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MONOTONICITY RULES FOR THE RATIO OF POWER SERIES WITH APPLICATIONS

*Zhong-Xuan Mao and Jing-Feng Tian**

In this paper, we present monotonicity rules for the ratio of two power series $x \mapsto \sum_{k=0}^{\infty} a_k x^k \big/ \sum_{k=0}^{\infty} b_k x^k$ under the assumption that the sequence $\{a_k/b_k\}_{k\geq 0}$ changes monotonicity twice. We also prove that the number of monotonicity changes of this function does not exceed that of the sequence $\{a_k/b_k\}_{k\geq 0}$. As an application, we establish a necessary and sufficient condition for the monotonicity of a function involving the confluent hypergeometric function of the first kind, $x \mapsto \frac{e^{-x}}{x+1} M(a, b, cx)$, where $a, b, c > 0$.

1. INTRODUCTION

Monotonicity rules play a foundational role in analysis and are widely applied in approximation theory, differential geometry, information theory, probability, and statistics. Notably, they serve as essential tools in the research of many special functions, such as the Gaussian hypergeometric function, the confluent hypergeometric function, elliptic integrals, and modified Bessel functions. In this paper, we primarily focus on monotonicity rules related to the ratio of two power series.

We highlight two important motivations for researching the ratio of power series. First, by Taylor's theorem, the ratio of any two functions can be expressed in terms of the ratio of power series. Second, the essential generating functions in

*Corresponding author. Jing-Feng Tian
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probability theory, number theory, and combinatorial mathematics often take the form of power series.

The origins of the monotonicity rules for the ratio of functions can be attributed to a lemma [**4**, Lemma 1] that was initially employed by Biernacki and Krzyż in their studies on differential geometry.

Monotonicity rule 1. *Let real power series* $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $\mathcal{B}(x) = \sum_{k=0}^{\infty} b_k x^k$ *converge on* $(-r, r)$ *with* $b_k > 0$. *If the sequence* $\{a_k/b_k\}_{k\geq 0}$ *is increasing, then the function*

$$x \mapsto \frac{\mathcal{A}(x)}{\mathcal{B}(x)} = \frac{\sum_{k=0}^{\infty} a_k x^k}{\sum_{k=0}^{\infty} b_k x^k} \tag{1}$$

is increasing on $(0, r)$.

After that, in 2010, Baricz [**2**] proposed an open problem.

Open problem 2. *If the sequence* $\{a_k/b_k\}_{k\geq 0}$ *is (strictly) convex (concave), does it follow that the function* $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ *is also (strictly) convex (concave) on* $(0, r)$*?*

Unfortunately, to date, no scholars have provided a definitive positive or negative answer to this question. As Baricz [**2**] noted, this problem is quite difficult to solve, yet solving it could yield simple and unique proofs for many other interesting open and known problems in the theory of special functions. Thus, it has motivated researchers to develop various monotonicity rules [**5, 6, 8, 9, 19, 21**]

, which have been actively applied to the study of special functions [**3, 12, 13, 14, 15, 16, 20, 22, 23, 24, 25, 26, 27**]. Some results also have been further employed in the qualitative analysis of partial differential equations : [**10, 17**].

From another perspective, in 2015, Yang, Chu and Wang [**18**] provided the following interesting monotonicity rule, which considers a more complex case that the monotonicity of $\{a_k/b_k\}_{k\geq 0}$ changes once.

Monotonicity rule 3. *Let real power series* $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $\mathcal{B}(x) = \sum_{k=0}^{\infty} b_k x^k$ *converge on* $(-r, r)$ *with* $b_k > 0$. *If there exists a given integer* $m \geq 1$ *such that the sequence* $\{a_k/b_k\}$ *is increasing (decreasing) for all* $0 \leq k \leq m$ *and decreasing (increasing) for all* $k \geq m$ *with neither* $\{a_k/b_k\}_{0\leq k\leq m}$ *nor* $\{a_k/b_k\}_{k\geq m}$ *is constant, then*

(1) *the function* $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ *is increasing (decreasing) on* $(0, r)$ *if and only if* $H_{\mathcal{A},\mathcal{B}}(r^-) \geq (\leq) 0$,

(2) *there exists* $x_0 \in (0, r)$ *such that the function* $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ *is increasing (decreasing) on* $(0, x_0]$ *and decreasing (increasing) on* $[x_0, r)$ *if* $H_{\mathcal{A},\mathcal{B}}(r^-) < (>) 0$,

where the function $H_{f,g}$ *is defined by* $H_{f,g} := \frac{f'}{g'} g - f$, *which is called Yang's* H*-function (named in [**11**]).*

According to the number of times the monotonicity of $\{a_k/b_k\}$ undergoes changes, a research motivation occurs: consider that case that such monotonicity changes $n(n \geq 2)$ times. The extension is both valuable and challenging. Its significance lies in enabling us to study more complex functions using the extended rules. Thus, the primary objective of this paper is to establish monotonicity rules for the ratio of power series when the monotonicity of the sequence $\{a_k/b_k\}$ changes twice, and to provide an application whose monotonicity can be analyzed using the proposed rules.

This paper is structured as follows: In Section 2, we will establish the monotonicity rules for

$$x \mapsto \frac{\mathcal{A}(x)}{\mathcal{B}(x)} = \frac{\sum_{k=0}^{\infty} a_k x^k}{\sum_{k=0}^{\infty} b_k x^k}, \tag{2}$$

under the case that the monotonicity of $\{a_k/b_k\}$ change twice. We also provide a local monotonicity rule in this section. In section 3, by employing monotonicity rules, we will establish the necessary and sufficient conditions for the function

$$R_{a,b,c}(x) := \frac{M(a,b,cx)}{M(2,1,x)} = \frac{e^{-x} M(a,b,cx)}{x+1}, \tag{3}$$

to be monotonic, where $M(a,b,x)$ is the confluent hypergeometric function of the first kind, and $a,b,c > 0$ are constants.

2. MONOTONICITY RULES FOR THE RATIO OF POWER SERIES

We begin with a complete introduction to Yang's H-function [**18**]. Let $-\infty \leq a < b \leq \infty$, functions F and G be differentiable on (a,b), and $G \neq 0$ on (a,b). Yang's H function is defined by

$$H_{F,G} = \frac{F'}{G'}G - F.$$

It is easy to check that the following two formulas hold

$$\left(\frac{F}{G}\right)' = \frac{G'}{G^2} H_{F,G},$$

and

$$H'_{F,G} = \left(\frac{F'}{G'}\right)' G,$$

where the second identity requires both F and G are twice differentiable. In what follows, we also denote that $[a,b]_{\mathbb{N}} := [a,b] \cap \mathbb{N}$, where $\mathbb{N} := \{0,1,2,\cdots\}$ is the set of natural numbers. Hereafter, $\mathcal{A}'$ and $\mathcal{B}'$ are the derivatives of functions $\mathcal{A}$ and $\mathcal{B}$, respectively. We also use the forms $\mathcal{A}^{(k)}$ and $\mathcal{B}^{(k)}$ to respectively represent the k-th order derivatives of $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}^{(0)}$ and $\mathcal{B}^{(0)}$ mean $\mathcal{A}$ and $\mathcal{B}$, respectively.

2.1 The monotonicity of $\{a_k/b_k\}$ changes twice

First, we establish the monotonicity rule for function (2), where the monotonicity of $\{a_k/b_k\}$ changes twice. For convenience, we introduce some notations

(i) arrows "$\nearrow$" and "$\searrow$" respectively denote "increasing" and "decreasing",

(ii) arrows "$\nearrow\searrow$" and "$\searrow\nearrow$" respectively denote "there exist $x_1 \in (0, r)$ such that the function increasing on $(0, x_1]$ and decreasing on $[x_1, r)$" and "there exist $x_1 \in (0, r)$ such that the function decreasing on $(0, x_1]$ and increasing on $[x_1, r)$",

(iii) arrows "$\nearrow\searrow\nearrow$" and "$\searrow\nearrow\searrow$" respectively denote "there exist $x_2, x_3 \in (0, r)$ such that the function increasing on $(0, x_2] \cup [x_3, r)$ and decreasing on $[x_2, x_3]$" and "there exist $x_2, x_3 \in (0, r)$ such that the function decreasing on $(0, x_2] \cup [x_3, r)$ and increasing on $[x_2, x_3]$".

Monotonicity rule 4. *Let real power series $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\mathcal{B}(x) = \sum_{k=0}^{\infty} b_k x^k$ converge on $(0, r)$ with $b_k > 0$. If there exists different integers $m_2 > m_1 \geq 1$ such that the sequence $\{a_k/b_k\}$ is increasing (decreasing) for all $k \in [0, m_1]_{\mathbb{N}} \cup [m_2, \infty)_{\mathbb{N}}$ and decreasing (increasing) for all $k \in [m_1, m_2]_{\mathbb{N}}$, as well as $\{a_k/b_k\}_{0 \leq k \leq m_1}$, $\{a_k/b_k\}_{m_1 \leq k \leq m_2}$, and $\{a_k/b_k\}_{k \geq m_2}$ are non-constant, then*

(C1) the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" ("$\searrow$") on $(0, r)$ if one of the following conditions holds

(i) $H_{\mathcal{A},\mathcal{B}}(r^-) \geq (\leq) 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) \leq (\geq) 0$;

(ii) $H_{\mathcal{A},\mathcal{B}}(r^-) > (<) 0$, $H_{\mathcal{A}',\mathcal{B}'}(r^-) > (<) 0$, and $H_{\mathcal{A},\mathcal{B}}(x) \geq (\leq) 0$ for all $x \in (0, r)$.

(C2) the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" ("$\searrow\nearrow$") on $(0, r)$ if one of the following conditions holds

(iii) $H_{\mathcal{A},\mathcal{B}}(r^-) < (>) 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) \leq (\geq) 0$;

(iv) $H_{\mathcal{A},\mathcal{B}}(r^-) \leq (\geq) 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > (<) 0$.

(C3) the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow\nearrow$" ("$\searrow\nearrow\searrow$") on $(0, r)$ if

(v) $H_{\mathcal{A},\mathcal{B}}(r^-) \geq (\leq) 0$, $H_{\mathcal{A}',\mathcal{B}'}(r^-) > (<) 0$, and there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A},\mathcal{B}}(x_0) < (>) 0$.

Proof. Without loss of generality, we consider the case that the sequence $\{a_k/b_k\}$ is increasing for all $k \in [0, m_1]_{\mathbb{N}} \cup [m_2, \infty)_{\mathbb{N}}$ and decreasing for all $k \in [m_1, m_2]_{\mathbb{N}}$. We will use mathematical induction on m_1 to complete the proof.

(1) When $m_1 = 1$, based on the conditions, the sequence $\{a_k/b_k\}$ is increasing for all $k \in \{0, 1\} \cup [m_2, \infty)_{\mathbb{N}}$ and decreasing for all $k \in [1, m_2]_{\mathbb{N}}$. Thus, the sequence $\{a_{k+1}/b_{k+1}\}$ is "$\nearrow\searrow$", and then we consider the monotonicity of the function

$$\frac{\mathcal{A}'(x)}{\mathcal{B}'(x)} = \frac{\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k}{\sum_{k=0}^{\infty} (k+1) b_{k+1} x^k}.$$

According to Monotonicity rule 3, we obtain the following two conclusions

(i) If $H_{\mathcal{A}',\mathcal{B}'}(r^-) \le 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\searrow$".

(ii) If $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\searrow\nearrow$".

From $H'_{\mathcal{A},\mathcal{B}} = (\mathcal{A}'/\mathcal{B}')'\mathcal{B}$ and $\mathcal{B} > 0$, we know that the monotonicity of function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is the same as that of the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$. Noting that

$$H_{\mathcal{A},\mathcal{B}}(0^+) = \frac{\mathcal{A}'(0^+)}{\mathcal{B}'(0^+)}\mathcal{B}(0^+) - \mathcal{A}(0^+) = b_0\Big(\frac{a_1}{b_1} - \frac{a_0}{b_0}\Big) > 0,$$

we obtain

(i) When $H_{\mathcal{A},\mathcal{B}}(r^-) \ge 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) \le 0$. We obtain that the function $H_{\mathcal{A},\mathcal{B}}(x)$ is decreasing and further know that $H_{\mathcal{A},\mathcal{B}}(x) \ge 0$ for all $x \in (0, r)$ from $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) \ge 0$. Based on the identity $(\mathcal{A}/\mathcal{B})' = \mathcal{B}'/\mathcal{B}^2 H_{\mathcal{A},\mathcal{B}}$, we obtain that the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is increasing on $(0, r)$.

(ii) When $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) \le 0$. We know that the function $H_{\mathcal{A},\mathcal{B}}(x)$ is decreasing. Combined with $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$, it follows that the function $H_{\mathcal{A},\mathcal{B}}(x)$ is positive first and then negative. Based on the identity $(\mathcal{A}/\mathcal{B})' = \mathcal{B}'/\mathcal{B}^2 H_{\mathcal{A},\mathcal{B}}$, we deduce that the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$".

(iii) When $H_{\mathcal{A},\mathcal{B}}(r^-) > 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$ as well as $H_{\mathcal{A},\mathcal{B}}(x) \ge 0$ for all $x \in (0, r)$. We know that the function $H_{\mathcal{A},\mathcal{B}}(x)$ is "$\searrow\nearrow$". However, since $H_{\mathcal{A},\mathcal{B}}(x) \ge 0$ holds for all x, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is increasing on $(0, r)$.

(iv) When $H_{\mathcal{A},\mathcal{B}}(r^-) > 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$ as well there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A},\mathcal{B}}(x_0) < 0$. We know that the function $H_{\mathcal{A},\mathcal{B}}(x)$ is "$\searrow\nearrow$". Combined with $H_{\mathcal{A},\mathcal{B}}(0) > 0$, $H_{\mathcal{A},\mathcal{B}}(r^-) > 0$ and $H_{\mathcal{A},\mathcal{B}}(x_0) < 0$, it follows that the function $H_{\mathcal{A},\mathcal{B}}(x)$ is initially positive, then becomes negative, and finally turns positive again. Furthermore, we obtain the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow\nearrow$".

(v) When $H_{\mathcal{A},\mathcal{B}}(r^-) \le 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$. We know that the function $H_{\mathcal{A},\mathcal{B}}(x)$ is "$\searrow\nearrow$". Combined with $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$, it follows that the function $H_{\mathcal{A},\mathcal{B}}(x)$ is positive first and then negative. Furthermore, we obtain the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$".

(2) Now we consider the case that $m_1 = 2$, namely, the sequence $\{a_k/b_k\}$ is increasing for all $k \in \{0, 1, 2\} \cup [m_2, \infty)_{\mathbb{N}}$ and decreasing for all $k \in [2, m_2]_{\mathbb{N}}$. Then the sequence $\{a_{k+1}/b_{k+1}\}$ is increasing for all $k \in \{0, 1\} \cup [m_2 - 1, \infty)_{\mathbb{N}}$ and decreasing for all $k \in [1, m_2 - 1]_{\mathbb{N}}$. According the conclusions in (1), we have

(C1') the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow$" on $(0,r)$ if one of the following conditions holds

(i) $H_{\mathcal{A}',\mathcal{B}'}(r^-) \geq (\leq)0$ and $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \leq (\geq)0$,

(ii) $H_{\mathcal{A}',\mathcal{B}'}(r^-) > (<)0$, $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > (<)0$ and $H_{\mathcal{A}',\mathcal{B}'}(x) \geq (\leq)0$ for all $x \in (0,r)$.

(C2') the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow$" on $(0,r)$ if one of the following conditions holds

(iii) $H_{\mathcal{A}',\mathcal{B}'}(r^-) < (>)0$ and $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \leq (\geq)0$,

(iv) $H_{\mathcal{A}',\mathcal{B}'}(r^-) \leq (\geq)0$ and $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > (<)0$.

(C3') the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow\nearrow$" on $(0,r)$ if

(v) $H_{\mathcal{A}',\mathcal{B}'}(r^-) \geq (\leq)0$, $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > (<)0$, and there exists $x_0 \in (0,r)$ such that $H_{\mathcal{A}',\mathcal{B}'}(x_0) < (>)0$.

We now proceed to consider the different cases separately.

(I) When $H_{\mathcal{A},\mathcal{B}}(r^-) \geq 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) < 0$.

(i) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow$". That is, the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is "$\nearrow\searrow$". Combined with $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) \geq 0$, we obtain that the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is always non-negative. Furthermore, we know that the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0,r)$.

(ii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \leq 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow$". This is similar to situation (I)-(i) and we know the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0,r)$.

By combining cases (i) and (ii), it can be concluded that when $H_{\mathcal{A},\mathcal{B}}(r^-) \geq 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) < 0$, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0,r)$.

(II) When $H_{\mathcal{A},\mathcal{B}}(r^-) \geq 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) = 0$.

(i) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow$". This is similar to situation (I)-(i) and we know the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0,r)$.

(ii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \leq 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow$". That is, the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is "$\nearrow$". Combined with $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) \geq 0$, we obtain that the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is always non-negative. Furthermore, we know that the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0,r)$.

By combining cases (i) and (ii), it can be concluded that when $H_{\mathcal{A},\mathcal{B}}(r^-) \geq 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) = 0$, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0,r)$.

(III) When $H_{\mathcal{A},\mathcal{B}}(r^-) > 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$.

(i) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$ and $H_{\mathcal{A}',\mathcal{B}'}(x) \geq 0$ for all $x \in (0, r)$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow$". This is similar to situation (II)-(ii) and we know the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0, r)$.

(ii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$ and there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A}',\mathcal{B}'}(x_0) < 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow\nearrow$". That is, the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is "$\nearrow\searrow\nearrow$". Combined with $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) > 0$, we obtain two cases

(1) If there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A},\mathcal{B}}(x_0) < 0$, then the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is negative first and then positive. Thus, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow\nearrow$" on $(0, r)$.

(2) If $H_{\mathcal{A},\mathcal{B}}(x_0) \geq 0$ for all $x \in (0, r)$, then the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is always non-negative. Thus, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0, r)$.

(iii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \leq 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow$". This is similar to situation (II)-(ii) and we know the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow$" on $(0, r)$.

According to the identity $(\mathcal{A}/\mathcal{B})' = \mathcal{B}'/\mathcal{B}^2 H_{\mathcal{A},\mathcal{B}}$, if there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A},\mathcal{B}}(x_0) < 0$, then the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is decreasing at x_0. This indicate that $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$, otherwise, we have $H_{\mathcal{A},\mathcal{B}}(x) \geq 0$ for all $x \in (0, r)$ according to the cases (iii), which leads to a contradiction. In the same way, there also exists $x_1 \in (0, r)$ such that $H_{\mathcal{A}',\mathcal{B}'}(x_1) < 0$, or it contradicts cases (i). Thus, we know that, under the assumptions $H_{\mathcal{A},\mathcal{B}}(r^-) > 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow\nearrow$" if there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A},\mathcal{B}}(x_0) < 0$, otherwise, it is monotonically increasing.

(IV) When $H_{\mathcal{A},\mathcal{B}}(r^-) = 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$.

(i) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$ and $H_{\mathcal{A}',\mathcal{B}'}(x) \geq 0$ for all $x \in (0, r)$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow$" and the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is also "$\nearrow$". However, since $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) = 0$, the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ can not be increasing, which leads to a contradiction. Therefore, under the assumption of (IV), the condition (IV)-(i) cannot be imposed additionally.

(ii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$ and there exist $x_0 \in (0, r)$ such that $H_{\mathcal{A}',\mathcal{B}'}(x_0) < 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow\nearrow$". That is, the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is "$\nearrow\searrow\nearrow$". Combined with $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) = 0$, we obtain that there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A},\mathcal{B}}(x_0) < 0$, and the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is positive first and then negative. Thus, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" on $(0, r)$.

(iii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \leq 0$, then the functions $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ and $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ are both "$\nearrow$", which is impossible.

By combining cases (i)–(iii), it can be concluded that when $H_{\mathcal{A},\mathcal{B}}(r^-) = 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$, only case (ii) can hold, and in this case, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" on $(0, r)$.

(V) When $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) < 0$.

(i) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is $\nearrow\searrow$. That is, the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is "$\nearrow\searrow$". Combined with $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$, we obtain that $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is positive first and then negative. Thus, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" on $(0, r)$.

(ii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \le 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is $\nearrow\searrow$. This is similar to situation (III)-(i) and we know the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" on $(0, r)$.

By combining cases (i) and (ii), if $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) < 0$, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" on $(0, r)$.

(VI) When $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) = 0$.

(i) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow$". This is similar to situation (V)-(i) and we know the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" on $(0, r)$.

(ii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \le 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow$". That is, the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is "$\nearrow$". However, it is impossible since $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$.

By combining cases (i) and (ii), if $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) = 0$, then $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$ and furthermore the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" on $(0, r)$.

(VII) When $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$.

(i) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$ and $H_{\mathcal{A}',\mathcal{B}'}(x) \ge 0$ for all $x \in (0, r)$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow$". However, it is impossible since $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$.

(ii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$ and there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A}',\mathcal{B}'}(x_0) < 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ is "$\nearrow\searrow\nearrow$". That is, the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is "$\nearrow\searrow\nearrow$". Combined with $H_{\mathcal{A},\mathcal{B}}(0^+) > 0$ and $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$, we obtain that the function $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ is positive first and then negative. Thus, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$".

(iii) If furthermore $H_{\mathcal{A}'',\mathcal{B}''}(r^-) \le 0$, then the function $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ and $x \mapsto H_{\mathcal{A},\mathcal{B}}(x)$ are both "$\nearrow$", which is also impossible.

By combining cases (i), (ii) and (iii), under the assumption $H_{\mathcal{A},\mathcal{B}}(r^-) < 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$, it can be concluded that $H_{\mathcal{A}'',\mathcal{B}''}(r^-) > 0$ and there exists $x_0 \in (0, r)$ such that $H_{\mathcal{A}',\mathcal{B}'}(x_0) < 0$. In this situation, the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is "$\nearrow\searrow$" on $(0, r)$.

By combining cases (I)-(VI), we prove that (C1)-(C3) hold for the case $m_1 = 2$.

(3) By repeating the above steps and using mathematical induction, we can prove that conclusions (C1)-(C3) hold for any natural number m_1 greater than 2. □

Remark 5. *If one of $\{a_k/b_k\}_{0\le k\le m_1}$, $\{a_k/b_k\}_{m_1\le k\le m_2}$, and $\{a_k/b_k\}_{m_2\le k}$ is a constant, then we regard the monotonicity of $\{a_k/b_k\}$ change once rather than twice. This monotonicity rule is also closely related to Open problem 2.*

Remark 6. *Noting that the condition $H_{\mathcal{A},\mathcal{B}}(r^-) = 0$ and $H_{\mathcal{A}',\mathcal{B}'}(r^-) > 0$ deduces there exist $x_0 \in (0, r)$ such that $H_{\mathcal{A},\mathcal{B}}(x_0) < 0$, we know that, for any given functions $\mathcal{A}$ and $\mathcal{B}$, one of (i), (ii), (iii), (iv), (v) in (C1), (C2), (C3) holds. And the conclusions in Monotonicity rule 4 could be simply represented in Table 1.*

Cases	$\{a_k/b_k\}$	$H_{\mathcal{A},\mathcal{B}}(r^-)$	$H_{\mathcal{A}',\mathcal{B}'}(r^-)$	$H_{\mathcal{A},\mathcal{B}}(x)$	$\mathcal{A}/\mathcal{B}$
1	$\nearrow\searrow\nearrow$	≥ 0	≤ 0		$\nearrow$
2	$\nearrow\searrow\nearrow$	> 0	> 0	≥ 0 for all $x \in (0, r)$	$\nearrow$
3	$\nearrow\searrow\nearrow$	< 0	≤ 0		$\nearrow\searrow$
4	$\nearrow\searrow\nearrow$	≤ 0	> 0		$\nearrow\searrow$
5	$\nearrow\searrow\nearrow$	≥ 0	> 0	exists $x_0 \in (0, r)$ such that < 0	$\nearrow\searrow\nearrow$
6	$\searrow\nearrow\searrow$	≤ 0	≥ 0		$\searrow$
7	$\searrow\nearrow\searrow$	< 0	< 0	≤ 0 for all $x \in (0, r)$	$\searrow$
8	$\searrow\nearrow\searrow$	> 0	≥ 0		$\searrow\nearrow$
9	$\searrow\nearrow\searrow$	≥ 0	< 0		$\searrow\nearrow$
10	$\searrow\nearrow\searrow$	≤ 0	< 0	exists $x_0 \in (0, r)$ such that > 0	$\searrow\nearrow\searrow$

Table 1: The Monotonicity of $\mathcal{A}/\mathcal{B}$ in Monotonicity rule 4

Let $r \to \infty$. Then we have the following monotonicity rule.

Monotonicity rule 7. *Let real power series $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\mathcal{B}(x) = \sum_{k=0}^{\infty} b_k x^k$ converge on $(0, \infty)$ with $b_k > 0$. If there exists two different integers $m_2 > m_1 \ge 1$ such that the sequence $\{a_k/b_k\}$ is increasing (decreasing) for all $k \in [0, m_1]_{\mathbb{N}} \cup [m_2, \infty)_{\mathbb{N}}$ and decreasing (increasing) for all $k \in [m_1, m_2]_{\mathbb{N}}$, as well as $\{a_k/b_k\}_{0\le k\le m_1}$, $\{a_k/b_k\}_{m_1\le k\le m_2}$, and $\{a_k/b_k\}_{k\ge m_2}$ are non-constant, then*

(C1) the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is increasing (decreasing) on $(0, \infty)$ if one of the following conditions holds:

(i) $H_{\mathcal{A},\mathcal{B}}(\infty) \ge (\le) 0$ and $H_{\mathcal{A}',\mathcal{B}'}(\infty) \le (\ge) 0$;

(ii) $H_{\mathcal{A},\mathcal{B}}(\infty) > (<) 0$, $H_{\mathcal{A}',\mathcal{B}'}(\infty) > (<) 0$, and $H_{\mathcal{A},\mathcal{B}}(x) \ge (\le) 0$ for all $x \in (0, \infty)$.

(C2) there exists $x_1 \in (0, \infty)$ such that the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is increasing (decreasing) on $(0, x_1]$ and decreasing (increasing) on $[x_1, \infty)$ if one of the following conditions holds:

(iii) $H_{\mathcal{A},\mathcal{B}}(\infty) < (>)0$ *and* $H_{\mathcal{A}',\mathcal{B}'}(\infty) \leq (\geq)0$*;*

(iv) $H_{\mathcal{A},\mathcal{B}}(\infty) \leq (\geq)0$ *and* $H_{\mathcal{A}',\mathcal{B}'}(\infty) > (<)0$.

(C3) there exists $x_2, x_3 \in (0,\infty)$ *such that the function* $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ *is increasing (decreasing) on* $(0,x_2] \cup [x_3, r)$ *and decreasing (increasing) on* $[x_2, x_3]$ *if:*

(v) $H_{\mathcal{A},\mathcal{B}}(\infty) \geq (\leq)0$, $H_{\mathcal{A}',\mathcal{B}'}(\infty) > (<)0$, *and there exists* $x_0 \in (0,\infty)$ *such that* $H_{\mathcal{A},\mathcal{B}}(x_0) < (>)0$.

Likewise, we present the monotonicity rule for the ratio of two polynomials.

Monotonicity rule 8. *Let* $\mathcal{A}_N(x) = \sum_{k=0}^{N} a_k x^k$ *and* $\mathcal{B}_N(x) = \sum_{k=0}^{N} b_k x^k$ *defined on* $(0,r)$ *with* $b_k > 0$. *If there exists different integers* $N > m_2 > m_1 \geq 1$ *such that the sequence* $\{a_k/b_k\}$ *is increasing (decreasing) for all* $k \in [0,m_1]_{\mathbb{N}} \cup [m_2,N]_{\mathbb{N}}$ *and decreasing (increasing) for all* $k \in [m_1,m_2]_{\mathbb{N}}$, *as well as* $\{a_k/b_k\}_{0\leq k\leq m_1}$, $\{a_k/b_k\}_{m_1\leq k\leq m_2}$, *and* $\{a_k/b_k\}_{m_2\leq k\leq N}$ *are non-constant, then*

(C1) the function $x \mapsto \mathcal{A}_N(x)/\mathcal{B}_N(x)$ *is increasing (decreasing) on* $(0,r)$ *if one of the following conditions holds:*

(i) $H_{\mathcal{A}_N,\mathcal{B}_N}(r^-) \geq (\leq)0$ *and* $H_{\mathcal{A}'_N,\mathcal{B}'_N}(r^-) \leq (\geq)0$*;*

(ii) $H_{\mathcal{A}_N,\mathcal{B}_N}(r^-) > (<)0$, $H_{\mathcal{A}'_N,\mathcal{B}'_N}(r^-) > (<)0$, *and* $H_{\mathcal{A}_N,\mathcal{B}_N}(x) \geq (\leq)0$ *for all* $x \in (0,r)$.

(C2) there exists $x_1 \in (0,r)$ *such that the function* $x \mapsto \mathcal{A}_N(x)/\mathcal{B}_N(x)$ *is increasing (decreasing) on* $(0,x_1]$ *and decreasing (increasing) on* $[x_1,r)$ *if one of the following conditions holds:*

(iii) $H_{\mathcal{A}_N,\mathcal{B}_N}(r^-) < (>)0$ *and* $H_{\mathcal{A}'_N,\mathcal{B}'_N}(r^-) \leq (\geq)0$*;*

(iv) $H_{\mathcal{A}_N,\mathcal{B}_N}(r^-) \leq (\geq)0$ *and* $H_{\mathcal{A}'_N,\mathcal{B}'_N}(r^-) > (<)0$.

(C3) there exists $x_2, x_3 \in (0,r)$ *such that the function* $x \mapsto \mathcal{A}_N(x)/\mathcal{B}_N(x)$ *is increasing (decreasing) on* $(0,x_2] \cup [x_3,r)$ *and decreasing (increasing) on* $[x_2,x_3]$ *if:*

(v) $H_{\mathcal{A}_N,\mathcal{B}_N}(r^-) \geq (\leq)0$, $H_{\mathcal{A}'_N,\mathcal{B}'_N}(r^-) > (<)0$, *and there exists* $x_0 \in (0,r)$ *such that* $H_{\mathcal{A}_N,\mathcal{B}_N}(x_0) < (>)0$.

Monotonicity rule 9. *Let* $\mathcal{A}_N(x) = \sum_{k=0}^{N} a_k x^k$ *and* $\mathcal{B}_N(x) = \sum_{k=0}^{N} b_k x^k$ *defined on* $(0,\infty)$ *with* $b_k > 0$. *If there exists different integers* $N > m_2 > m_1 \geq 1$ *such that the sequence* $\{a_k/b_k\}$ *is increasing (decreasing) for all* $k \in [0,m_1]_{\mathbb{N}} \cup [m_2,N]_{\mathbb{N}}$ *and decreasing (increasing) for all* $k \in [m_1,m_2]_{\mathbb{N}}$, *as well as* $\{a_k/b_k\}_{0\leq k\leq m_1}$, $\{a_k/b_k\}_{m_1\leq k\leq m_2}$, *and* $\{a_k/b_k\}_{m_2\leq k\leq N}$ *are non-constant, then*

(C1) the function $x \mapsto \mathcal{A}_N(x)/\mathcal{B}_N(x)$ *is increasing (decreasing) on* $(0,\infty)$ *if one of the following conditions holds:*

(i) $H_{\mathcal{A}_N,\mathcal{B}_N}(\infty) \geq (\leq)0$ *and* $H_{\mathcal{A}'_N,\mathcal{B}'_N}(\infty) \leq (\geq)0$*;*

(ii) $H_{\mathcal{A}_N,\mathcal{B}_N}(\infty) > (<)0$, $H_{\mathcal{A}'_N,\mathcal{B}'_N}(\infty) > (<)0$, *and* $H_{\mathcal{A}_N,\mathcal{B}_N}(x) \geq (\leq)0$ *for all* $x \in (0,\infty)$.

(C2) there exists $x_1 \in (0,\infty)$ *such that the function* $x \mapsto \mathcal{A}_N(x)/\mathcal{B}_N(x)$ *is increasing (decreasing) on* $(0, x_1]$ *and decreasing (increasing) on* $[x_1,\infty)$ *if one of the following conditions holds:*

(iii) $H_{\mathcal{A}_N,\mathcal{B}_N}(\infty) < (>)0$ *and* $H_{\mathcal{A}'_N,\mathcal{B}'_N}(\infty) \leq (\geq)0$;

(iv) $H_{\mathcal{A}_N,\mathcal{B}_N}(\infty) \leq (\geq)0$ *and* $H_{\mathcal{A}'_N,\mathcal{B}'_N}(\infty) > (<)0$.

(C3) there exists $x_2, x_3 \in (0,\infty)$ *such that the function* $x \mapsto \mathcal{A}_N(x)/\mathcal{B}_N(x)$ *is increasing (decreasing) on* $(0, x_2] \cup [x_3,\infty)$ *and decreasing (increasing) on* $[x_2, x_3]$ *if:*

(v) $H_{\mathcal{A}_N,\mathcal{B}_N}(\infty) \geq (\leq)0$, $H_{\mathcal{A}'_N,\mathcal{B}'_N}(\infty) > (<)0$, *and there exists* $x_0 \in (0,\infty)$ *such that* $H_{\mathcal{A}_N,\mathcal{B}_N}(x_0) < (>)0$.

Analytic functions can be represented by their Maclaurin series in a neighborhood of the origin. Thus we obtain the following corollary, which considers the monotonicity of the ratio of two Maclaurin series

Corollary 10. *Assume that* $\mathcal{A}(x)$ *and* $\mathcal{B}(x)$ *are analytic functions and* $\mathcal{B}^{(k)}(0) > 0$ *for all* $k \geq 0$. *If there exists different integers* $m_2 > m_1 \geq 1$ *such that the sequence* $\left\{\frac{\mathcal{A}^{(k)}(0)}{\mathcal{B}^{(k)}(0)}\right\}$ *is increasing (decreasing) for all* $k \in [0, m_1]_{\mathbb{N}} \cup [m_2,\infty)_{\mathbb{N}}$ *and decreasing (increasing) for all* $k \in [m_1, m_2]_{\mathbb{N}}$, *then the conclusions (C1), (C2), and (C3) in Theorem 4 hold.*

Now, we consider some special cases by choosing different functions for $\mathcal{B}$. Let $\mathcal{B}(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ in the monotonicity rule 4. This leads to the following corollary.

Corollary 11. *Let real power series* $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ *converge on* $(0,\infty)$. *If there exists different integers* $m_2 > m_1 \geq 1$ *such that the sequence* $\{k!a_k\}$ *is increasing (decreasing) on* $k \in [0, m_1]_{\mathbb{N}} \cup [m_2,\infty)_{\mathbb{N}}$ *and decreasing (increasing) on* $k \in [m_1, m_2]_{\mathbb{N}}$, *then*

(C1) the function $x \mapsto e^{-x}\mathcal{A}(x)$ *is increasing (decreasing) if one of the following conditions holds:*

(i) $\mathcal{A}'(\infty) \geq (\leq)\mathcal{A}(\infty)$ *and* $\mathcal{A}''(\infty) \leq (\geq)\mathcal{A}'(\infty)$;

(ii) $\mathcal{A}'(\infty) > (<)\mathcal{A}(\infty)$, $\mathcal{A}''(\infty) > (<)\mathcal{A}'(\infty)$, *and* $\mathcal{A}'(x) - \mathcal{A}(x) \geq 0$ *for all* $x \in (0,\infty)$.

(C2) there exists $x_1 \in (0,\infty)$ *such that the function* $x \mapsto e^{-x}\mathcal{A}(x)$ *is increasing (decreasing) on* $(0, x_1]$ *and decreasing (increasing) on* $[x_1,\infty)$ *if one of the following conditions holds:*

(iii) $\mathcal{A}'(\infty) < (>)\mathcal{A}(\infty)$ *and* $\mathcal{A}''(\infty) \leq (\geq)\mathcal{A}'(\infty)$;

(iv) $\mathcal{A}'(\infty) \le (\ge)\mathcal{A}(\infty)$ *and* $\mathcal{A}''(\infty) > (<)\mathcal{A}'(\infty)$.

(C3) *there exists* $x_2, x_3 \in (0,\infty)$ *such that the function* $x \mapsto e^{-x}\mathcal{A}(x)$ *is increasing (decreasing) on* $(0,x_2] \cup [x_3,\infty)$ *and decreasing (increasing) on* $[x_2,x_3]$ *if:*

(v) $\mathcal{A}'(\infty) \ge (\le)\mathcal{A}(\infty)$, $\mathcal{A}''(\infty) > (<)\mathcal{A}'(\infty)$, *and there exists* $x_0 \in (0,\infty)$ *such that* $\mathcal{A}'(x_0) < (>)\mathcal{A}(x_0)$.

Taking $\mathcal{B}(x) = \frac{1}{(1-x)^d} = \sum_{k=0}^{\infty} \frac{(d)_k}{k!} x^k$, then we have the following corollary, where $d > 0$ and $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol.

Corollary 12. *Let real power series* $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ *converge on* $(0,1)$ *and* $d > 0$. *If there exists different integers* $m_2 > m_1 \ge 1$ *such that the sequence* $\{k!a_k/(d)_k\}$ *is increasing (decreasing) on* $k \in [0,m_1]_{\mathbb{N}} \cup [m_2,\infty)_{\mathbb{N}}$ *and decreasing (increasing) on* $k \in [m_1,m_2]_{\mathbb{N}}$*, then*

(C1) *the function* $x \mapsto (1-x)^{-d}\mathcal{A}(x)$ *is increasing (decreasing) if one of the following conditions holds:*

(i) $\lim_{x\to1}\left(\mathcal{A}'(x)\frac{1-x}{d} - \mathcal{A}(x)\right) \ge (\le)0$ *and* $\lim_{x\to1}\left(\mathcal{A}''(x)\frac{1-x}{d+1} - \mathcal{A}'(x)\right) \le (\ge)0$;

(ii) $\lim_{x\to1}\left(\mathcal{A}'(x)\frac{1-x}{d} - \mathcal{A}(x)\right) > (<)0$, $\lim_{x\to1}\left(\mathcal{A}''(x)\frac{1-x}{d+1} - \mathcal{A}'(x)\right) > (<)0$, *and* $\mathcal{A}'(x)\frac{1-x}{d} - \mathcal{A}(x) \ge 0$ *for all* $x \in (0,1)$.

(C2) *there exists* $x_1 \in (0,1)$ *such that the function* $x \mapsto (1-x)^{-d}\mathcal{A}(x)$ *is increasing (decreasing) on* $(0,x_1]$ *and decreasing (increasing) on* $[x_1,1)$ *if one of the following conditions holds:*

(iii) $\lim_{x\to1}\left(\mathcal{A}'(x)\frac{1-x}{d} - \mathcal{A}(x)\right) < (>)0$ *and* $\lim_{x\to1}\left(\mathcal{A}''(x)\frac{1-x}{d+1} - \mathcal{A}'(x)\right) \le (\ge)0$;

(iv) $\lim_{x\to1}\left(\mathcal{A}'(x)\frac{1-x}{d} - \mathcal{A}(x)\right) \le (\ge)0$ *and* $\lim_{x\to1}\left(\mathcal{A}''(x)\frac{1-x}{d+1} - \mathcal{A}'(x)\right) > (<)0$.

(C3) *there exists* $x_2, x_3 \in (0,1)$ *such that the function* $x \mapsto (1-x)^{-d}\mathcal{A}(x)$ *is increasing (decreasing) on* $(0,x_2] \cup [x_3,1)$ *and decreasing (increasing) on* $[x_2,x_3]$ *if:*

(v) $\lim_{x\to1}\left(\mathcal{A}'(x)\frac{1-x}{d} - \mathcal{A}(x)\right) \ge (\le)0$, $\lim_{x\to1}\left(\mathcal{A}''(x)\frac{1-x}{d+1} - \mathcal{A}'(x)\right) > (<)0$, *and there exists* $x_0 \in (0,1)$ *such that* $\mathcal{A}'(x_0)\frac{1-x_0}{d} - \mathcal{A}(x_0) < (>)0$.

In particular, taking $d = 1$, namely, $\mathcal{B}(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, then we have the following corollary.

Corollary 13. *Let real power series* $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ *converge on* $(0,1)$. *If there exists different integers* $m_2 > m_1 \ge 1$ *such that the sequence* $\{a_k\}$ *is increasing (decreasing) on* $k \in [0,m_1]_{\mathbb{N}} \cup [m_2,\infty)_{\mathbb{N}}$ *and decreasing (increasing) on* $k \in [m_1,m_2]_{\mathbb{N}}$*, then*

(C1) *the function* $x \mapsto (1-x)\mathcal{A}(x)$ *is increasing (decreasing) if one of the following conditions holds:*

(i) $\lim_{x\to 1}\big(\mathcal{A}'(x)(1-x)-\mathcal{A}(x)\big) \geq (\leq) 0$ *and* $\lim_{x\to 1}\big(\mathcal{A}''(x)\frac{1-x}{2}-\mathcal{A}'(x)\big) \leq (\geq) 0$*;*

(ii) $\lim_{x\to 1}\big(\mathcal{A}'(x)(1-x)-\mathcal{A}(x)\big) > (<) 0$, $\lim_{x\to 1}\big(\mathcal{A}''(x)\frac{1-x}{2}-\mathcal{A}'(x)\big) > (<) 0$, *and* $\mathcal{A}'(x)(1-x)-\mathcal{A}(x) \geq 0$ *for all* $x \in (0,1)$.

(C2) *there exists* $x_1 \in (0,1)$ *such that the function* $x \mapsto (1-x)\mathcal{A}(x)$ *is increasing (decreasing) on* $(0, x_1]$ *and decreasing (increasing) on* $[x_1, 1)$ *if one of the following conditions holds:*

(iii) $\lim_{x\to 1}\big(\mathcal{A}'(x)(1-x)-\mathcal{A}(x)\big) < (>) 0$ *and* $\lim_{x\to 1}\big(\mathcal{A}''(x)\frac{1-x}{2}-\mathcal{A}'(x)\big) \leq (\geq) 0$*;*

(iv) $\lim_{x\to 1}\big(\mathcal{A}'(x)(1-x)-\mathcal{A}(x)\big) \leq (\geq) 0$ *and* $\lim_{x\to 1}\big(\mathcal{A}''(x)\frac{1-x}{2}-\mathcal{A}'(x)\big) > (<) 0$.

(C3) *there exists* $x_2, x_3 \in (0,1)$ *such that the function* $x \mapsto (1-x)\mathcal{A}(x)$ *is increasing (decreasing) on* $(0, x_2] \cup [x_3, 1)$ *and decreasing (increasing) on* $[x_2, x_3]$ *if:*

(v) $\lim_{x\to 1}\big(\mathcal{A}'(x)(1-x)-\mathcal{A}(x)\big) \geq (\leq) 0$, $\lim_{x\to 1}\big(\mathcal{A}''(x)\frac{1-x}{2}-\mathcal{A}'(x)\big) > (<) 0$, *and there exists* $x_0 \in (0,1)$ *such that* $\mathcal{A}'(x_0)(1-x_0)-\mathcal{A}(x_0) < (>) 0$.

Taking $\mathcal{B}(x) = -\ln(1-dx) = \sum_{k=1}^{\infty}\frac{d^k}{(k)!}x^k$, then we have the following corollary.

Corollary 14. *Let real power series* $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ *converge on* $(0, 1/d)$*, where* $d > 0$*. If there exists different integers* $m_2 > m_1 \geq 1$ *such that the sequence* $\{k! a_k / d^k\}$ *is increasing (decreasing) on* $k \in [0, m_1]_{\mathbb{N}} \cup [m_2, \infty)_{\mathbb{N}}$ *and decreasing (increasing) on* $k \in [m_1, m_2]_{\mathbb{N}}$*, then*

(C1) *the function* $x \mapsto -\frac{\mathcal{A}(x)-a_0}{\ln(1-dx)}$ *is increasing (decreasing) if one of the following conditions holds:*

(i) $\lim_{x\to\frac{1}{d}}\big(-\mathcal{A}'(x)\frac{(1-dx)\ln(1-dx)}{d}-\mathcal{A}(x)\big) \geq (\leq) 0$ *and* $\lim_{x\to\frac{1}{d}}(\mathcal{A}''(x)\frac{1-dx}{d}-\mathcal{A}'(x)) \leq (\geq) 0$*;*

(ii) $\lim_{x\to\frac{1}{d}}\big(-\mathcal{A}'(x)\frac{(1-dx)\ln(1-dx)}{d}-\mathcal{A}(x)\big) > (<) 0$, $\lim_{x\to\frac{1}{d}}\big(\mathcal{A}''(x)\frac{1-dx}{d}-\mathcal{A}'(x)\big) > (<) 0$, *and* $-\mathcal{A}'(x)\frac{(1-dx)\ln(1-dx)}{d}-\mathcal{A}(x) \geq 0$ *for all* $x \in (0, \frac{1}{d})$.

(C2) *there exists* $x_1 \in (0, \frac{1}{d})$ *such that the function* $x \mapsto -\frac{\mathcal{A}(x)-a_0}{\ln(1-dx)}$ *is increasing (decreasing) on* $(0, x_1]$ *and decreasing (increasing) on* $[x_1, \frac{1}{d})$ *if one of the following conditions holds:*

(iii) $\lim_{x\to\frac{1}{d}}\big(-\mathcal{A}'(x)\frac{(1-dx)\ln(1-dx)}{d}-\mathcal{A}(x)\big) < (>) 0$ *and* $\lim_{x\to\frac{1}{d}}(\mathcal{A}''(x)\frac{1-dx}{d}-\mathcal{A}'(x)) \leq (\geq) 0$*;*

(iv) $\lim_{x\to\frac{1}{d}}\left(-\mathcal{A}'(x)\frac{(1-dx)\ln(1-dx)}{d}-\mathcal{A}(x)\right)\le(\ge)0$ *and* $\lim_{x\to\frac{1}{d}}(\mathcal{A}''(x)\frac{1-dx}{d}-\mathcal{A}'(x))>(<)0$.

(C3) there exists $x_2,x_3\in(0,\frac{1}{d})$ *such that the function* $x\mapsto-\frac{\mathcal{A}(x)-a_0}{\ln(1-dx)}$ *is increasing (decreasing) on* $(0,x_2]\cup[x_3,\frac{1}{d})$ *and decreasing (increasing) on* $[x_2,x_3]$ *if:*

(v) $\lim_{x\to\frac{1}{d}}\left(-\mathcal{A}'(x)\frac{(1-dx)\ln(1-dx)}{d}-\mathcal{A}(x)\right)\ge(\le)0$, $\lim_{x\to\frac{1}{d}}\left(\mathcal{A}''(x)\frac{1-dx}{d}-\mathcal{A}'(x)\right)>(<)0$, *and there exists* $x_0\in(0,\frac{1}{d})$ *such that* $-\mathcal{A}'(x_0)(1-dx_0)\frac{\ln(1-dx_0)}{d}-\mathcal{A}(x_0)<(>)0$.

Taking $\mathcal{B}(x)=\sinh(dx)=\sum_{k=0}^{\infty}\frac{d^{2k+1}}{(2k+1)!}x^{2k+1}$ and $\mathcal{B}(x)=\cosh(dx)=\sum_{k=0}^{\infty}\frac{d^{2k}}{(2k)!}x^{2k}$, respectively, we have the following two corollaries.

Corollary 15. *Let real power series* $\mathcal{A}(x)=\sum_{k=0}^{\infty}a_kx^{2k+1}$ *converge on* $(0,\infty)$ *and* $d>0$. *If there exists different integers* $m_2>m_1\ge1$ *such that the sequence* $\{(2k+1)!a_k/d^{2k+1}\}$ *is increasing (decreasing) on* $k\in[0,m_1]_{\mathbb{N}}\cup[m_2,\infty)_{\mathbb{N}}$ *and decreasing (increasing) on* $k\in[m_1,m_2]_{\mathbb{N}}$, *then*

(C1) the function $x\mapsto\frac{\mathcal{A}(x)}{\sinh(dx)}$ *is increasing (decreasing) if one of the following conditions holds:*

(i) $\lim_{x\to\infty}\left(\mathcal{A}'(x)\frac{\tanh(dx)}{d}-\mathcal{A}(x)\right)\ge(\le)0$ *and* $\lim_{x\to\infty}\left(\mathcal{A}''(x)\frac{\coth(dx)}{d}-\mathcal{A}'(x)\right)\le(\ge)0$;

(ii) $\lim_{x\to\infty}\left(\mathcal{A}'(x)\frac{\tanh(dx)}{d}-\mathcal{A}(x)\right)>(<)0$, $\lim_{x\to\infty}\left(\mathcal{A}''(x)\frac{\coth(dx)}{d}-\mathcal{A}'(x)\right)>(<)0$, *and* $\mathcal{A}'(x)\frac{\tanh(dx)}{d}-\mathcal{A}(x)\ge0$ *for all* $x\in(0,\infty)$.

(C2) there exists $x_1\in(0,\infty)$ *such that the function* $x\mapsto\frac{\mathcal{A}(x)}{\sinh(dx)}$ *is increasing (decreasing) on* $(0,x_1]$ *and decreasing (increasing) on* $[x_1,\infty)$ *if one of the following conditions holds:*

(iii) $\lim_{x\to\infty}\left(\mathcal{A}'(x)\frac{\tanh(dx)}{d}-\mathcal{A}(x)\right)<(>)0$ *and* $\lim_{x\to\infty}\left(\mathcal{A}''(x)\frac{\coth(dx)}{d}-\mathcal{A}'(x)\right)\le(\ge)0$;

(iv) $\lim_{x\to\infty}\left(\mathcal{A}'(x)\frac{\tanh(dx)}{d}-\mathcal{A}(x)\right)\le(\ge)0$ *and* $\lim_{x\to\infty}\left(\mathcal{A}''(x)\frac{\coth(dx)}{d}-\mathcal{A}'(x)\right)>(<)0$.

(C3) there exists $x_2,x_3\in(0,\infty)$ *such that the function* $x\mapsto\frac{\mathcal{A}(x)}{\sinh(dx)}$ *is increasing (decreasing) on* $(0,x_2]\cup[x_3,\infty)$ *and decreasing (increasing) on* $[x_2,x_3]$ *if:*

(v) $\lim_{x\to\infty}\left(\mathcal{A}'(x)\frac{\tanh(dx)}{d}-\mathcal{A}(x)\right)\ge(\le)0$, $\lim_{x\to\infty}\left(\mathcal{A}''(x)\frac{\coth(dx)}{d}-\mathcal{A}'(x)\right)>(<)0$, *and there exists* $x_0\in(0,\infty)$ *such that* $\mathcal{A}'(x_0)\frac{\tanh(dx_0)}{d}-\mathcal{A}(x_0)<(>)0$.

Corollary 16. *Let real power series $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^{2k}$ converge on $(0,\infty)$ and $d > 0$. If there exists different integers $m_2 > m_1 \geq 1$ such that the sequence $\{(2k)!a_k/d^{2k}\}$ is increasing (decreasing) on $k \in [0,m_1]_{\mathbb{N}} \cup [m_2,\infty)_{\mathbb{N}}$ and decreasing (increasing) on $k \in [m_1,m_2]_{\mathbb{N}}$, then*

(C1) the function $x \mapsto \frac{\mathcal{A}(x)}{\cosh(dx)}$ is increasing (decreasing) if one of the following conditions holds:

(i) $\lim_{x\to\infty}\big(\mathcal{A}'(x)\frac{\coth(dx)}{d} - \mathcal{A}(x)\big) \geq (\leq) 0$ and $\lim_{x\to\infty}\big(\mathcal{A}''(x)\frac{\tanh(dx)}{d} - \mathcal{A}'(x)\big) \leq (\geq) 0$;

(ii) $\lim_{x\to\infty}\big(\mathcal{A}'(x)\frac{\coth(dx)}{d} - \mathcal{A}(x)\big) > (<) 0$, $\lim_{x\to\infty}\big(\mathcal{A}''(x)\frac{\tanh(dx)}{d} - \mathcal{A}'(x)\big) > (<) 0$, and $\mathcal{A}'(x)\frac{\coth(dx)}{d} - \mathcal{A}(x) \geq 0$ for all $x \in (0,\infty)$.

(C2) there exists $x_1 \in (0,\infty)$ such that the function $x \mapsto \frac{\mathcal{A}(x)}{\cosh(dx)}$ is increasing (decreasing) on $(0,x_1]$ and decreasing (increasing) on $[x_1,\infty)$ if one of the following conditions holds:

(iii) $\lim_{x\to\infty}\big(\mathcal{A}'(x)\frac{\coth(dx)}{d} - \mathcal{A}(x)\big) < (>) 0$ and $\lim_{x\to\infty}\big(\mathcal{A}''(x)\frac{\tanh(dx)}{d} - \mathcal{A}'(x)\big) \leq (\geq) 0$;

(iv) $\lim_{x\to\infty}\big(\mathcal{A}'(x)\frac{\coth(dx)}{d} - \mathcal{A}(x)\big) \leq (\geq) 0$ and $\lim_{x\to\infty}\big(\mathcal{A}''(x)\frac{\tanh(dx)}{d} - \mathcal{A}'(x)\big) > (<) 0$.

(C3) there exists $x_2, x_3 \in (0,\infty)$ such that the function $x \mapsto \frac{\mathcal{A}(x)}{\cosh(dx)}$ is increasing (decreasing) on $(0,x_2] \cup [x_3,\infty)$ and decreasing (increasing) on $[x_2,x_3]$ if:

(v) $\lim_{x\to\infty}\big(\mathcal{A}'(x)\frac{\coth(dx)}{d} - \mathcal{A}(x)\big) \geq (\leq) 0$, $\lim_{x\to\infty}\big(\mathcal{A}''(x)\frac{\tanh(dx)}{d} - \mathcal{A}'(x)\big) > (<) 0$, and there exists $x_0 \in (0,\infty)$ such that $\mathcal{A}'(x_0)\frac{\coth(dx_0)}{d} - \mathcal{A}(x_0) < (>) 0$.

2.2 Other type monotonicity rules

The following monotonicity rule shows that the monotonicity of $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ near point $x = 0$ is determined by the monotonicity of $\{a_k/b_k\}$ near point $k = 0$, which named local monotonicity rule in this paper.

Monotonicity rule 17. *Let real power series $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\mathcal{B}(x) = \sum_{k=0}^{\infty} b_k x^k$ converge on $(0,r)$ with $b_k > 0$. If there exists integer $m \geq 1$ such that the sequence $\{a_k/b_k\}$ is strictly increasing (decreasing) for all $0 \leq k \leq m$, then there exists $x_0 \in (0,r]$ such that the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is strictly increasing (decreasing) on $(0,x_0)$.*

Proof. From

$$H_{\mathcal{A},\mathcal{B}}(0^+) = \frac{\mathcal{A}'(0^+)}{\mathcal{B}'(0^+)}\mathcal{B}(0^+) - \mathcal{A}(0^+) = b_0\Big(\frac{a_1}{b_1} - \frac{a_0}{b_0}\Big) > 0,$$

and the identity $(\mathcal{A}/\mathcal{B})' = \mathcal{B}'/\mathcal{B}^2 H_{\mathcal{A},\mathcal{B}}$, we obtain that there exists $x_0 \in (0, r]$ such that $H_{\mathcal{A},\mathcal{B}}(x) > 0$ for all $x \in (0, x_0)$ and the function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ is strictly increasing (decreasing) on $(0, x_0)$. □

Let the monotonicity of $\{a_k/b_k\}$ changes $n(n \geq 0)$ times and the monotonicity of function $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ changes $\tau_{\mathcal{A},\mathcal{B}}(n)$ times. The following monotonicity rule shows $\tau_{\mathcal{A},\mathcal{B}}(n) \leq n$.

Monotonicity rule 18. *Let real power series $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\mathcal{B}(x) = \sum_{k=0}^{\infty} b_k x^k$ converge on $(0, r)$ with $b_k > 0$. If the monotonicity of $\{a_k/b_k\}$ changes $n(n \geq 0)$ times, then the monotonicity of $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ changes no more than $n(n \geq 0)$ times.*

Proof. We complete the proof by using mathematical induction. According to Monotonicity rule 1, Monotonicity rule 3, and Monotonicity rule 4, the conclusion is true for $n = 0, 1, 2$, respectively. Suppose it is true for $n = q$, now we consider $n = q + 1$.

In this case, the monotonicity of the sequence $\{a_k/b_k\}$ changes $q + 1$ times. Without loss of generality, we suppose that there exist integer $m \geq 1$, which can't be bigger, such that the sequence $\{a_k/b_k\}$ is increasing for all $0 \leq k \leq m$. Thus, the monotonicity of the sequence $\{a_{k+1}/b_{k+1}\}$ changes q times if $m = 1$ and changes $q + 1$ times if $m \geq 2$.

(I) When $m = 1$, the monotonicity of $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ changes q times with decreasing first, same as $H_{\mathcal{A},\mathcal{B}}$ due to identity $H'_{\mathcal{A},\mathcal{B}} = (\mathcal{A}'/\mathcal{B}')'\mathcal{B}$. According to $H_{\mathcal{A},\mathcal{B}}(0^+) \geq 0$, we obtain that the monotonicity of $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ changes no more than $q + 1$ times.

(II) Suppose it holds for $m = p$, we consider $m = p + 1$. The monotonicity of $x \mapsto \mathcal{A}'(x)/\mathcal{B}'(x)$ changes $q+1$ times with increasing first, same as $H_{\mathcal{A},\mathcal{B}}$. According to $H_{\mathcal{A},\mathcal{B}}(0^+) \geq 0$, we obtain that the monotonicity of $x \mapsto \mathcal{A}(x)/\mathcal{B}(x)$ changes no more than $q + 1$ times.

Thus, we complete the proof. □

Corollary 19. *Let real power series $\mathcal{A}(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\mathcal{B}(x) = \sum_{k=0}^{\infty} b_k x^k$ converge on $(0, r)$ with $b_k > 0$ as well as $l \in \mathbb{N}$. If the monotonicity of $\{a_k/b_k\}$ changes $n(n \geq 0)$ times, then the monotonicity of $x \mapsto \mathcal{A}^{(l)}(x)/\mathcal{B}^{(l)}(x)$ changes no more than n times.*

Proof. Clearly, the monotonicity of $\{a_{k+l}/b_{k+l}\}$ changes no more than n times. Thus, the monotonicity of $x \mapsto \mathcal{A}^{(l)}(x)/\mathcal{B}^{(l)}(x)$ changes no more than n times. □

3. APPLICATIONS IN CONFLUENT HYPERGEOMETRIC FUNCTIONS OF THE FIRST KIND

First, we briefly introduce the confluent hypergeometric function of the first kind, commonly denoted by $M(a,b,x)$, $\Phi(a,b,x)$ or ${}_1F_1(a,b,x)$, also referred to as Kummer's function. It is defined as the solution to the following second-order ordinary differential equation (i.e., Kummer's equation) [**1**, p. 504]

$$xZ''(x) + (b-x)Z'(x) - aZ(x) = 0.$$

It is easy to verify that $M(a,b,x)$ has the following series representation (see [**1**, p. 504])

$$M(a,b,x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}, \tag{4}$$

where

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad k = 1, 2, \cdots$$

is the Pochhammer symbol. Furthermore, for large $x \to \infty$, the following asymptotic formula holds

$$M(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \Big(1 + \frac{(1-a)(b-a)}{x} + O(x^{-2})\Big). \tag{5}$$

In this section, we will focus on studying the monotonicity of the function (3). It is noteworthy that this function exhibits rather complex monotonicity behavior for different parameters $a, b, c > 0$. For example,

(i) when $a = 1$, $b = 1/3$, and $c = 4/3$, the function exhibits monotonicity behavior of the form "$\nearrow$";

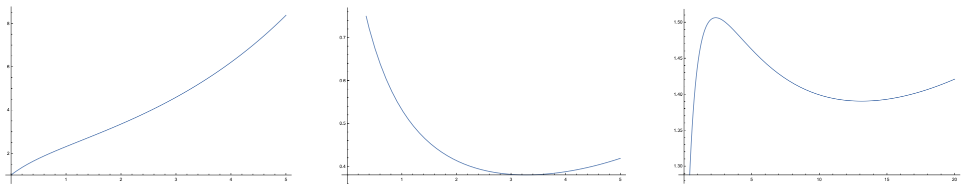
(ii) when $a = 1$, $b = 4/3$, and $c = 4/3$, the function exhibits monotonicity behavior of the form "$\searrow\nearrow$";

(iii) when $a = 1$, $b = 1/3$, and $c = 1.02$, the function exhibits monotonicity behavior of the form "$\nearrow\searrow\nearrow$",

as shown in Figure 1. A natural question is to determine the conditions on the parameters a, b, c under which the function exhibits distinct monotonicity behavior.

Fortunately, based on the series representation (4), we can rewrite the function (3) as

$$\frac{M(a,b,cx)}{M(2,1,x)} = \frac{\sum_{k=0}^{\infty} \frac{c^k (a)_k}{(b)_k} \frac{x^k}{k!}}{\sum_{k=0}^{\infty} \frac{(2)_k}{(1)_k} \frac{x^k}{k!}},$$



(a) Graph of the function $R_{1,1/3,4/3}(x)$

(b) Graph of the function $R_{1,4/3,4/3}(x)$

(c) Graph of the function $R_{1,1/3,1.02}(x)$

Figure 1: Monotonicity of the function (3) with different parameters

which is the ratio of two power series. The monotonicity rule we provided in the previous section offers a theoretical tool for its study. Note that the ratio of the coefficients of the power series

$$\left\{ \frac{c^k (a)_k}{(b)_k} \Big/ \frac{(2)_k}{(1)_k} \right\}_{k \geq 0}$$

changes monotonicity at most twice (as we will explain in detail in Lemma 22). According to the monotonicity rule 18, we know that the monotonicity of the function (3) changes at most twice, which means that the monotonicity pattern of the function (3) must be one of "$\nearrow$", "$\searrow$", "$\nearrow\searrow$", "$\searrow\nearrow$", "$\nearrow\searrow\nearrow$", and "$\searrow\nearrow\searrow$". Next, we will classify the region $\{(a, b, c) : a, b, c > 0\}$ so that each class corresponds to a specific monotonicity pattern in Theorem 24. This helps us better study the properties of the confluent hypergeometric function of the first kind, partly because some tight bounds can be easily obtained if the monotonicity is known.

Next, we introduce two lemmas for calculating Yang's H-function.

Lemma 20. *Let $a, b, c > 0$ and $(a, b) \neq (2, 1)$. If $c \neq 1$, then*

$$H_{M(a,b;cx),M(2,1;x)}(\infty) > (<)0$$

if and only if $c > (<)1$. If $c = 1$, then

$$H_{M(a,b;x),M(2,1;x)}(\infty) > (<)0$$

if and only if $a > (<)b + 1$. Moreover, if $c = 1$ and $a = b + 1$, then

$$H_{M(a,b;x),M(2,1;x)}(\infty) > (<)0$$

if and only if $b < (>)1$.

Proof. The conclusion follows directly from formula (5), and the detailed derivation is therefore omitted. □

Lemma 21. *Let $a, b, c > 0$ and $(a, b) \neq (2, 1)$. If $c \neq 1$, then*

$$H_{M'(a,b;cx),M'(2,1;x)}(\infty) > (<)0$$

if and only if $c > (<)1$. If $c = 1$, then

$$H_{M'(a,b;x),M'(2,1;x)}(\infty) > (<)0$$

if and only if $a > (<)b + 1$. Moreover, if $c = 1$ and $a = b + 1$, then

$$H_{M'(a,b;x),M'(2,1;x)}(\infty) > (<)0$$

if and only if $b < (>)1$.

Proof. The conclusion follows directly from formula (5), and the detailed derivation is therefore omitted. □

We now study the monotonicity of the ratio of the coefficients of the power series. For convenience, for a non-identically-zero sequence a_k, we introduce the following notation:

(i) If $a_k \geq (\leq)0$ for all $k \in \mathbb{N}$, we say a_k is "$+(-)$".

(ii) If there exists $m_1 \in \mathbb{N}$ such that $a_k \geq (\leq)0$ for all $k \in [0, m_1]_{\mathbb{N}}$ and $a_k \leq (\geq)0$ for all $k \in [m_1, \infty)_{\mathbb{N}}$, and furthermore, a_k is not identically equal to 0 on both $[0, m_1]_{\mathbb{N}}$ and $[m_1, \infty)_{\mathbb{N}}$, then we say a_k is "$+-(-+)$".

(iii) If there exist $m_2, m_3 \in \mathbb{N}$ such that $a_k \geq (\leq)0$ for all $k \in [0, m_2]_{\mathbb{N}} \cup [m_3, \infty)_{\mathbb{N}}$ and $a_k \leq (\geq)0$ for all $k \in [m_2, m_3]_{\mathbb{N}}$, and furthermore, a_k is not identically equal to 0 on $[0, m_2]_{\mathbb{N}}$, $[m_2, m_3]_{\mathbb{N}}$, and $[m_3, \infty)_{\mathbb{N}}$, then we say a_k is "$+-+(-+-)$".

Lemma 22. *Let $a, b, c > 0$ and $(a, b) \neq (2, 1)$. Then*

(I) the sequence

$$\left\{\frac{c^k (a)_k (1)_k}{(b)_k (2)_k}\right\}_{k \geq 0} \tag{6}$$

is "$\nearrow (\searrow)$" if one of the following conditions holds:

(i) $a = b + 1$, $b < (>)1$, $c = 1$;

(ii) $a > (<)b + 1$, $a \geq (\leq)2b$, $c = 1$;

(iii) $ac = 2b$, $ac - b + 2c - 3 \geq (\leq)0$, $c > (<)1$;

(iv) $ac - 2b > (<)0$, $ac - b + c - 2 < (>)0$, $\min_{k \in \mathbb{N}} L(k) \geq (\leq)0$, $c > (<)1$;

(v) $ac - 2b > (<)0$, $ac - b + c - 2 \geq (\leq)0$.

(II) the sequence (6) is "$\nearrow\searrow (\searrow\nearrow)$" if one of the following conditions holds:

(i) $a < (>) b + 1$, $a > (<) 2b$, $c = 1$;

(ii) $ac - 2b > (<) 0$, $c < (>) 1$;

(iii) $ac = 2b$, $ac - b + 2c - 3 > (<) 0$, $c < (>) 1$.

(III) the sequence (6) *is "*$\nearrow\searrow\nearrow$ $(\searrow\nearrow\searrow)$*" if the following condition holds:* $ac - 2b > (<) 0$, $ac - b + c - 2 < (>) 0$, $\min_{k\in\mathbb{N}} L(k) < (>) 0$, $c > (<) 1$,

where $L(k) := (c-1)k^2 + (ac - b + c - 2)k + ac - 2b$.

Proof. By taking the difference, we obtain

$$\frac{c^{k+1}(a)_{k+1}(1)_{k+1}}{(b)_{k+1}(2)_{k+1}} - \frac{c^k(a)_k(1)_k}{(b)_k(2)_k} = L(k)\frac{c^k(a)_k(1)_k}{(b)_{k+1}(2)_{k+1}},$$

where $L(k)$ is a polynomial sequence of degree at most 2. Next, we analyze the sign of this polynomial.

Case 1 If $c = 1$, then $L(k) = (a - b - 1)k + a - 2b$. In this case, if $a = b + 1$, the sequence $L(k)$ is "$+(-)$" if and only if $b < (>) 1$. If $a > (<) b+1$ and $a \geq (\leq) 2b$, then the sequence $L(k)$ is "$+(-)$"; assume $a > (<) b + 1$ and $a < (>) 2b$, then the sequence $L(k)$ is "$- + (+-)$".

Case 2 If $c > (<) 1$, then the function $x \mapsto L(x)$ is a quadratic function opening upwards (downwards), and $L(0) = ac - 2b$.

(i) If $L(0) = ac - 2b < (>) 0$, the sequence $L(k)$ is necessarily "$- + (+-)$";

(ii) If $L(0) = ac - 2b = 0$ and $L(1) = ac - b + 2c - 3 < (>) 0$, the sequence $L(k)$ is "$- + (+-)$";

(iii) If $L(0) = ac - 2b = 0$ and $L(1) = ac - b + 2c - 3 \geq (\leq) 0$, the sequence $L(k)$ is "$+(-)$";

(iv) If $L(0) = ac - 2b > (<) 0$ and $ac - b + c - 2 < (>) 0$ and $\min_{k\in\mathbb{N}} L(k) < (>) 0$, the sequence $L(k)$ is "$+ - +(- + -)$";

(v) If $L(0) = ac - 2b > (<) 0$ and $ac - b + c - 2 < (>) 0$ and $\min_{k\in\mathbb{N}} L(k) \geq (\leq) 0$, the sequence $L(k)$ is "$+(-)$";

(vi) If $L(0) = ac - 2b > (<) 0$ and $ac - b + c - 2 \geq (\leq) 0$, the sequence $L(k)$ is "$+(-)$".

□

Remark 23. *When* $c = \frac{4}{3}$, *we illustrate the conclusions of the above lemma in Figure 2. According to Lemma 22, we have the following conclusions:*

(I) The sequence (6) *is "*$\nearrow$*" if* $3b \leq 2a$, $9b + 1 \leq 8a$

(II) The sequence (6) *is "*$\searrow\nearrow$*" if one of the following conditions holds:*

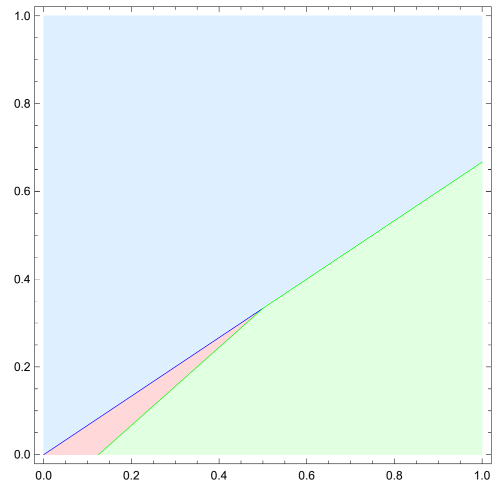


Figure 2: Monotonicity patterns of the sequence (6) for different a and b when $c = \frac{4}{3}$ (horizontal axis represents a, vertical axis represents b)

> *(i)* $3b > 2a$;
>
> *(ii)* $3b = 2a$, $a < \frac{1}{2}$,

(III) The sequence (6) *is "*$\nearrow\searrow\nearrow$*" if* $3b < 2a$ *and* $9b + 1 > 8a$.

This conclusion can also be demonstrated in Figure 2, where the sequence (6) *has "*$\nearrow$*" monotonicity in the green region or on the green lines, "*$\searrow\nearrow$*" monotonicity in the blue region or on the blue lines, and "*$\nearrow\searrow\nearrow$*" monotonicity in the red region. For example, when* $a = 0.1$ *and* $b = 0.5$*, the sequence* (6) *is "*$\nearrow\searrow\nearrow$*".*

Proof. Note that under the condition $c = 4/3$ and $3b < 2a$, we can determine $\min_{k\in\mathbb{N}} L(k) = L(1)$ because the axis of symmetry of the quadratic function $x \mapsto L(x)$ is $x = \frac{3}{2}b - 2a + 1 < 1 - a < 1$. □

Next, we can determine the monotonicity of the function (3).

Theorem 24. *Let* $a, b, c > 0$ *and* $(a, b) \neq (2, 1)$*. Then*

(I) The function (3) *has "*$\nearrow$ ($\searrow$)*" monotonicity if one of the following conditions holds:*

> *(i)* $a = b + 1$, $b < (>)1$, $c = 1$;
>
> *(ii)* $a > (<)b + 1$, $a \geq (\leq)2b$, $c = 1$;
>
> *(iii)* $ac = 2b$, $ac - b + 2c - 3 \geq (\leq)0$, $c > (<)1$;
>
> *(iv)* $ac - 2b > (<)0$, $ac - b + c - 2 < (>)0$, $\min_{k\in\mathbb{N}} L(k) \geq (\leq)0$, $c > (<)1$.
>
> *(v)* $ac - 2b > (<)0$, $ac - b + c - 2 \geq (\leq)0$.

(II) The function (3) *has "$\nearrow\searrow$ ($\searrow\nearrow$)" monotonicity if one of the following conditions holds:*

(i) $a < (>) b+1$, $a > (<) 2b$, $c = 1$;

(iii) $ac - 2b > (<) 0$, $c < (>) 1$;

(iv) $ac = 2b$, $ac - b + 2c - 3 > (<) 0$, $c < (>) 1$.

(III) The sequence (6) *is "$\nearrow\searrow\nearrow$ ($\searrow\nearrow\searrow$)" if the following conditions hold:* $ac - 2b > (<) 0$, $ac - b + c - 2 < (>) 0$, $\min_{k\in\mathbb{N}} L(k) < (>) 0$, $c > (<) 1$, *and there exists* $x_0 \in (0,\infty)$ *such that* $H_{M(a,b;x),M(2,1,x)}(x_0) < (>) 0$.

Proof. By applying the monotonicity rules 1, 3, 4 and Lemmas 20, 21, we can easily obtain the desired conclusions. □

Remark 25. *For the remaining case, if* $ac - 2b > (<) 0$, $ac - b + c - 2 < (>) 0$, $\min_{k\in\mathbb{N}} L(k) < (>) 0$, $c > (<) 1$ *and we have* $H_{M(a,b;x),M(2,1,x)}(x) \geq (\leq) 0$ *for any* $x \in (0,\infty)$, *then the function* (3) *is increasing (decreasing).*

Finally, by Kummer's transformation $M(a,b,x) = e^x M(b-a,b,-x)$, we derive the following corollaries.

Corollary 26. *Let* $a, b > 0$ *and* $(a,b) \neq (2,1)$. *Then the function*

$$x \mapsto \frac{M(a,b,x)}{M(2,1,x)} = \frac{e^{-x}M(a,b,x)}{x+1} = \frac{M(b-a,b,-x)}{x+1} \tag{7}$$

is "$\nearrow$ ($\searrow$)" if $a = b+1$, $b < (>) 1$ *or* $a > (<) b+1$, $a \geq (\leq) 2b$; *the function* (7) *is "$\nearrow\searrow$ ($\searrow\nearrow$)" if* $a < (>) b+1$, $a > (<) 2b$.

Noting that $M(a,2a,x) = e^{x/2}(x/4)^{1/2-a}\Gamma(a+1/2)I_{a-1/2}(x/2)$, where $I_\nu(x)$ is the modified Bessel function of the first kind, we have the following corollary.

Corollary 27. *Let* $\nu > -\frac{1}{2}$. *Then the function*

$$x \mapsto \frac{e^{-x}x^{-\nu}I_\nu(x)}{2x+1}$$

is monotonically decreasing, and we have

$$0 < \frac{2^\nu e^{-x}\Gamma(\nu+1)x^{-\nu}I_\nu(x)}{2x+1} < 1.$$

For $a = 1, b = \frac{3}{2}$, we have $M(a,b,x) = M(1,\frac{3}{2},x) = \frac{\sqrt{\pi}e^x \operatorname{erf}(\sqrt{x})}{2\sqrt{x}}$, and we derive the following corollary, where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the error function.

Corollary 28. *The function*

$$x \mapsto \frac{\operatorname{erf}\left(\sqrt{x}\right)}{\sqrt{x}(x+1)}$$

is monotonically decreasing, and we have

$$0 < \frac{\sqrt{\pi}\operatorname{erf}\left(\sqrt{x}\right)}{2\sqrt{x}(x+1)} < 1.$$

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Zhong-Xuan Mao
Department of Mathematics and Physics,
North China Electric Power University,
Baoding, P. R. China
E-mail: *maozhongxuan000@gmail.com*
https://orcid.org/0000-0001-5089-301X

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Jing-Feng Tian
Department of Mathematics and Physics,
North China Electric Power University,
Baoding, P. R. China
E-mail: *tianjf@ncepu.edu.cn*
https://orcid.org/0000-0002-0631-038X