic, and also affine-periodic solutions by making dichotomy assumption on the systems. In the last part of this manuscript, we turn the spotlight on summable dichotomy perception which is introduced by M. Pinto in [**28**] and we obtain some technical results regarding affine-periodicity and boundedness. As the final task, we introduce an abstract form a discrete dynamical system that covers some well-known models in the applied sciences. Due to the summable dichotomy, we propose a mapping for the application of fixed point theory, and we get sufficient conditions for the existence of affine-periodic solutions of the introduced equation.

In the light of above-given literature review and discussions, we improve the qualitative theory of discrete dynamical systems and contribute the already established literature regarding affine-periodic solutions of difference equations. We organize the next section which contains the main outcomes as follows: At the beginning, we give some preliminary content about affine-periodic functions for the readership, and we represent our results in two phases. In the first phase, Subsection 2.1, we prove two landmark results, namely Massera's theorem and Floquet's theorem, for linear discrete dynamical systems with respect to affine-periodicity. Subsequently, Subsection 2.2 is devoted to summable dichotomy, and its utilization in existence results for discrete dynamical systems.

2. MAIN OUTCOMES

Throughout the manuscript, $\mathbb{Z}$ stands for the set of integers, $\mathbb{Z}_+$ indicates the set of positive integers, $\mathbb{N}_0$ notates the set of nonnegative integers and $\mathbb{Z}_-$ stands for the set of negative integers so that $\mathbb{Z} = \mathbb{N}_0 \cup \mathbb{Z}_-$.

As a preparation for the representation of the main results of the manuscript, we will consider the affine-periodicity and affine-symmetry definitions as the first task of this section. The following definitions can be found in [**24**] (see also [**16**]).

Definition 1. *A matrix-valued function $A : \mathbb{Z} \to \mathbb{R}^{n\times n}$ is said to be (Q, T)-affine periodic if there is a fixed $T \in \mathbb{Z}_+$ so that*

$$A(t+T) = QA(t)Q^{-1} \text{ for } Q \in GL(\mathbb{R}^n), \quad t \in \mathbb{Z},$$

where $GL(\mathbb{R}^n)$ indicates an n-dimensional linear group over $\mathbb{R}$. Besides, the linear discrete dynamical system

$$x(t+1) = A(t)x(t), \ t \in \mathbb{Z} \tag{1}$$

is called (Q, T)-affine periodic system if A is (Q, T)-affine periodic matrix function.

Definition 2. *A vector-valued function $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be (Q, T)-affine symmetric if there are $Q \in GL(\mathbb{R}^n)$ and $T \in \mathbb{Z}_+$ such that*

$$f(t+T, x) = Qf\left(t, Q^{-1}x\right) \text{ for all } t \in \mathbb{Z} \text{ and } x \in \mathbb{R}^n. \tag{2}$$

Moreover, the nonlinear system

$$x(t+1) = f(t, x(t)), \ t \in \mathbb{Z}$$

is called (Q, T)-affine periodic if $f(\cdot, \cdot)$ is (Q, T)-affine symmetric.

Definition 3. *A solution $x(\cdot)$ of the system (1) (or (2)) is called (Q, T)-affine periodic if*

$$x(t+T) = Qx(t) \text{ for all } t \in \mathbb{Z}.$$

It should be noted that, according to the particular choices of the matrix Q, the (Q,T)-affine periodicity notion coincides with the well-known concepts in the existing literature. We can address the linkage between (Q,T)-affine periodicity and conventional periodicity concepts by providing the following list:

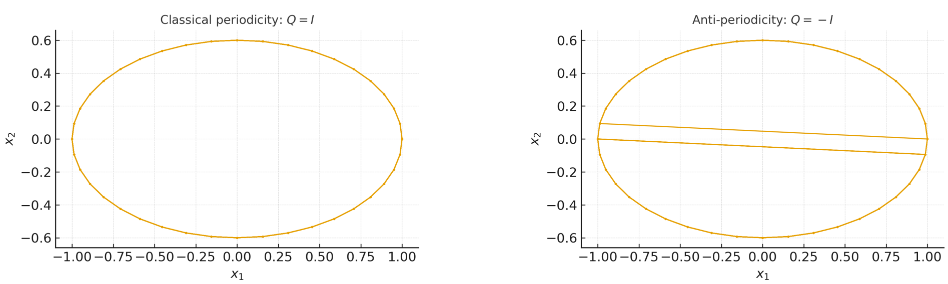
(i) If $Q = I_{n\times n}$, where $I_{n\times n}$ is the $n \times n$ identity matrix, then (Q,T)-affine periodicity turns into regular periodicity.

(ii) If $Q = -I_{n\times n}$, then (Q,T)-affine periodicity is equivalent to anti-periodicity.

(iii) If $Q \in O(n)$, that is Q is an orthogonal matrix, then (Q,T)-affine periodicity turns into rotating-periodicity (see [**20**], [**32**], [**34**]).

(iv) If $Q^N = I_{n\times n}$, then any (Q,T)-affine periodic function is indeed harmonic (see the discussion given in [**7**]).

(v) If $Q \in SO(n)$, where $SO(n)$ is the special orthogonal group of matrices with determinant 1, then (Q,T)-affine periodicity turns into quasi-periodicity (see the discussion given in [**7**]).



(a) Classical periodicity $(Q = I)$.

(b) Anti-periodicity $(Q = -I)$.

Figure 1: Geometrical interpretation of affine periodicity for two representative cases. (A) Classical periodicity: orbit repeats exactly after T steps. (B) Anti-periodicity: trajectory flips sign every T steps and is classically $2T$-periodic.

In addition to the two representative cases shown in Figure 1, it is instructive to consider a separate detailed example where Q is a rotation matrix. This case not only illustrates affine periodicity but also provides an explicit numerical construction that can be simulated directly. We present this as a standalone example below.

Example 1. *Consider the linear discrete system*

$$x(t+1) = Bx(t), \qquad t \in \mathbb{Z}, \tag{3}$$

where

$$B = \begin{bmatrix} \cos\frac{\pi}{2T} & -\sin\frac{\pi}{2T} \\ \sin\frac{\pi}{2T} & \cos\frac{\pi}{2T} \end{bmatrix}.$$

Then B represents a rotation by $\frac{\pi}{2T}$ radians in the plane. It follows that

$$B^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} =: Q,$$

the rotation matrix through $90°$. Hence the solution of (3) satisfies

$$x(t+T) = Q\,x(t), \qquad t \in \mathbb{Z},$$

which shows that the trajectory is (Q,T)-affine periodic. Geometrically, the orbit repeats its shape after T steps, rotated by $90°$. After four such periods we recover the starting position, i.e. $x(t+4T) = x(t)$ (see Figure 2).

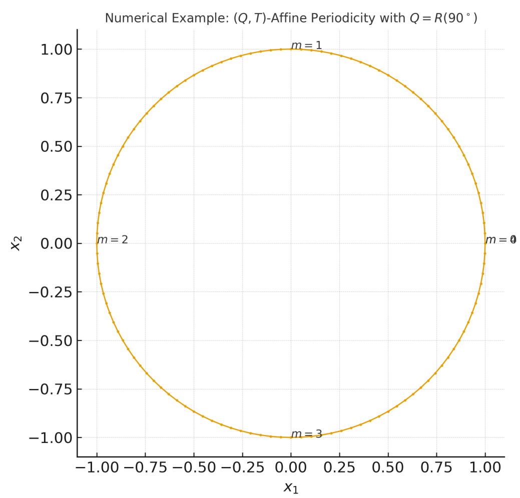


Figure 2: Numerical simulation of the system (3). The orbit is (Q,T)-affine periodic with $Q = R(90°)$. Labels $m = 0,1,2,3,4$ indicate successive period blocks.

Now, we are ready to exhibit the main results of the manuscript:

2.1 Two fundamental results

Discrete counterpart of Massera's theorem based on affine-periodicity

The main objective of this part is to propose a Massera type criterion for linear nonhomogeneous discrete dynamical systems. By introducing an alternative boundedness notion so-called $m_{(Q,T)}$-boundedness, we investigate the relationship between the existence of $m_{(Q,T)}$-bounded solutions and (Q,T)-affine periodic solutions for linear systems. First of all, we will slightly modify [**17**, Definition 1]:

Definition 4. *Let $T \in \mathbb{Z}_+$ be fixed, let $Q \in GL(\mathbb{R}^n)$ and let*

$$(4) \qquad m_T^{(t)} = \begin{cases} \max\{n \in \mathbb{N}_0 : t \geq nT\}, & t \in \mathbb{N}_0 \\ \max\{n \in \mathbb{Z}_- : t \geq nT\}, & t \in \mathbb{Z}_- \end{cases}.$$

Then, $x : \mathbb{Z} \to \mathbb{R}^n$ is called $m_{(Q,T)}$-bounded if

$$\sup_{t\in\mathbb{Z}} \left| Q^{-m_T^{(t)}} x(t) \right| < +\infty.$$

The next result plays a crucial role in the proof of the discrete Massera type theorem regarding affine-periodicity.

Theorem 1 (Brouwer's fixed point theorem)**.** *Every continuous function that maps a compact convex subset of an Euclidian space into itself has a fixed point.*

In the sequel, we consider the following discrete dynamical system:

$$x(t+1) = A(t)\,x(t) + f(t)\,,\ t \in \mathbb{Z}, \tag{5}$$

where A and $f(\cdot)$ are (Q,T)-affine periodic functions, i.e., $A(t+T) = QA(t)\,Q^{-1}$ and $f(t+T) = Qf(t)$ for all $t \in \mathbb{Z}$.

Theorem 2. *[A generalized discrete Massera theorem]The discrete dynamical system* (5) *has a (Q,T)-affine periodic solution if and only if it has an $m_{(Q,T)}$-bounded solution.*

Proof. Suppose that x is a (Q,T)-affine periodic solution of (5) and $M = \sup_{s\in[0,T)\cap\mathbb{Z}} |x(s)|$ for a fixed $T \in \mathbb{Z}_+$. Any arbitrary integer t can be represented as $t = m_T^{(t)} T + s$ due to (4), where $s \in [0,T) \cap \mathbb{Z}$. To deduce the $m_{(Q,T)}$-boundedness of $x(\cdot)$, we focus on

$$\left| Q^{-m_T^{(t)}} x(t) \right| = \left| Q^{-m_T^{(t)}} x\left(m_T^{(t)} T + s \right) \right| = \left| Q^{-m_T^{(t)}} Q^{m_T^{(t)}} x(s) \right| \le M.$$

This completes the proof of the necessity part.

On the other hand, assume that the discrete dynamical system (5) has an $m_{(Q,T)}$-bounded solution. Then the unique solution of (5) with the initial data $x(0) = x_0$ is given by the variation of constants formula (see [**15**]):

$$x(t, x_0) = X(t)\,x_0 + \sum_{k=0}^{t-1} X(t)\,X^{-1}(k+1)\,f(k)\,,$$

where X is the principal fundamental matrix solution for the homogeneous part of (5); that is,

$$X(t+1) = A(t)\,X(t) \ \text{ with } X(0) = I_{n\times n}.$$

As the setup for the utilization of the Brouwer's fixed point theorem, we introduce the following non-empty, bounded and closed set

$$\Lambda := \left\{ x_0 \in \mathbb{R}^n : |x_0| \le M \text{ and } \left| Q^{-m_T^{(t)}} x(t, x_0) \right| \le M \right\}. \tag{6}$$

Obviously, Λ is a compact set and the convexity of Λ can be easily shown. Pick some $\alpha \in [0,1]$ and choose $x_1,\ x_2 \in \Lambda$. Then $|\alpha x_1 + (1-\alpha) x_2| \leq M$ and

$$\begin{aligned}
&\left|Q^{-m_T^{(t)}} x\left(t, \alpha x_1 + (1-\alpha) x_2\right)\right| \\
&= \left|Q^{-m_T^{(t)}} \left(X(t)\left(\alpha x_1 + (1-\alpha) x_2\right) + \sum_{k=0}^{t-1} X(t) X^{-1}(k+1) f(k)\right)\right| \\
&= \left|Q^{-m_T^{(t)}} \left(\alpha \left(X(t) x_1 + \sum_{k=0}^{t-1} X(t) X^{-1}(k+1) f(k)\right)\right.\right. \\
&\left.\left. + (1-\alpha)\left(X(t) x_2 + \sum_{k=0}^{t-1} X(t) X^{-1}(k+1) f(k)\right)\right)\right| \\
&\leq \alpha \left|Q^{-m_T^{(t)}} x(t, x_1)\right| + (1-\alpha)\left|Q^{-m_T^{(t)}} x(t, x_2)\right| \\
&\leq M.
\end{aligned}$$

This implies that Λ is a convex set.

Next, we define the mapping $\Psi : \Lambda \to \mathbb{R}^n$ by

$$\Psi(x_0) := Q^{-1} x(T, x_0).$$

Note that for any $x_0 \in \Lambda$, we have $\Psi(x_0) \leq M$. At this point, we aim to show that Ψ maps Λ into itself. To achieve this task, we write

$$\begin{aligned}
&\left|Q^{-m_T^{(t)}} x\left(t, \Psi(x_0)\right)\right| \\
&= \left|Q^{-m_T^{(t)}} \left(X(t)\Psi(x_0) + \sum_{k=0}^{t-1} X(t) X^{-1}(k+1) f(k)\right)\right| \\
&= \left|Q^{-m_T^{(t)}} \left(X(t) Q^{-1} x(T, x_0) + \sum_{k=0}^{t-1} X(t) X^{-1}(k+1) f(k)\right)\right| \\
&= \left|Q^{-m_T^{(t)}} \left(X(t) Q^{-1}\left(X(T) x_0 + \sum_{k=0}^{T-1} X(T) X^{-1}(k+1) f(k)\right)\right.\right. \\
&\left.\left. + \sum_{k=0}^{t-1} X(t) X^{-1}(k+1) f(k)\right)\right|.
\end{aligned}$$

Here, we need to exhibit a helpful technical result to pursue the rest of the proof. We shall underline that a similar approach is already used in the proof of [**12**, Theorem 2]. Set $Y(t) := Q^{-1} X(t+T) X^{-1}(T) Q$. Then we obtain

$$\begin{aligned}
Y(t+1) &= Q^{-1} X(t+T+1) X^{-1}(T) Q \\
&= Q^{-1} A(t+T) X(t+T) X^{-1}(T) Q \\
&= Q^{-1} Q A(t) Q^{-1} X(t+T) X^{-1}(T) Q \\
&= A(t) Y(t).
\end{aligned}$$

By the uniqueness of the solutions, we get $Y(t) = X(t) = Q^{-1}X(t+T)X^{-1}(T)Q$, and consequently,

$$Q^{-1}X(t+T) = X(t)Q^{-1}X^{-1}(T). \tag{7}$$

Using (7), we arrive at

$$\begin{aligned}
&\left|Q^{-m_T^{(t)}}x(t,\Psi(x_0))\right| \\
&= \left|Q^{-m_T^{(t)}}\left(Q^{-1}X(t+T)x_0 + Q^{-1}X(t+T)\sum_{k=0}^{T-1}X^{-1}(k+1)f(k)\right.\right. \\
&\left.\left.+X(t)\sum_{k=T}^{t+T-1}X^{-1}(k-T+1)f(k-T)\right)\right|.
\end{aligned}$$

Notice that

$$X^{-1}(k-T+1) = Q^{-1}X(T)X^{-1}(k+1)Q$$

due to (7); since $f(\cdot)$ is (Q,T)-affine periodic, we finally get

$$f(k-T) = Q^{-1}f(k).$$

Thus,

$$\begin{aligned}
&\left|Q^{-m_T^{(t)}}x(t,\Psi(x_0))\right| \\
&= \left|Q^{-m_T^{(t)}}\left(Q^{-1}X(t+T)x_0 + Q^{-1}X(t+T)\sum_{k=0}^{T-1}X^{-1}(k+1)f(k)\right.\right. \\
&\left.\left.+Q^{-1}X(t+T)\sum_{k=T}^{t+T-1}X^{-1}(k+1)f(k)\right)\right| \\
&= \left|Q^{-m_T^{(t)}-1}\left[X(t+T)x_0 + X(t+T)\sum_{k=0}^{t+T-1}X^{-1}(k+1)f(k)\right]\right| \\
&= \left|Q^{-m_T^{(t)}-1}x(t+T,x_0)\right| \\
&\le M,
\end{aligned}$$

according to (6). This results in $\Psi : \Lambda \to \Lambda$. As a consequence of Brouwer's fixed point theorem, Ψ has a fixed point $x_0 \in \Lambda$ so that

$$\Psi(x_0) = Q^{-1}x(T,x_0) = x_0.$$

Now, we need to show $x(\cdot)$ is a (Q,T)-affine periodic solution of (5). We define

$$y(t) \equiv Q^{-1}x(t+T) - x(t);$$

then it is obvious that $y(0) = 0$. We have

$$\begin{aligned} y(t+1) &= Q^{-1}x(t+T+1) - x(t+1) \\ &= Q^{-1}\left(A(t+T)x(t+T) + f(t+T)\right) - A(t)x(t) - f(t) \\ &= Q^{-1}\left(QA(t)Q^{-1}x(t+T) + Qf(t)\right) - A(t)x(t) - f(t) \\ &= A(t)\left(Q^{-1}x(t+T) - x(t)\right) \\ &= A(t)y(t), \end{aligned}$$

so that $y(t) \equiv 0$. Consequentially, $x(\cdot)$ is (Q,T)-affine periodic and this completes the proof. □

Remark 1. *We shall highlight that Theorem 2 can be considered as a discretization of [**17**, Theorem 4]. When $Q = I$, Theorem 2 coincides with the landmark result of Massera (see [**23**]) on discrete time domains. Moreover, Theorem 2 can be related with the papers [**5**] and [**27**] in the particular cases $Q \in O(n)$ and $Q \in SO(n)$ on discrete time domains, respectively.*

Floquet's decomposition theorem based on affine periodicity

In this part, we adapt the notion of (Q,T)-affine periodicity to the Floquet theory, and propose an affine-periodic Floquet's theorem. We consider the affine-periodic linear discrete dynamical system

$$x(t+1) = A(t)x(t), \ t \in \mathbb{Z}, \tag{8}$$

which has the principal fundamental matrix solution X so that $X(0) = I_{n\times n}$. In order to develop (Q,T)-affine periodic Floquet decomposition for the linear system (8), we borrow the following results from the monograph [**15**]:

Lemma 1. *If $X(t)$ is a fundamental matrix solution for (8), then $Z(t)$ is another fundamental matrix if and only if there is a regular matrix C such that*

$$Z(t) = X(t)C \text{ for all } t \in \mathbb{Z}.$$

Lemma 2. *If C is a regular matrix and $T \in \mathbb{Z}_+$, then there is a regular matrix B such that $B^T = C$.*

Now, we ready to present a generalized Floquet decomposition.

Theorem 3. *[Affine periodic Floquet decomposition]Let X be the principal fundamental matrix of the affine periodic linear discrete dynamical system (8). Then $Z(t) = Q^{-1}X(t+T)$ is also a fundamental matrix for (8) and*

$$Z(t) = X(t)C \text{ for all } t \in \mathbb{Z},$$

where

$$C = Q^{-1}A(T-1)A(T-2)\cdots A(0).$$

Additionally, there exist a (Q, T)*-affine periodic matrix function* R *and a non-singular matrix* B *so that*

$$X(t) = R(t) QB^t.$$

Proof. We set $Z(t) = Q^{-1}X(t+T) = X(t) Q^{-1}X(T)$, where we used (7). We have

$$\begin{aligned} Z(t+1) &= Q^{-1}X(t+T+1) \\ &= Q^{-1}A(t+T) X(t+T) \\ &= Q^{-1}QA(t) Q^{-1}QX(t) Q^{-1}X(T) \\ &= A(t) X(t) Q^{-1}X(T) \\ &= A(t) Z(t), \end{aligned}$$

which shows that Z is a fundamental matrix for (8). Employing Lemma 1, we deduce that there exists a regular matrix C so that

$$Z(t) = Q^{-1}X(t+T) = X(t) C \text{ for all } t \in \mathbb{Z}. \tag{9}$$

On the other hand, if we iterate (8), then we obtain

$$X(T) = A(T-1) A(T-2) \cdots A(0).$$

Besides, we fix $t = 0$ in (9), and we get $X(T) = QC$. This yields to

$$C = Q^{-1}A(T-1) A(T-2) \cdots A(0).$$

In the sequel, we use the fact that there exists a regular matrix B such that $B^T = C$ as a consequence of Lemma 2. We introduce

$$R(t) = X(t) B^{-t}Q^{-1},$$

and focus on

$$\begin{aligned} R(t+T) &= X(t+T) B^{-t-T}Q^{-1} \\ &= QX(t) CB^{-T}B^{-t}Q^{-1} \\ &= QX(t) B^{-t}Q^{-1} \\ &= QR(t) Q^{-1}. \end{aligned}$$

Thus, $R(t)$ is (Q, T)-affine periodic, which completes the proof. □

Remark 2. *It is worth noting that Theorem 3 is the first of its kind which gathers affine periodicity and Floquet theory. However, we shall emphasize that one may establish a linkage between Theorem 3 and the papers [***26, 35***] in the particular case* $Q \in SO(n)$ *on discrete time domains.*

2.2 Existence and uniqueness results

Summable dichotomy and its consequences

Let us focus on the linear homogeneous discrete dynamical system

$$x(t+1) = A(t)\,x(t), \ \ t \in \mathbb{Z}, \tag{10}$$

with the fundamental matrix (or principal fundamental matrix) X. Then, the Green's function of the homogeneous system is given by

$$G(t,s) = \begin{cases} X(t)\,PX^{-1}(s+1), & \text{for } t \geq s+1 \\ X(t)\,(I-P)\,X^{-1}(s+1), & \text{for } t < s+1 \end{cases}, \tag{11}$$

for a projection P.

Subsequently, we give the following dichotomy definition which is crucial for the rest of the paper.

Definition 5 (Summable dichotomy [**3**, Definition 1])**.** *The linear discrete dynamical system* (10) *is said to admits a summable dichotomy on* $\mathbb{Z}$ *with the data* (μ, P) *if there is a projection* P *so that*

$$\sup_{t\in\mathbb{Z}} \sum_{k=-\infty}^{\infty} |G(t,k)| = \mu < \infty.$$

It should be noted that summable dichotomy can be regarded as an extension of the exponential dichotomy. If the system in (10) admits an exponential dichotomy, then actually it has a summable dichotomy. However, the converse statement may not be true. For an elaborative discussion about summable dichotomies, we refer to [**2**] and [**3**].

In the next result, we recall an important property about summable dichotomies.

Lemma 3 ([**3**, Proposition 6])**.** *If the linear discrete dynamical system* (10) *admits a summable dichotomy with the data* (μ, P), *then the projection* P *is unique.*

At this point, we are ready to present our initial result for this subsection which gathers summable dichotomy and affine-periodicity notions.

Lemma 4. *Suppose that the linear discrete dynamical system* (10) *is* (Q,T)*-affine periodic system, and it admits a summable dichotomy. Then the Green's function of* (10) *is* (Q,T)*-bi-affine periodic, that is*

$$G(t+T, s+T) = QG(t,s)\,Q^{-1} \ \textit{for all} \ (t,s) \in \mathbb{Z}\times\mathbb{Z}.$$

Proof. Assume that (10) is a (Q,T)-affine periodic system and (10) possesses a summable dichotomy with projection P. Set $\hat{P} \equiv Q^{-1}X(T)\,PX^{-1}(T)\,Q$. Then we get

$$\begin{aligned} X(t)\,\hat{P}X^{-1}(s+1) &= X(t)\,Q^{-1}X(T)\,PX^{-1}(T)\,QX^{-1}(s+1) \\ &= Q^{-1}X(t+T)\,PX^{-1}(s+T+1)\,Q \end{aligned}$$

and

$$\begin{aligned}
&X(t)\left(I-\hat{P}\right)X^{-1}(s+1)\\
&= X(t)\Big(I-Q^{-1}X(T)PX^{-1}(T)Q\Big)X^{-1}(s+1)\\
&= X(t)X^{-1}(s+1)-X(t)Q^{-1}X(T)PX^{-1}(T)QX^{-1}(s+1)\\
&= Q^{-1}X(t+T)X^{-1}(T)QQ^{-1}X(T)X^{-1}(s+T+1)Q\\
&- Q^{-1}X(t+T)PX^{-1}(s+T+1)Q\\
&= Q^{-1}X(t+T)(I-P)X^{-1}(s+T+1)Q,
\end{aligned}$$

where we utilized (7). Consequentially, we have a summable dichotomy for (10) with the projection $\hat{P}$ since $X(t+T)$ solves (10). Subsequently, we deduce that $\hat{P}=P$ due to Lemma 3. To conclude, we focus on

$$\begin{aligned}
X(t)PX^{-1}(s+1) &= X(t)\hat{P}X^{-1}(s+1)\\
&= X(t)Q^{-1}X(T)PX^{-1}(T)QX^{-1}(s+1)\\
&= Q^{-1}X(t+T)PX^{-1}(s+T+1)Q;
\end{aligned}$$

this yields to

$$QX(t)PX^{-1}(s+1)Q^{-1}=X(t+T)PX^{-1}(s+T+1).$$

Similarly,

$$QX(t)(I-P)X^{-1}(s+1)Q^{-1}=X(t+T)(I-P)X^{-1}(s+T+1).$$

Thus, the Green's function is (Q,T)-bi-affine periodic and the proof is complete. □

In the next result, we motivated by [**17**, Theorem 2], and we highlight the relationship between the summable dichotomy and the $m_{(Q,T)}$-bounded solutions for the nonhomogeneous (Q,T)-affine periodic discrete dynamical system (5).

Theorem 4. *Suppose that the homogeneous part of the* (Q,T)*-affine periodic system* (5) *admits a summable dichotomy with data* (μ,P). *Then* (5) *has an* $m_{(Q,T)}$*-bounded solution if the nonhomogeneous term* $f(\cdot)$ *of* (5) *is bounded, i.e.* $\sup_{t\in\mathbb{Z}}|f(t)|<\infty$.

Proof. Suppose that the function $f(\cdot)$ in (5) is bounded with $\sup_{t\in\mathbb{Z}}|f(t)|\le\beta$ for a positive constant β. Additionally, assume that the homogeneous part of (5) admits a summable dichotomy with data (μ,P). Then, it is obvious that the solution $x(\cdot)$ of (5) has a representation

$$x(t)=\sum_{k=-\infty}^{\infty}G(t,k+1)f(k). \tag{12}$$

On the other hand, we use (4) and express any $t \in \mathbb{Z}$ as $t = m_T^{(t)} T + s$ for $s \in [0, T) \cap \mathbb{Z}$. Thence, we rewrite (12) as follows:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} G\left(m_T^{(t)} T + s, k+1\right) f(k) \\ &= \sum_{k=-\infty}^{\infty} G\left(m_T^{(t)} T + s, m_T^{(t)} T + k + 1\right) f\left(m_T^{(t)} T + k\right). \end{aligned}$$

By Lemma 4, we already observed that G is (Q, T)-bi-affine periodic, and therefore it satisfies

$$G\left(m_T^{(t)} T + s, m_T^{(t)} T + k + 1\right) = Q^{m_T^{(t)}} G(s, k+1) Q^{-m_T^{(t)}}.$$

Accordingly, we obtain

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} Q^{m_T^{(t)}} G(s, k+1) Q^{-m_T^{(t)}} Q^{m_T^{(t)}} f(k) \\ &= Q^{m_T^{(t)}} \sum_{k=-\infty}^{\infty} G(t, k+1) f(k), \end{aligned} \tag{13}$$

where we also utilized (Q, T)-affine periodicity of $f(\cdot)$. To conclude, we multiply both sides of (13) with $Q^{-m_T^{(t)}}$ from the left, and write

$$\left| Q^{-m_T^{(t)}} x(t) \right| \leq \sum_{k=-\infty}^{\infty} |G(t, k+1)| \, |f(k)| \leq \mu\beta.$$

This implies that (5) has an $m_{(Q,T)}$-bounded solution. □

A certain kind of discrete dynamical system

Let $AP_{(Q,T)}$ stands for the set of all affine-periodic functions, which is explicitly given by

$$AP_{(Q,T)} := \{x : \mathbb{Z} \to \mathbb{R}^n : x(t+T) = Qx(t) \text{ for all } t \in \mathbb{Z}\}.$$

Then $AP_{(Q,T)}$ is a Banach space endowed with the norm

$$\|x\|_{AP_{(Q,T)}} = \sup_{t \in [0,T] \cap \mathbb{Z}} |x(t)|.$$

Inspired by [**4**], we propose the following discrete dynamical system

$$x(t+1) = A(t) x(t) + \sum_{j=-\infty}^{t-1} U_1(t, j, x(j)) + \sum_{j=t}^{\infty} U_2(t, j, x(j)), \tag{14}$$

where $A:\mathbb{Z}\to\mathbb{R}^{n\times n}$ and $U_{1,2}:\mathbb{Z}\times\mathbb{Z}\times\mathbb{R}^n\to\mathbb{R}^n$. It should be noted that it is reasonable to study such an abstract form of a discrete dynamical system since it covers some important models under particular choice of the coefficients. For instance, if we set $U_2=0$ and $U_1(t,j,x(j))=B(t,j)\,x(j)$, then (14) turns into a Volterra difference system with infinite delay

$$x(t+1)=A(t)\,x(t)+\sum_{j=-\infty}^{t-1}\Omega(t,j)\,x(j),$$

which is extensively studied in the existing literature (see [**9**]).

We make the following assumptions on (14) for the construction of our existence results:

C1 The homogeneous part of (14) admits a summable dichotomy with data (μ,P).

C2 A is a (Q,T)-affine periodic matrix function.

C3 $U_{1,2}$ are (Q,T)-bi-affine symmetric functions; that is,

$$U_{1,2}(t+T,s+T,x)=QU_{1,2}\left(t,s,Q^{-1}x\right)\text{ for all }(t,s)\in\mathbb{Z}\times\mathbb{Z}.$$

C4 For all $x,\ y\in\mathbb{R}^n$, the following inequalities

$$\begin{aligned}|U_1(t,s,x)-U_1(t,s,y)|&\leq w_1(t,s)\,|x-y|,\\|U_2(t,s,x)-U_2(t,s,y)|&\leq w_2(t,s)\,|x-y|\end{aligned}$$

hold with

$$\sup_{t\in\mathbb{Z}}\sum_{j=-\infty}^{t-1}w_1(t,j)=W_1<\infty$$

and

$$\sup_{t\in\mathbb{Z}}\sum_{j=t}^{\infty}w_2(t,j)=W_2<\infty.$$

At this stage, we propose the following results regarding the existence and uniqueness of the (Q,T)-affine periodic solutions of (14) under certain conditions by using classical tools of the fixed point theory.

Theorem 5. *Suppose that* ***(C1-C4)*** *hold. If*

$$\mu(W_1+W_2)=\gamma<1,\tag{15}$$

then the discrete dynamical system (14) *has a unique* (Q,T)*-affine periodic solution.*

Proof. Assume that the conditions (**C1-C4**) and the inequality (15) are satisfied. First of all, we introduce the mapping

$$(Hx)(t) := \sum_{k=-\infty}^{\infty} G(t, k+1) V(k, x(k)),$$

where G is as in (11), and V is given by

$$V(k, x(k)) := \sum_{j=-\infty}^{k-1} U_1(k, j, x(j)) + \sum_{j=k}^{\infty} U_2(k, j, x(j)). \tag{16}$$

Initially, we observe that V is (Q,T)-affine symmetric. To see this, we write

$$\begin{aligned} V(k+T, x) &= \sum_{j=-\infty}^{k+T-1} U_1(k+T, j, x) + \sum_{j=k+T}^{\infty} U_2(k+T, j, x) \\ &= \sum_{j=-\infty}^{k-1} U_1(k+T, j+T, x) + \sum_{j=k}^{\infty} U_2(k+T, j+T, x). \end{aligned}$$

By (**C3**), we have

$$\begin{aligned} V(k+T, x) &= \sum_{j=-\infty}^{k-1} QU_1\left(k, j, Q^{-1}x\right) + \sum_{j=k}^{\infty} QU_2\left(k, j, Q^{-1}x\right) \\ &= QV\left(k, Q^{-1}x\right). \end{aligned}$$

This results in $H : AP_{(Q,T)} \to AP_{(Q,T)}$ as an implementation of Lemma 4. To apply the Banach fixed point theorem, it remains to show that H is a contraction. To succeed this task, we pick $x,\ y \in AP_{(Q,T)}$; then we have

$$\begin{aligned} &\|(Hx)(t) - (Hy)(t)\|_{AP_{(Q,T)}} \\ &= \left\| \sum_{k=-\infty}^{\infty} G(t, k+1)\left(V(k, x(k)) - V(k, y(k))\right) \right\|_{AP_{(Q,T)}} \\ &\le \sup_{t \in [0,T] \cap \mathbb{Z}} \sum_{k=-\infty}^{\infty} |G(t, k+1)|\, |V(k, x(k)) - V(k, y(k))|. \end{aligned}$$

In particular, it can be easily seen that

$$\begin{aligned}|V(k,x)-V(k,y)| &\leq \sum_{j=-\infty}^{k-1}|U_1(k,j,x)-U_1(k,j,y)|\\ &+\sum_{j=k}^{\infty}|U_2(k,j,x)-U_2(k,j,y)|\\ &\leq \sum_{j=-\infty}^{k-1}w_1(k,j)|x-y|+\sum_{j=k}^{\infty}w_2(k,j)|x-y|\\ &\leq (W_1+W_2)|x-y|.\end{aligned}$$

Thus we get

$$\begin{aligned}\|(Hx)(t)-(Hy)(t)\|_{AP_{(Q,T)}} &\leq \mu(W_1+W_2)\|x-y\|_{AP_{(Q,T)}}\\ &=\gamma\|x-y\|_{AP_{(Q,T)}}.\end{aligned}$$

The Banach fixed point theorem implies that H has a unique fixed point, and consequentially, the discrete dynamical system (14) has a unique (Q,T)-affine periodic solution. □

Afterwards, we recall the Schauder's fixed point theorem in order to develop our second existence result.

Theorem 6 (Schauder)**.** *Let $\mathbb{B}$ be a Banach space. Assume that K is a closed, bounded and convex subset of $\mathbb{B}$. If $T: K \to K$ is a compact operator, then it has a fixed point in K.*

Let us introduce the following set

$$\Theta_M := \left\{x \in AP_{(Q,T)} : \|x\|_{AP_{(Q,T)}} \leq M\right\},$$

which is a bounded, closed and convex subset of $AP_{(Q,T)}$ for a fixed positive constant M.

Theorem 7. *In addition to (**C1-C4**), suppose also that*

C5 $U_1(t,s,0)=U_2(t,s,0)=0$ *for all* $(t,s)\in\mathbb{Z}\times\mathbb{Z}$.

Then, the system (14) *has a (Q,T)-affine periodic solution in Θ_M if*

$$\mu(W_1+W_2)\leq 1. \tag{17}$$

Proof. Assume that (**C1-C5**) and (17) are satisfied. By Theorem 5, we know that $H: AP_{(Q,T)} \to AP_{(Q,T)}$ under the conditions (**C1-C4**). Hence, showing $H:\Theta_M\to\Theta_M$ is equivalent to proving $\|Hx\|_{AP_{(Q,T)}}\leq M$. We shall underline that

$$|U_1(t,s,x)|\leq w_1(t,s)|x|,$$

and

$$|U_2(t,s,x)| \leq w_2(t,s)|x|$$

due to (**C4**) and (**C5**). This indicates

$$\begin{aligned}|V(k,x)| &\leq \sum_{j=-\infty}^{k-1} |U_1(k,j,x)| + \sum_{j=k}^{\infty} |U_2(k,j,x)| \\ &\leq \sum_{j=-\infty}^{k-1} w_1(k,j)|x| + \sum_{j=k}^{\infty} w_2(k,j)|x| \\ &\leq (W_1+W_2)|x|,\end{aligned}$$

where V is as in (16). Subsequently, we choose $x \in \Theta_M$ and consider

$$\begin{aligned}\|Hx\|_{AP_{(Q,T)}} &= \sup_{t\in[0,T]\cap\mathbb{Z}} \sum_{k=-\infty}^{\infty} |G(t,k+1)|\,|V(k,x(k))| \\ &\leq \sup_{t\in[0,T]\cap\mathbb{Z}} \sum_{k=-\infty}^{\infty} |G(t,k+1)|(W_1+W_2)|x(k)| \\ &\leq \mu(W_1+W_2)M \leq M.\end{aligned}$$

So $H : \Theta_M \to \Theta_M$.

Next, we have to show the continuity of H. We take $x,\ y \in \Theta_M$; if $\|x-y\|_{AP_{(Q,T)}} < \delta$, then we obtain

$$\begin{aligned}\|Hx-Hy\|_{AP_{(Q,T)}} &\leq \sup_{t\in[0,T]\cap\mathbb{Z}} \sum_{k=-\infty}^{\infty} |G(t,k+1)|\,|V(k,x(k)) - V(k,y(k))| \\ &\leq \sup_{t\in[0,T]\cap\mathbb{Z}} \sum_{k=-\infty}^{\infty} |G(t,k+1)|(W_1+W_2)|x(k)-y(k)| \\ &\leq \mu(W_1+W_2)\|x-y\|_{AP_{(Q,T)}}.\end{aligned}$$

By setting $\delta = \frac{\varepsilon}{\mu(W_1+W_2)}$, we prove the continuity of H.

Finally, we will prove the pre-compactness of the set $H(\Theta_M)$ by diagonalization process. Indeed, this part of the proof is exactly identical with the corresponding part of the proof of [**16**, Theorem 4]. Basically, if we pick a sequence $\{x^n\}_{n\in\mathbb{Z}_+} \in \Theta_M$, then we have a convergent subsequence $\{x^n(t_k)\}$. If this procedure is repeated for all $n \in \mathbb{Z}_+$, then one may easily construct a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ in Θ_M. The continuity of H results in the sequence $H\{x^n\}$ has a convergent subsequence in $H\{\Theta_M\}$; therefore, $H\{\Theta_M\}$ is precompact. The proof is complete by Schauder's theorem. □

3. APPLICATIONS

Example 2. *Let us revisit [**16**, Example 1], and consider the Keynesian cross economic model with lagged income*

$$D(t) = C(t) + I(t-1) + G(t-1) \tag{18}$$

$$C(t) = cx(t) \tag{19}$$

$$x(t+1) = \delta D(t+1) + x(t)(1-\delta), \tag{20}$$

*where c is a nonnegative constant, D is aggregate demand, x is aggregate income, C is aggregate consumption, I is aggregate investment, G is government spent and $\delta < 1$ is speed of adjustment term (see [**10**, Page 23]). We shall emphasize that, we suppose that the demand is affected by the previous period's investment and government spent, and also there is no fixed consumption. As it is done in [**10**], we rewrite (18-20) as*

$$x(t+1) = \frac{1-\delta}{1-\delta c} x(t) + \frac{\delta}{1-\delta c}(I(t) + G(t)) \tag{21}$$

by assuming $\delta c \neq 1$. When $c \neq 1$, the homogeneous part of equation (21), admits a summable dichotomy. By choosing I and G are both (Q,T)-affine periodic, we employ Lemma 4 and observe that (21) has a (Q,T)-affine periodic solution. In conclusion, Theorem 2 implies that Keynesian Cross economic model has an $m_{(Q,T)}$-bounded solution.

Remark 3. *In the Keynesian framework of Example 2, the state variables represent macroeconomic aggregates such as output, consumption, and investment. Classical periodicity would correspond to these variables repeating exactly after each cycle, which is often too restrictive for real economies. Affine periodicity provides a more flexible interpretation: after each cycle of length T, the system returns not to the same state but to a transformed state $Qx(t)$.*

Economically, this means that business cycles may reappear in a modified form. For instance, if $Q = -I$, the economy alternates between expansion and contraction in successive cycles (anti-periodicity). If Q is a rotation-type transformation, then the cycle re-emerges with variables playing shifted roles, e.g. investment and consumption exchanging their relative positions. In this way, affine periodicity generalizes the notion of purely periodic cycles and captures more realistic recurrent behavior in Keynesian models.

Example 3. *Consider the following linear discrete dynamical system*

$$\begin{cases} \Delta x_1(t) = -k_1 x_1(t) + f_1(t) \\ \Delta x_2(t) = k_1 x_1(t) - k_2 x_2(t) + f_2(t) \end{cases}, \ t \in \mathbb{Z}, \tag{22}$$

where Δ is the forward difference operator, and $k_{1,2}$ are nonzero constants. We rewrite the system (22) as follows

$$x(t+1) = \begin{bmatrix} 1-k_1 & 0 \\ k_1 & 1-k_2 \end{bmatrix} x(t) + f(t), \ t \in \mathbb{Z} \tag{23}$$

so that

$$x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \text{ and } f(t) := \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

Note that (22) is a perturbation of the linear discrete dynamical system

$$\begin{cases} \Delta x_1(t) = -k_1 x_1(t) \\ \Delta x_2(t) = k_1 x_1(t) - k_2 x_2(t) \end{cases}, \ t \in \mathbb{Z},$$

*which can be utilized to model blood alcohol level on discrete time by taking x_1 is the concentration of alcohol in stomach and x_2 is the concentration of alcohol in blood (see [**22**]). Thus, it is reasonable to focus on the system (22) since it has a medical background.*

As the setup of our example, we assume that

$$f_1(t+T) = c_1 f_1(t) \text{ and } f_2(t+T) = c_2 f_2(t) \text{ for all } t \in \mathbb{Z},$$

and this implies f is (Q,T)-affine periodic with the matrix

$$Q = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.$$

*Moreover, we assume that f is bounded, and this assumption is quite logical when we take the medical motivation into account. On the other hand, the coefficient matrix of the system (23) admits an exponential dichotomy (consequentially summable dichotomy) since none of its eigenvalues lie on the unit circle (see [**1**, Section 5.8]). To conclude, we deduce that all conditions of Theorem 4 are satisfied, and the affine-periodic linear system (22) has an $m_{(Q,T)}$-bounded solution.*

Example 4. *Consider the following Volterra difference system with delay*

$$x(t+1) = A(t)x(t) + \sum_{j=-\infty}^{t-1} \Omega(t,j)x(j), \ t \in \mathbb{Z}, \tag{24}$$

where A and $\Omega(t,j) = [\Omega_{ik}(t,j)]$ are $n \times n$ matrix functions. If we compare (24) with (14), then we observe that

$$U_1(t,j,x) = \Omega(t,j)x, \tag{25}$$

and

$$U_2(t,j,x) = 0.$$

We make the following assumptions on (24):

C6 *A is (Q,T)-affine periodic matrix function,*

C7 *The homogeneous part of (24) admits a summable dichotomy with data (μ, P),*

C8 *The matrix function $\Omega(t,j)$ is (Q,T)-bi-affine periodic, that is*

$$\Omega(t+T, j+T) = Q\Omega(t,j)\,Q^{-1},$$

C9 $\sup_{t\in\mathbb{Z}} \sum_{j=-\infty}^{t-1} |\Omega(t,j)| = W_\Omega < \infty.$

*Obviously, $U_1(t,j,x)$ given in (25) is (Q,T)-bi-affine symmetric due to (**C8**). To see this, we write*

$$\begin{aligned} U_1(t+T, j+T, x) &= \Omega(t+T, j+T)\,x \\ &= Q\Omega(t,j)\,Q^{-1}x \\ &= QU_1\left(t, j, Q^{-1}x\right). \end{aligned}$$

Then, the infinite delayed Volterra system (24) has a unique (Q,T)-affine periodic solution as a consequence of Theorem 5 whenever $\mu W_\Omega < 1$.

To be more specific, we give the next example.

Example 5. *Let us consider the following 2-D discrete dynamical system*

$$\begin{aligned} x(t+1) = \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \sum_{j=-\infty}^{t-1} \left(\begin{bmatrix} \frac{1}{9}2^{j-t} & 0 \\ 0 & \frac{1}{7}3^{j-t} \end{bmatrix} \begin{bmatrix} x_1(j) \\ x_2(j) \end{bmatrix} + \begin{bmatrix} e^{-t}\cos(\pi j) \\ 0 \end{bmatrix} \right). \end{aligned} \tag{26}$$

By comparing (26) with (14), we get that

$$A(t) = \frac{1}{2} I_{2\times 2},$$

and

$$U_1(t,j,x) = \begin{bmatrix} e^{-t}\cos(\pi j) + \frac{1}{9}2^{j-t}x_1 \\ \frac{1}{7}3^{j-t}x_2 \end{bmatrix}.$$

Then, it can be easily verified that $U_1(t,j,x)$ is $(Q,2)$-bi-affine symmetric for $Q = e^{-2}I_{2\times 2}$. Then, we write

$$\begin{aligned} |U_1(t,j,x) - U_1(t,j,y)| &= \left| \begin{bmatrix} \frac{1}{9}2^{j-t}(x_1 - y_1) \\ \frac{1}{7}3^{j-t}(x_2 - y_2) \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} \frac{1}{9}2^{j-t} & 0 \\ 0 & \frac{1}{7}3^{j-t} \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} \right| \\ &\le W_1 |x - y| \end{aligned}$$

for any $x, y \in \mathbb{R}^2$. Here, we obtain $W_1 = \frac{1}{9}$ by using the matrix norm

$$\|N\| = \max_{1\le i\le n} \sum_{j=1}^{n} |n_{ij}|.$$

On the other hand, the homogeneous part of (26) admits a summable dichotomy with the data $(3, I_{2\times 2})$. *This implies that all conditions of Theorem 5 hold, and (26) has a unique* $(Q, 2)$*-affine periodic solution.*

4. CONCLUSIONS AND FINAL REMARKS

This study provided a detailed analysis based on affine-periodic solutions of discrete dynamical systems, and it improved and contributed to the qualitative theory of discrete dynamical systems. Indeed, the present work can be considered as a continuation of the papers[**16**] and [**24**] which initiated the discussion of the affine-periodicity notion for difference equations. In our analysis, we revisited two classical results, Massera's theorem and Floquet's theorem, by proving versions of them for linear systems with respect to affine-periodicity. Also, a specific form of difference systems are handled, and the sufficient conditions for the existence of their affine-periodic solutions are investigated. For the construction of the solutions of systems, we utilized summable dichotomy and obtained some important results which may play an important role for future research. The existence theorems are proved due to the classical tools of fixed point theory, namely contraction mapping principle and Schauder's fixed point theorem, and this enabled us to obtain the technical conditions behind the existence results swiftly and elementarily.

Below, we list our concluding remarks, and future directions:

(A) Floquet theory plays an important role in the theory of dynamical systems. In this work, we only give an affine-periodic Floquet decomposition for the sake of brevity. Establishing a complete affine-periodic Floquet theory involving stability analysis can be an interesting future research topic. The same can be said idea can be extended to dynamical systems on continuous time-domains.

(B) We shall highlight that the reverse of the statement in Theorem 4 is not clear. Existence of an $m_{(Q,T)}$-bounded solution of (5) may not imply that homogeneous part of (5) possesses a summable dichotomy. It may be a nice task to clarify this ambiguity in future works.

(C) Summable dichotomy can be fruitful research subject for mathematicians working in applied mathematics. To the best of our knowledge, the sufficient conditions for its existence and uniqueness have not been investigated yet. This will make an interesting research project that will narrow the gap concerning the qualitative theory of discrete dynamical systems.

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