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ENUMERATION OF LATTICE PATHS USING VIETA POLYNOMIALS

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Dyck-type lattice paths, consisting of up and down diagonal steps on the integer lattice, can be either unconfined – if the only restriction is to remain in the non-negative half-plane – or confined, if, in addition, they are not allowed to pass above a fixed horizontal line. In this paper, we establish a formula for the number of unconfined Dyck-type lattice paths of a given length, and also investigate the enumeration of confined Dyck-type paths, deriving recurrence relations involving Vieta-Fibonacci polynomials.

1. INTRODUCTION

Lattice paths are a fundamental concept in mathematics and a useful tool to model a variety of real-world scenarios where movement or decision-making follows a discrete set of possibilities. They are applied in various areas such as combinatorics and graph theory [**2, 3, 31**], statistics and probabilities [**20, 26, 28**] or reliability theory [**8, 17, 18**].

Let $\mathbb{Z}^2$ be the set of lattice points in plane. A lattice path in $\mathbb{Z}^2$ with steps in the set $S \subset \mathbb{Z}^2$ is a sequence of lattice points $v_0, v_1, \ldots, v_k \in \mathbb{Z}^2$ such that each consecutive difference $v_i - v_{i-1}$ lies in S [**31**]. Lattice paths represent an important instrument for studying various combinatorial objects. For instance, the number of lattice paths with steps in the set $S = \{(1,0),(0,1)\}$ from the origin

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to a point $(a, b) \in \mathbb{N}^2$ is given by the binomial coefficient $\binom{a+b}{a}$ (see [**5**]). Another classical result is related to the *Dyck paths* - the lattice paths with steps in the set $S = \{(1, 1), (1, -1)\}$, joining the origin $(0, 0)$ to the point $(2n, 0)$ and never passing below the x-axis. The number of Dyck paths is equal to the Catalan number: $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see for instance [**32**], Theorem 1.5.1). We remark that sometimes the Dyck paths are considered to be lattice paths with steps in the set $S = \{(1, 0), (0, 1)\}$, joining the origin to (n, n) and never passing below the straight line $y = x$; obviously, the two approaches are equivalent. A more general result (see [**30**]) states that the lattice paths with steps in $\{(1, 0), (0, 1)\}$ joining the origin to (n, kn) without crossing the line $y = kx$ or, equivalently, the k-Dyck paths of length $(k+1)n$ – lattice paths with steps in the set $\{(1, 1), (1, -k)\}$, starting at $(0, 0)$, ending at $((k+1)n, 0)$ and staying above the x-axis (see [**15**]) – are counted by the Fuss-Catalan numbers, $\frac{1}{kn+1}\binom{(k+1)n}{n}$. We also highlight that enumeration results related to k-generalized Dyck paths can produce new combinatorial identities and new expressions for Catalan numbers (see [**25**]).

Our interest in lattice path combinatorics was sparked by a practical problem in the domain of reliability theory, namely - the calculation of the reliability polynomial of hammock networks. Introduced by Moore and Shannon in [**27**], the hammock networks are a regular type of two-terminal networks highly appreciated for their robustness. But, in spite of their regularity, the computation of the reliability polynomial of large-size hammock networks is a difficult task (see [**10**] for a comprehensive presentation of hammock networks). Fortunately, an efficient approximation is realized by the full Hermite interpolation polynomial (see [**9**]), which provides a better approximation if the first two non-zero coefficients of the reliability polynomial are known. And now, the lattice paths come into play, as the first non-zero coefficient of a hammock network of length l and width w is related to the number of lattice paths of length l with steps in the set $S = \{(1, 1), (1, -1)\}$, starting on the y-axis and never passing below the x-axis, or above the horizontal line $y = w$.

We refer to *Dyck-type lattice paths* as lattice paths with steps in the set $S = \{(1, 1), (1, -1)\}$, starting on the y-axis and never passing below the x-axis. If $y \geq 0$ is the only constraint, then we discuss about *unconfined* Dyck-type lattice paths. If, in addition, the paths are also constrained by $y \leq w$, then they are said to be *confined* Dyck-type lattice paths (also known as *corridor lattice paths* [**2, 3, 11**] or lattice paths in strips along the x-axis [**6, 13**]).

In the present paper, we investigate both classes of Dyck-type lattice paths, with particular emphasis on the confined ones. Our main goal is to analyse their enumerative properties and structural features. In particular, we show that the number of such confined paths is intimately related to the Vieta polynomials, a connection that appears to be new in the literature. This relationship provides a new perspective on the combinatorial structure of these paths and constitutes one of the main novel contributions of the present work.

The outline of the paper is as follows. In Section 2, we find and prove an explicit formula for the number $\tilde{x}_{i,l}$ of unconfined Dyck-type lattice paths of length

l starting at the point $(0, i)$ (or, equivalently, ending up at the point (l, i)). The triangular sequence obtained for $0 \leq i \leq l$, is the sequence A375659 introduced by the first author in the On-line Encyclopedia of Integer Sequences [**29**].

In Section 3, using the reflection principle, we first calculate, for $l \leq w+1$, the number $x_{i,l}$ of lattice paths of length l confined in the strip $0 \leq y \leq w$ and starting at the point $(0, i)$, $i = 0, \ldots, w$, and the total number $N(w, l)$ of confined Dyck-type lattice paths. Then, we analyze the case $l > w + 1$ by using the instruments of linear algebra (see also [**8**]). A novel idea of our paper is to use the Cayley-Hamilton theorem in order to find a recurrence relation of order $w + (w \bmod 2)$ which applies to every sequence $\{x_{i,l}\}_{l \geq 0}$ (for all $i = 0, \ldots, w$). We prove that the characteristic polynomial of the recurrence relation is the Vieta-Fibonacci polynomial $V_{w+2}(x)$.

In Section 4, we present the main properties of the Vieta-Fibonacci $V_n(x)$ and Vieta-Lucas polynomials $v_n(x)$ (introduced in [**16**]) and also highlight several features of the related polynomials $\Delta V_n(x)$ and $\Sigma V_n(x)$. All of these polynomial sequences are orthogonal polynomials defined by the same recurrence relation (of second order), but with different initial values. A detailed presentation of the deep connection between lattice paths enumeration and the theory of orthogonal polynomials can be found in [**21**]. We note that Vieta-Fibonacci polynomials $V_n(x)$ and the related polynomials $\Delta V_n(x)$, $\Delta^2 V_n(x)$ also appear in the recurrence relations arising in the enumeration of meaningful compositions of higher-order differential operators on the space $\mathbb{R}^n$ (see [**22, 23, 24**]).

Finally, in Section 5, we show how can be used the symmetry (w.r.t. the line $y = \frac{w}{2}$) in order to obtain halved-order recurrence relations whose characteristic polynomial is either the Vieta-Lucas polynomial $v_{n+1}(x)$ if $w = 2n$, or the Δ–Vieta-Fibonacci polynomial $\Delta V_n(x)$, if $w = 2n - 1$. As initial values for these recurrent sequences, we use the numbers $\tilde{x}_{i,l}$ of unconfined Dyck-type lattice paths.

2. UNCONFINED DYCK-TYPE LATTICE PATHS

Definition 1. A lattice path with steps in the set $S = \{(1, 1), (1, -1)\}$, starting on the y-axis and never passing below the x-axis is said to be an unconfined Dyck-type lattice path.

Definition 2. A lattice path with steps in the set $S = \{(1, 1), (1, -1)\}$, starting on the y-axis and never passing below the x-axis or above the line $y = w$ is said to be a Dyck-type lattice path confined in the interval $[0, w]$.

Let $n^{(l)}(i, j)$ denote the number of unconfined Dyck-type lattice paths joining the points $(0, i)$ and (l, j). We say that a lattice point (x, y) is even (odd) if $x + y$ is even (odd, respectively). Obviously, the points of a Dyck-type lattice path are all odd, or all even, so $n^{(l)}(i, j) = 0$ if $i \not\equiv l + j \pmod 2$. We also remark that $n^{(l)}(i, j) = 0$ if $|i - j| > l$.

For every $i, l \geq 0$, we denote by $\tilde{x}_{i,l} = \sum_{j \geq 0} n^{(l)}(i,j)$ the number of unconfined Dyck-type lattice paths of length l, starting at the point $(0,i)$ (or, equivalently, the number of unconfined Dyck-type lattice paths ending up at the point (l,i)). For $l = 0$, we consider $\tilde{x}_{i,l} = 1$, for all $i \geq 0$. The following theorem states an explicit formula for the numbers $\tilde{x}_{i,l}$. We consider $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$.

Theorem 1. *The number $\tilde{x}_{i,l}$ of unconfined Dyck-type lattice paths of length l, starting at the point $(0,i)$ is given by the formula*

$$\tilde{x}_{i,l} = \sum_{k=\lfloor \frac{l-i}{2} \rfloor}^{\lfloor \frac{l+i}{2} \rfloor} \binom{l}{k}, \text{ for all } i, l \geq 0. \tag{1}$$

Proof. Let $(0,i)$ and (l,j) be two points such that $i \equiv l + j \pmod 2$. The *total* number of lattice paths with steps $(1,1)$ or $(1,-1)$, joining these points (including the ones passing below the x-axis) is given by the binomial coefficient

$$\binom{l}{\frac{l+i-j}{2}}. \tag{2}$$

To count how many of these lattice paths are crossing (and passing below) the x-axis, we use the *reflection principle* (see [**21, 26**]). Any such lattice path has at least one point of intersection with the line $y = -1$. Let $(x_0, -1)$ be the rightmost point of intersection. By reflecting w.r.t. the line $y = -1$ the part of the lattice path between $(x_0, -1)$ and (l, j), we obtain a lattice path with steps $(1,1)$ or $(1,-1)$, joining the points $(0,i)$ and $(l, -j-2)$. Thus, we have a one-to-one correspondence between the lattice paths from $(0,i)$ to $(l,-j-2)$ and the lattice paths from $(0,i)$ to (l,j) that pass below the x-axis. Hence, using (2), the number of unconfined Dyck-type lattice paths from $(0,i)$ to (l,j) is

$$n^{(l)}(i,j) = \binom{l}{\frac{l+i-j}{2}} - \binom{l}{\frac{l+i+j+2}{2}}, \tag{3}$$

and the total number of unconfined Dyck-type lattice paths of length l starting from $(0,i)$ is

$$\tilde{x}_{i,l} = \sum_{j=0}^{i+l} n^{(l)}(i,j) = \sum_{j=0}^{i+l} \left(\binom{l}{\frac{l+i-j}{2}} - \binom{l}{\frac{l+i+j+2}{2}} \right), \tag{4}$$

where the terms of the series above are considered to be 0 whenever $j \not\equiv l + i \pmod 2$. We set $k = \frac{l+i-j}{2}$ so the equation (4) becomes:

$$\tilde{x}_{i,l} = \sum_{k=0}^{\lfloor \frac{l+i}{2} \rfloor} \binom{l}{k} - \sum_{k=0}^{\lfloor \frac{l+i}{2} \rfloor} \binom{l}{k-i-1}.$$

Furthermore, in the second sum, we let $r = k - i - 1$, so we obtain

$$\tilde{x}_{i,l} = \sum_{k=0}^{\lfloor \frac{l+i}{2} \rfloor} \binom{l}{k} - \sum_{r=0}^{\lfloor \frac{l-i}{2} \rfloor - 1} \binom{l}{r}, \tag{5}$$

and the formula (1) follows. □

Remark 1. If $i \geq l$ then the second sum in (5) is empty, so we have

$$\tilde{x}_{i,l} = 2^l, \text{ for all } i \geq l.$$

Thus, for fixed values of the length l, every sequence $\{\tilde{x}_{i,l}\}_{i \geq 0}$ becomes constant for $i \geq l$. This can be noticed in Table 1, which gives the first terms of the infinite matrix $\{\tilde{x}_{i,l}\}_{i,l \geq 0}$. The triangular sequence $\{\tilde{x}_{i,l}\}_{0 \leq i \leq l}$ contains the *significant terms* of the matrix (the boldfaced numbers) and corresponds to the sequence A375659, while the infinite matrix $\{\tilde{x}_{i,l}\}_{i,l \geq 0}$ is the sequence A368175 in the On-line Encyclopedia of Integer Sequences [**29**].

Table 1: The number of unconfined Dyck-type lattice paths $\tilde{x}_{i,l}$, $i, l \geq 0$

$i \backslash l$	0	1	2	3	4	5	6	…
0	**1**	**1**	**2**	**3**	**6**	**10**	**20**	…
1	1	**2**	**3**	**6**	**10**	**20**	**35**	…
2	1	2	**4**	**7**	**14**	**25**	**50**	…
3	1	2	4	**8**	**15**	**30**	**56**	…
4	1	2	4	8	**16**	**31**	**62**	…
5	1	2	4	8	16	**32**	**63**	…
6	1	2	4	8	16	32	**64**	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Remark 2. It can be easily shown that the numbers $\tilde{x}_{i,l}$ satisfy the recurrence relation

$$\tilde{x}_{i,l} = \tilde{x}_{i-1,l-1} + \tilde{x}_{i+1,l-1} \text{ for all } i \geq 0 \text{ and } l \geq 1,$$

where $\tilde{x}_{i,0} = 1$ for all $i \geq 0$ and $\tilde{x}_{-1,l} = 0$ for all $l \geq 0$.

3. CONFINED DYCK-TYPE LATTICE PATHS

We consider in this section the Dyck-type lattice paths of width w. We denote by $n^{(w,l)}(i,j)$ the number of Dyck-type lattice paths confined in the interval $[0, w]$, joining the points $(0, i)$ and (l, j), where $i, j \in \{0, 1, \ldots, w\}$. For every $l \geq 0$ and

$i = 0, \dots, w$, we denote by $x_{i,l} = \sum_{j=0}^{w} n^{(w,l)}(i,j)$ the number of confined Dyck-type lattice paths of length l, starting at the point $(0,i)$ (or, equivalently, the number of confined Dyck-type lattice paths ending up at the point (l,i)). If $l = 0$, we consider $x_{i,0} = 1$, for all $i = 0, \dots, w$. Like in the previous section, we look for an explicit formula for the numbers $x_{i,l}$.

We remark that the numbers $n^{(w,l)}(i,j)$ are expressed by the following trigonometric sums (see [**20**], Eq. (5)):

$$n^{(w,l)}(i,j) = \frac{2}{w+2}\sum_{k=1}^{w+1}\left(2\cos\frac{k\pi}{w+2}\right)^l \sin\frac{k(i+1)\pi}{w+2}\sin\frac{k(j+1)\pi}{w+2}.$$

We aim to find an expression of $n^{(w,l)}(i,j)$ based on binomial coefficients (similar to the formula (3) obtained for unconfined lattice paths). Thus, for every $j \in \{0, 1, \dots, w\}$ with the property that $j \equiv l + i \pmod 2$, the number of Dyck-type lattice paths from $(0,i)$ to (l,j) confined in the interval $[0,w]$ is equal to the difference between the total number of Dyck-type lattice paths, $\binom{l}{\frac{l+i-j}{2}}$, and the number of lattice paths that violates at least one of the restrictions $y \geq 0$ and $y \leq w$. If $l \leq w + 1$, then only one of the restrictions can be violated (see also [**9**]). As stated in the proof of Theorem 1, the number of lattice paths that pass below the x-axis is equal to the number of all lattice paths from $(0,i)$ to $(l,-j-2)$. In a similar way, we can conclude that the number of lattice paths passing above the straight line $y = w$ is equal to the number of all lattice paths from $(0,i)$ to $(l, 2w+2-j)$. Thus, we have

$$n^{(w,l)}(i,j) = \binom{l}{\frac{l+i-j}{2}} - \binom{l}{\frac{l+i+j+2}{2}} - \binom{l}{\frac{l+i+j}{2} - w - 1},$$

so the number of confined Dyck-type lattice paths of length l, starting at the point $(0,i)$ is

$$x_{i,l} = \sum_{j=0}^{w} n^{(w,l)}(i,j) = \sum_{j=0}^{w}\left(\binom{l}{\frac{l+i-j}{2}} - \binom{l}{\frac{l+i+j+2}{2}} - \binom{l}{\frac{l+i+j}{2} - w - 1}\right), \tag{6}$$

where the binomial coefficients above are considered to be 0 whenever $j \not\equiv l + i$ (mod 2). We denote:

$$k = \frac{l+i-j}{2},\ r = \frac{l+i+j+2}{2},\ \text{and } s = \frac{l+i+j-2w-2}{2},$$

so the equation (6) becomes:

$$x_{i,l} = \sum_{k=\lfloor\frac{l+i-w+1}{2}\rfloor}^{\lfloor\frac{l+i}{2}\rfloor}\binom{l}{k} - \sum_{r=\lfloor\frac{l+i+3}{2}\rfloor}^{\lfloor\frac{l+i+w+2}{2}\rfloor}\binom{l}{r} - \sum_{s=\lfloor\frac{l+i-2w-1}{2}\rfloor}^{\lfloor\frac{l+i-w-2}{2}\rfloor}\binom{l}{s}.$$

If $l \le w+1$, then $\left\lfloor \frac{l+i+w+2}{2} \right\rfloor \ge l$ and $\left\lfloor \frac{l+i-2w-1}{2} \right\rfloor \le 0$, so we have:

$$\begin{aligned} x_{i,l} &= \sum_{k=\left\lfloor \frac{l+i-w+1}{2} \right\rfloor}^{\left\lfloor \frac{l+i}{2} \right\rfloor} \binom{l}{k} - \sum_{r=\left\lfloor \frac{l+i+3}{2} \right\rfloor}^{l} \binom{l}{r} - \sum_{s=0}^{\left\lfloor \frac{l+i-w-2}{2} \right\rfloor} \binom{l}{s} \\ &= 2^l - \sum_{k=0}^{\left\lfloor \frac{l+i-w-1}{2} \right\rfloor} \binom{l}{k} - \sum_{k=\left\lfloor \frac{l+i+2}{2} \right\rfloor}^{l} \binom{l}{k} - \sum_{r=\left\lfloor \frac{l+i+3}{2} \right\rfloor}^{l} \binom{l}{r} - \sum_{s=0}^{\left\lfloor \frac{l+i-w-2}{2} \right\rfloor} \binom{l}{s} \end{aligned}$$

and the next theorem follows.

Theorem 2. *If $l \le w+1$, then the number of Dyck-type lattice paths of length l, confined in the interval $[0,w]$ and starting at the point $(0,i)$ is given by the formula*

$$x_{i,l} = 2^l - \sum_{k=0}^{\left\lfloor \frac{l+i-w-1}{2} \right\rfloor} \binom{l}{k} - \sum_{k=0}^{\left\lfloor \frac{l-i-1}{2} \right\rfloor} \binom{l}{k} - \sum_{k=0}^{\left\lfloor \frac{l-i-2}{2} \right\rfloor} \binom{l}{k} - \sum_{k=0}^{\left\lfloor \frac{l+i-w-2}{2} \right\rfloor} \binom{l}{k}, \tag{7}$$

for all $i = 0, 1, \ldots, w$.

Remark 3. We remark that for $i, l \le \left\lfloor \frac{w}{2} \right\rfloor$ the numbers of confined and unconfined lattice paths are equal:

$$x_{i,l} = \tilde{x}_{i,l} = \sum_{k=\left\lfloor \frac{l-i}{2} \right\rfloor}^{\left\lfloor \frac{l+i}{2} \right\rfloor} \binom{l}{k}, \text{ for all } i, l \le \left\lfloor \frac{w}{2} \right\rfloor. \tag{8}$$

Remark 4. It can be also shown (using (7)) that for any fixed value of l, the sequence $\{x_{i,l}\}_{i=0,\ldots,w}$ is symmetric, that is:

$$x_{i,l} = x_{w-i,l}, \text{ for all } i = 0, \ldots, w. \tag{9}$$

Moreover, if we denote by $S_k(i)$, $k = 1, 2, 3, 4$ the four sums in (7), then we can see that

$$S_1(w-i) = S_2(i), \text{ and } S_3(w-i) = S_4(i), \text{ for all } i = 0, \ldots, w. \tag{10}$$

We denote by $N(w,l)$ the number of Dyck-type lattice paths of length l confined in the strip $0 \le y \le w$:

$$N(w,l) = \sum_{i=0}^{w} x_{i,l}.$$

The following theorem establishes the expression of $N(w,l)$ for any $l \le w+1$. We remark that in [**9**, Theorem 5] the number of even ($N_e(w,l)$) and odd ($N_o(w,l)$) Dyck-type lattice paths respectively, were investigated (as the main interest was to find the first nonzero coefficients of some reliability polynomials). The formula we give in Theorem 3 for the total number of lattice paths $N(w,l) = N_e(w,l) + N_o(w,l)$ matches the results found in [**9**], and the proof is considerably simplified.

Theorem 3. *For any positive integers l and w such that $l \leq w+1$, the total number of Dyck-type lattice paths of length l, confined in the strip $0 \leq y \leq w$ is*

$$N(w,l) = (w+2)\cdot 2^l - (2l+1+l \bmod 2)\binom{l}{\lfloor \frac{l}{2} \rfloor} \tag{11}$$

Remark 5. The sequence $\{a_n\}_{n\geq 1}$ defined by

$$a_n = (2n+1+n \bmod 2)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

is the sequence A152548 in the On-line Encyclopedia of Integer Sequences [**29**].

Proof of Theorem 3. Using the formulas (7) and (10) we obtain that the number of Dyck-type confined lattice paths is

$$N(w,l) = 2^l(w+1) - 2\sum_{i=0}^{w} S_2(i) - 2\sum_{i=0}^{w} S_3(i) \tag{12}$$

We compute the two sums in the equation (12) and prove that they are not depending on w.

First, we suppose that the length l is even: $l = 2m$. Then,

$$\begin{aligned}
\sum_{i=0}^{w} S_2(i) &= \sum_{i=0}^{w} \sum_{k=0}^{\lfloor \frac{2m-i-1}{2} \rfloor} \binom{2m}{k} = 2\sum_{k=0}^{m-1}(m-k)\binom{2m}{k} \\
&= 2m\sum_{k=0}^{m-1}\binom{2m}{k} - 2\sum_{k=1}^{m-1} 2m\binom{2m-1}{k-1} \\
&= m\left(2^{2m} - \binom{2m}{m}\right) - 2m\left(2^{2m-1} - \binom{2m-1}{m} - \binom{2m-1}{m}\right) \\
&= m\binom{2m}{m},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{w} S_3(i) &= \sum_{i=0}^{w} \sum_{k=0}^{\lfloor \frac{2m-i-2}{2} \rfloor} \binom{2m}{k} \\
&= 2\sum_{k=0}^{m-1}(m-k)\binom{2m}{k} - \sum_{k=0}^{m-1}\binom{2m}{k} \\
&= S_2(i) - \tfrac{1}{2}\left(2^{2m} - \binom{2m}{m}\right) \\
&= (m+\tfrac{1}{2})\binom{2m}{m} - 2^{2m-1}.
\end{aligned}$$

Hence, the relation (12) becomes:

$$N(w,2m) = 2^{2m}(w+2) - (4m+1)\binom{2m}{m},$$

so the formula (11) holds in this case.

If l is odd, $l = 2m+1$, then

$$\begin{aligned}
\sum_{i=0}^{w} S_2(i) &= \sum_{i=0}^{w} \sum_{k=0}^{\lfloor \frac{2m-i}{2} \rfloor} \binom{2m+1}{k} \\
&= 2\sum_{k=0}^{m}(m+1-k)\binom{2m+1}{k} - \sum_{k=0}^{m}\binom{2m+1}{k} \\
&= 2(m+1)\sum_{k=0}^{m}\binom{2m+1}{k} - 2\sum_{k=1}^{m}(2m+1)\binom{2m}{k-1} - 2^{2m} \\
&= (2m+1)2^{2m} - (2m+1)\left(2^{2m} - \binom{2m}{m}\right) \\
&= (2m+1)(m+1)\binom{2m}{m} = (m+1)\binom{2m+1}{m},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{w} S_3(i) &= \sum_{i=0}^{w} \sum_{k=0}^{\lfloor \frac{2m-i-1}{2} \rfloor} \binom{2m+1}{k} = 2\sum_{k=0}^{m-1}(m-k)\binom{2m+1}{k} \\
&= S_2(i) - \sum_{k=0}^{m}\binom{2m+1}{k} \\
&= (m+1)\binom{2m+1}{m} - 2^{2m}.
\end{aligned}$$

Hence, the relation (12) becomes:

$$N(w,2m+1) = 2^{2m+1}(w+2) - (4m+4)\binom{2m+1}{m},$$

so the formula (11) holds also for $l = 2m+1$. □

Both Theorem 2 and Theorem 3 hold only under the restriction $l \leq w+1$. In the following, we aim to find a recurrence relation which allows the computation of $x_{i,l}$, $i = 0, \ldots, w$, for any length l, starting from the initial values obtained in Theorem 2.

First of all, we remark that for every $i = 1, \dots, w$ and $l = 1, 2, \dots$, one can write:

$$x_{i,l} = \begin{cases} x_{1,l-1} & \text{if } i = 0, \\ x_{i-1,l-1} + x_{i+1,l-1} & \text{if } i = 1, \dots, w-1, \\ x_{w-1,l-1} & \text{if } i = w. \end{cases} \tag{13}$$

Let $A = (a_{i,j})_{i,j=0,\dots,w}$ be the square matrix of order $w + 1$ defined by:

$$a_{i,j} = \begin{cases} 1, & \text{if } |i-j| = 1 \\ 0, & \text{if } |i-j| \neq 1 \end{cases} \qquad i, j = 0, 1, \dots, w. \tag{14}$$

If we denote by $X^{(l)}$ the (column) vector of components $x_{i,l}$, $i = 0, \dots, w$, then the next recurrence relation follows by (13) :

$$X^{(l)} = AX^{(l-1)}, \quad \text{for all } l = 1, 2, \dots. \tag{15}$$

By (15) we obtain that

$$X^{(l)} = A^l X^{(0)} = A^l U, \quad \text{for all } l = 1, 2, \dots, \tag{16}$$

where $U = X^{(0)} = (1, 1, \dots, 1)^T$, and the next theorem directly follows.

Theorem 4. *The number of lattice paths with steps in $S = \{(1, 1), (1, -1)\}$ joining points on y-axis to points on the line $x = l$ and never passing below the x-axis or above the straight line $y = w$ is*

$$N(w, l) = \sum_{j=0}^{w} x_{l,j} = U^T A^l U.$$

where $A = (a_{i,j})_{i,j=0,\dots,w}$ is the square matrix defined by (14) *and $U = (1, 1, \dots, 1)^T$ is the column vector of dimension $w + 1$ with all entries equal to 1.*

Let us have a better look at the matrix A defined by (14):

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{17}$$

One can remark that A is the adjacency matrix of the path graph $\mathbf{P}_n = (V, E)$ (where $n = w + 1$) having the set of vertices $V = \{0, 1, \dots, w\}$ and the set of edges $E = \{(0, 1), (1, 2), \dots, (w - 1, w)\}$.

Several types of lattice paths in a strip can be interpreted as walks on some path-like graphs (see, for example, [**12**]). In our case, for every $i, j = 0, \ldots, w$, the element $a_{i,j}^{(k)}$ of the matrix A^k is equal to the number of walks of length k on the path graph $\mathbf{P}_n$ joining the vertex i to the vertex j [**4**, Lemma 2.5]. Obviously, this number also counts the Dyck-type lattice paths confined in the strip $[0, w]$ and connecting the points $(0, i)$ and (k, j) (it is 0 whenever $i + j + k$ is odd). We also remark that if $i = j$ then the element $a_{i,i}^{(k)}$ of the matrix A^k is equal to the number of closed walks of length k, evaluated in [**7**].

We compute the characteristic polynomial of the adjacency matrix $A = A_n$ (that is, the characteristic polynomial of the path graph $\mathbf{P}_n$):

$$P_n(\lambda) = \det(\lambda I_n - A_n) = \begin{vmatrix} \lambda & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & \lambda & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & \lambda & -1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & \lambda & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \lambda & -1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & \lambda \end{vmatrix}$$
$$= \lambda P_{n-1}(\lambda) - P_{n-2}(\lambda), \text{ for all } n = 3, 4, \ldots.$$

This recurrence relation together with the initial conditions $P_1(\lambda) = \lambda$ and $P_2(\lambda) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1$ defines the *Vieta-Fibonacci polynomials* V_{n+1} introduced in [**16**]. The next section is dedicated to this sequence of polynomials and other related sequences. We shall see (equation (21)) that the explicit expression of the characteristic polynomial $P_n(x)$ is

$$P_n(\lambda) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \lambda^{n-2k} = V_{n+1}(\lambda). \tag{18}$$

4. VIETA POLYNOMIALS

Closely related to Chebyshev, Fibonacci, and Lucas polynomials, *Vieta polynomials* have been extensively studied in [**16, 19**]. In this paper we emphasize the connection between this class of polynomials and the enumeration of confined Dyck-type lattice paths. Another notable combinatorial problem in which Vieta polynomials arise is the enumeration of meaningful compositions of higher-order differential operators on the space $\mathbb{R}^n$ (see [**22, 23, 24**]). Interestingly, in recent years, these polynomials have found remarkable and innovative applications in the field of fractional calculus (see for instance [**1, 14**]).

Before deepening the relation between this family of polynomials and the enumeration of confined Dyck-type lattice paths, we shortly present their most important properties.

Vieta-Fibonacci $V_n(x)$ and *Vieta-Lucas polynomials* $v_n(x)$ are defined by the same recurrence relation, but with different initial values:

$$V_n(x) = xV_{n-1}(x) - V_{n-2}(x),\ n = 2, 3, \ldots \qquad (19)$$
$$V_0(x) = 0,\ V_1(x) = 1;$$

$$v_n(x) = xv_{n-1}(x) - v_{n-2}(x),\ n = 2, 3, \ldots \qquad (20)$$
$$v_0(x) = 2,\ v_1(x) = x.$$

The expressions of the Vieta-Fibonacci and Vieta-Lucas polynomials are as follows:

$$V_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-k-1}{k} x^{n-2k-1},\ \text{for } n \geq 1,\ V_0(x) = 0, \qquad (21)$$

$$v_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k},\ \text{for } n \geq 1,\ v_0(x) = 2, \qquad (22)$$

and their generating functions are:

$$\sum_{n\geq 0} V_n(x)t^n = \frac{t}{1 - xt + t^2}, \quad \sum_{n\geq 0} v_n(x)t^n = \frac{2 - xt}{1 - xt + t^2}.$$

The first terms of the Vieta polynomial sequences are listed below:

$V_0(x) = 0$	$v_0(x) = 2$
$V_1(x) = 1$	$v_1(x) = x$
$V_2(x) = x$	$v_2(x) = x^2 - 2$
$V_3(x) = x^2 - 1$	$v_3(x) = x^3 - 3x$
$V_4(x) = x^3 - 2x$	$v_4(x) = x^4 - 4x^2 + 2$
$V_5(x) = x^4 - 3x^2 + 1$	$v_5(x) = x^5 - 5x^3 + 5x$
$V_6(x) = x^5 - 4x^3 + 3x$	$v_6(x) = x^6 - 6x^4 + 9x^2 - 2$
$V_7(x) = x^6 - 5x^4 + 6x^2 - 1$	$v_7(x) = x^7 - 7x^5 + 14x^3 - 7x$
....	....

It can be noticed that Vieta-Fibonacci polynomials (21) are the alternating-sign version of Fibonacci polynomials $F_n(x)$, while Vieta-Lucas polynomials (22) correspond to Lucas polynomials $L_n(x)$:

$$F_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^{n-2k-1},\ \text{for } n \geq 1,\ F_0(x) = 0,$$

$$L_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}, \text{ for } n \geq 1, \ L_0(x) = 2.$$

We also note their relations with Chebyshev polynomials of the second kind $U_n(x)$ and of the first kind $T_n(x)$, respectively:

$$V_n(x) = U_{n-1}\left(\tfrac{x}{2}\right),$$
$$v_n(x) = 2T_n\left(\tfrac{x}{2}\right).$$

The characteristic equation corresponding to both recurrences (19) and (20) is

$$\lambda^2 - x\lambda + 1 = 0,$$

with the roots

$$\alpha = \frac{x + \sqrt{x^2-4}}{2}, \quad \beta = \frac{x - \sqrt{x^2-4}}{2} = \frac{1}{\alpha}.$$

Hence the Vieta polynomials can be written in the *Binet form* as:

$$V_n(x) = \frac{\alpha^n - \alpha^{-n}}{\sqrt{x^2-4}}, \quad \text{for } x \neq \pm 2, \tag{23}$$

$$v_n(x) = \alpha^n + \alpha^{-n}. \tag{24}$$

For $x = 2$, one has $V_n(2) = n$ and $v_n(2) = 2$, while $x = -2$ implies $V_n(-2) = n(-1)^{n+1}$, $v_n(-2) = 2(-1)^n$.

Using the Binet form (23), (24), and the relations $\alpha + \alpha^{-1} = x$, $\alpha - \alpha^{-1} = \sqrt{x^2-4}$, the following results can be easily proved (see [**16**]):

Proposition 1. *For every $n \geq 1$ one has:*

$$V_{2n}(x) = V_n(x)\ v_n(x)$$
$$V_{2n+1}(x) = V_{n+1}^2(x) - V_n^2(x),$$
$$v_n = V_{n+1} - V_{n-1}$$

Proposition 2. *The zeros of the Vieta-Fibonacci polynomials $V_n(x)$, $n \geq 2$, are given by*

$$x_k = 2\cos\frac{k\pi}{n}, \quad k = 1, 2, \ldots, n-1,$$

while the roots of the Vieta-Lucas polynomials $v_n(x)$, $n \geq 1$, are

$$\tilde{x}_k = 2\cos\frac{(2k+1)\pi}{2n}, \quad k = 0, 1, \ldots, n-1.$$

The (first-order) Δ-Vieta-Fibonacci and Σ-Vieta-Fibonacci polynomials are defined as follows:

$$\Delta V_n = V_{n+1} - V_n, \tag{25}$$

$$\Sigma V_n = V_{n+1} + V_n, \tag{26}$$

By Proposition 1 we can write the following factorization of the Vieta-Fibonacci polynomials of odd order (and even degree):

$$V_{2n+1}(x) = \Delta V_n(x) \cdot \Sigma V_n(x).$$

It can be easily shown that the polynomial sequences $\Delta V_n(x)$ and $\Sigma V_n(x)$ defined by (25) and (26) respectively, satisfy the same recurrence relation as $V_n(x)$ and their generating functions are:

$$\sum_{n\geq 0} \Delta V_n(x) t^n = \frac{1-t}{1-xt+t^2}, \quad \sum_{n\geq 0} \Sigma V_n(x) t^n = \frac{1+t}{1-xt+t^2}.$$

We list below the first terms of these sequences.

$\Delta V_0(x) = 1$
$\Delta V_1(x) = x - 1$
$\Delta V_2(x) = x^2 - x - 1$
$\Delta V_3(x) = x^3 - x^2 - 2x + 1$
$\Delta V_4(x) = x^4 - x^3 - 3x^2 + 2x + 1$
$\Delta V_5(x) = x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\Sigma V_0(x) = 1$
$\Sigma V_1(x) = x + 1$
$\Sigma V_2(x) = x^2 + x - 1$
$\Sigma V_3(x) = x^3 + x^2 - 2x - 1$
$\Sigma V_4(x) = x^4 + x^3 - 3x^2 - 2x + 1$
$\Sigma V_5(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

We notice that the polynomials $\Delta V_n(x)$ and $\Sigma V_n(x)$ have the same coefficients, but with different signs. The explicit expression of these polynomials is stated by the next proposition (we remark that the coefficients of the polynomials $\Delta V_n(x)$ form the triangular sequence A108299 in the On-line Encyclopedia of Integer Sequences [**29**]).

Proposition 3. *For every $n \geq 1$, the polynomials $\Delta V_n(x)$ and $\Sigma V_n(x)$ are given by the following formulas:*

$$\Delta V_n(x) = \sum_{k=0}^{n} (-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor} \binom{n - \left\lfloor \frac{k+1}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} x^{n-k}. \tag{27}$$

$$\Sigma V_n(x) = \sum_{k=0}^{n} (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{n - \left\lfloor \frac{k+1}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} x^{n-k}. \tag{28}$$

Proof. From the relation (21) we can write:

$$\begin{aligned}\Delta V_n(x) &= V_{n+1} - V_n \\ &= \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} + \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^{k+1} \binom{n-k-1}{k} x^{n-2k-1}\end{aligned}$$

$$= \sum_{\substack{j=0 \\ j\ even}}^{n} (-1)^{\frac{j}{2}} \binom{n-\frac{j}{2}}{\frac{j}{2}} x^{n-j} + \sum_{\substack{j=0 \\ j\ odd}}^{n} (-1)^{\frac{j+1}{2}} \binom{n-\frac{j+1}{2}}{\frac{j}{2}} x^{n-j}$$

$$= \sum_{j=0}^{n} (-1)^{\left\lfloor \frac{j+1}{2} \right\rfloor} \binom{n-\left\lfloor \frac{j+1}{2} \right\rfloor}{\left\lfloor \frac{j}{2} \right\rfloor} x^{n-j},$$

and so (27) is proved. The relation (28) follows in a similar way. □

Proposition 4. *For every $n \geq 1$, one has:*

$$\Delta V_n(x) = \frac{\alpha^{n+1} + \alpha^{-n}}{\alpha + 1} \tag{29}$$

$$\Sigma V_n(x) = \frac{\alpha^{n+1} - \alpha^{-n}}{\alpha - 1} \tag{30}$$

Proof. Using the Binet form (23) for the polynomials V_{n+1} and V_n, we obtain:

$$\begin{aligned} \Delta V_n(x) &= \frac{\alpha^{n+1} - \alpha^{-n-1} - \alpha^n + \alpha^{-n}}{\alpha - \alpha^{-1}} \\ &= \frac{\alpha^n(\alpha - 1) + \alpha^{-n}(1 - \alpha^{-1})}{\alpha - \alpha^{-1}} \\ &= \frac{\alpha^n \cdot \alpha}{\alpha + 1} + \frac{\alpha^{-n}}{\alpha + 1}, \end{aligned}$$

and (29) directly follows. The formula (30) can be proved in a similar way. □

Proposition 5. *The roots of the polynomials $\Delta V_n(x)$ are*

$$x_k = 2\cos\frac{(2k-1)\pi}{2n+1}, \quad k = 1, 2, \ldots, n,$$

while the roots of the polynomials $\Sigma V_n(x)$ are

$$\tilde{x}_k = 2\cos\frac{2k\pi}{2n+1}, \quad k = 1, 2, \ldots, n.$$

Proof. Using (29), we obtain that $\Delta V_n(x) = 0$ if and only if $\alpha^{2n+1} = -1$, so

$$\alpha = \cos\frac{(2k-1)\pi}{2n+1} + i\sin\frac{(2k-1)\pi}{2n+1}, \ k = 1, \ldots, 2n+1,$$

and the roots of ΔV_n are of the form $x = 2\cos\frac{(2k-1)\pi}{2n+1}$. We notice that $x \neq -2$, so $k \neq n+1$ and since ΔV_n has at most n distinct roots, it follows that $k = 1, \ldots, n$.

Similarly, $\Sigma V_n(x) = 0$ if and only if $\alpha^{2n+1} = 1$, so

$$\alpha = \cos\frac{2k\pi}{2n+1} + i\sin\frac{2k\pi}{2n+1}, \ k = 0, \ldots, 2n,$$

and the roots of ΣV_n are of the form $\tilde{x} = 2\cos\frac{2k\pi}{2n+1}$. We notice that $x \neq 2$, so $k \neq 0$ and since ΣV_n has at most n distinct roots, it follows that $k = 1, \ldots, n$. □

Remark 6. We can also define higher-order Δ-Vieta-Fibonacci and Σ-Vieta-Fibonacci polynomials, for $n \geq 2$:

$$\Delta^n V_n = \Delta^{n-1} V_{n+1} - \Delta^{n-1} V_n,$$
$$\Sigma^n V_n = \Sigma^{n-1} V_{n+1} + \Sigma^{n-1} V_n.$$

Using Proposition 4, it can be easily proved that

$$\Delta^2 V_n = (x-2) V_{n+1},$$
$$\Sigma^2 V_n = (x+2) V_{n+1},$$

so one can write, for every $k, n \geq 1$:

$$\Delta^{2k} V_n = (x-2)^k V_{n+k},\ \Delta^{2k+1} V_n = (x-2)^k \Delta V_{n+k},$$
$$\Sigma^{2k} V_n = (x+2)^k V_{n+k},\ \Sigma^{2k+1} V_n = (x+2)^k \Sigma V_{n+k}.$$

In the next section we explore the close connection between Vieta polynomials and the enumeration of Dyck-type lattice paths. Before concluding this section, however, we would like to draw attention to the appearance of these polynomials in the recurrence relations associated with the enumeration of meaningful compositions of higher-order differential operators on the space $\mathbb{R}^n$ ([**22, 23, 24**]). Although the studies cited do not refer directly to Vieta polynomials, the recurrence coefficients computed therein correspond to those of the Vieta-Fibonacci and related polynomial families. Specifically, if n is even, $n = 2m$, then the characteristic polynomial of the corresponding recurrence is the Vieta-Fibonacci polynomial $V_{m+2}(x)$, whereas for n odd, $n = 2m+1$, the characteristic polynomial is $\Delta V_{m+1}(x)$ (see [**23**, Lemma 2]). If the Gateaux directional derivative is also included, then the characteristic polynomial of the recurrence relation is $\Delta V_{m+1}(x)$ if $n = 2m$, and $\Delta^2 V_m(x) = (x-2)V_{m+1}(x)$, if $n = 2m+1$ (see [**24**, Lemma 4.4]).

5. RECURRENCE RELATIONS USING VIETA POLYNOMIALS

In Section 3, we discussed how can be counted the Dyck-type lattice paths confined in the strip $0 \leq y \leq w$, using the powers of the matrix (17) (the adjacency matrix of the path graph $\mathbf{P}_{w+1}$ (on $w+1$ vertices). Since the characteristic polynomial of the matrix $A = A_{w+1}$ is the Vieta-Fibonacci polynomial V_{w+2} (see (18)), by Cayley-Hamilton Theorem we obtain that $V_{w+2}(A) = O_{w+1}$, so we have:

$$A^{w+1} = \sum_{k=1}^{\lfloor \frac{w+1}{2} \rfloor} (-1)^{k-1} \binom{w-k+1}{k} A^{w-2k+1}.$$

Hence, for every $l \geq w+1$, we can write:

$$A^{l} = \sum_{k=1}^{\lfloor \frac{w+1}{2} \rfloor} (-1)^{k-1} \binom{w-k+1}{k} A^{l-2k}.$$

We multiply this equation by the column vector $U = (1, 1, \ldots, 1)^T$ of dimension $w+1$, and the next theorem follows by equation (16).

Theorem 5. *If $X^{(l)}$ is the column vector whose components $x_{i,l}$, $i = 0, \dots, w$ count the Dyck-type lattice paths of length l confined in the strip $0 \le y \le w$, starting on the y-axis and ending up at the point (l, i), then the following recurrence relation holds for all $l \ge w + 1$:*

$$X^{(l)} = \sum_{k=1}^{\lfloor \frac{w+1}{2} \rfloor} (-1)^{k-1} \binom{w-k+1}{k} X^{(l-2k)}. \tag{31}$$

The total number of confined Dyck-type lattice paths $N(w, l)$ satisfies, for a fixed w and for every $l \ge w + 1$, the recurrence relation

$$N(w,l) = \sum_{k=1}^{\lfloor \frac{w+1}{2} \rfloor} (-1)^{k-1} \binom{w-k+1}{k} N(w, l-2k). \tag{32}$$

We remark that (32) is a recurrence of order w when w is even, and of order $w+1$ if w is odd. The initial values $N(w, l)$, $l = 1, \dots, w+1$ are given by Theorem 3.

As an example, the vectors $X^{(l)}$ and the total numbers $N(w, l)$ obtained for $w = 3$ and $l = 0, 1, \dots, 8$ are presented in Table 2.

Table 2: The number of Dyck-type confined lattice paths of width $w = 3$.

l	0	1	2	3	4	5	6	7	8
$x_{0,l}$	1	1	2	3	5	8	13	21	34
$x_{1,l}$	1	2	3	5	8	13	21	34	55
$x_{2,l}$	1	2	3	5	8	13	21	34	55
$x_{3,l}$	1	1	2	3	5	8	13	21	34
$N(3,l)$	4	6	10	16	26	42	68	110	178

For $w = 3$, the recurrence relation (31) is written

$$X^{(l)} = 3X^{(l-2)} - X^{(l-4)}, \ l = 4, 5, \dots$$

Obviously, looking at the table above, this relation is fulfilled. But Fibonacci numbers catch the eye, and we can also see another recurrence (of order 2, instead of 4):

$$X^{(l)} = X^{(l-1)} + X^{(l-2)}, \ l = 2, 3, \dots$$

In the same way, we present in Table 3 the vectors $X^{(l)}$ and the total numbers $N(w, l)$ obtained for $w = 5$ and $l = 0, 1, \dots, 8$:

It can be noticed, besides the recurrence of order 6 stated in Theorem 5,

$$X^{(l)} = 5X^{(l-2)} - 6X^{(l-4)} + X^{(l-6)}, \ l = 6, 7, \dots$$

another recurrence relation of order 3:

$$X^{(l)} = X^{(l-1)} + 2X^{(l-2)} - X^{(l-3)}, \ l = 3, 4, \dots$$

Table 3: The number of Dyck-type confined lattice paths of width $w = 5$.

l	0	1	2	3	4	5	6	7	8
$x_{0,l}$	1	1	2	3	6	10	19	33	61
$x_{1,l}$	1	2	3	6	10	19	33	61	108
$x_{2,l}$	1	2	4	7	13	23	42	75	136
$x_{3,l}$	1	2	4	7	13	23	42	75	136
$x_{4,l}$	1	2	3	6	10	19	33	61	108
$x_{5,l}$	1	1	2	3	6	10	19	33	61
$N(5,l)$	6	10	18	32	58	104	188	338	610

All these suggest that a recurrence relation of a halved order can be obtained using the symmetry of $X^{(l)}$.

Case I. We suppose that w is odd, $w = 2n - 1$.

Because of the symmetry of the vector $X^{(l)}$, we can study only its first n entries. We denote by $\bar{X}^{(l)} = (x_{0,l}, \ldots, x_{n-1,l})^T$ the first half of the vector $X^{(l)}$ and by $\bar{U} = (1, 1, \ldots, 1)^T = \bar{X}^{(0)}$ the n-dimensional column vector with all entries equal to 1. Then

$$\bar{X}^{(l+1)} = \bar{A}\bar{X}^{(l)},\ l = 0, 1, \ldots,$$

where $\bar{A}$ is the square matrix of order n:

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix} \tag{33}$$

We note that $\bar{A} = \bar{A}_n$ is the adjacency matrix of a path graph on n vertices which has also a loop at vertex n.

Theorem 6. *The characteristic polynomial of the matrix $\bar{A}$ defined by* (33) *is the Δ-Vieta-Fibonacci polynomial:*

$$\Delta V_n(\lambda) = \sum_{k=0}^{n} (-1)^{\lfloor \frac{k+1}{2} \rfloor} \binom{n - \lfloor \frac{k+1}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} \lambda^{n-k}.$$

Proof. The characteristic polynomial of the matrix $\bar{A}$ is:

$$\bar{P}_n(\lambda) = \det(\lambda I_n - \bar{A}) = \begin{vmatrix} \lambda & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & \lambda & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \lambda & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 1)V_n(\lambda) - V_{n-1}(\lambda)$$
$$= V_{n+1}(\lambda) - V_n(\lambda) = \Delta V_n(\lambda), \text{ for all } n = 2, 3, \dots .$$

□

By Cayley-Hamilton Theorem one can write:

$$\bar{A}^l = \sum_{k=1}^{n} (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{n - \left\lfloor \frac{k+1}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} \bar{A}^{l-k}, \text{ for all } l \geq n.$$

Since $\bar{X}^{(l)} = \bar{A}^l \bar{U}$, for every $l \geq 0$, by multiplying with the vector $\bar{U}$ the relation above we obtain:

$$\bar{X}^{(l)} = \sum_{k=1}^{n} (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{n - \left\lfloor \frac{k+1}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} \bar{X}^{(l-k)}$$

and the next corollary follows.

Corollary 1. *Let $w = 2n - 1$ be a fixed odd number and $l \geq 1$. If $x_{l,i}$, $i = 0, \dots, w$ count the Dyck-type lattice paths of length l, confined in the strip $0 \leq y \leq w$, starting on the y-axis and ending up at the point (l, i), then the following recurrence relation holds for every $i = 0, \dots, w$ and $l \geq n$:*

$$x_{i,l} = \sum_{k=1}^{n} (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{n - \left\lfloor \frac{k+1}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} x_{i,l-k}. \tag{34}$$

The total number $N(w, l)$ of Dyck-type lattice paths of length l confined in the strip $0 \leq y \leq w$ satisfies, for all $l \geq n$, the same recurrence relation:

$$N(w, l) = \sum_{k=1}^{n} (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{n - \left\lfloor \frac{k+1}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} N(w, l-k).$$

Case II. We suppose that w is even, $w = 2n$.

Because of the symmetry of the vector $X^{(l)}$, we can study only the first $n+1$ values of $X^{(l)}$. As an example, we present in Table 4 the vectors $X^{(l)}$ and the total numbers $N(w, l)$ obtained for $w = 6$ and $l = 0, 1, \dots, 8$.

Table 4: The number of Dyck-type confined lattice paths of width $w = 6$.

l	0	1	2	3	4	5	6	7	8
$x_{0,l}$	1	1	2	3	6	10	20	34	68
$x_{1,l}$	1	2	3	6	10	20	34	68	116
$x_{2,l}$	1	2	4	7	14	24	48	82	164
$x_{3,l}$	1	2	4	8	14	28	48	96	164
$x_{4,l}$	1	2	4	7	14	24	48	82	164
$x_{5,l}$	1	2	3	6	10	20	34	68	116
$x_{6,l}$	1	1	2	3	6	10	20	34	68
$N(6,l)$	7	12	22	40	74	136	252	464	860

We denote by $\tilde{X}^{(l)} = (x_{0,l}, \dots, x_{n,l})^T$ the column vector formed with the first $n+1$ components of the vector $X^{(l)}$, and by $\tilde{U} = \tilde{X}^{(0)} = (1, 1, \dots, 1)^T$ the $n+1$-dimensional column vector with all entries equal to 1. Then

$$\tilde{X}^{(l+1)} = \tilde{A}\tilde{X}^{(l)}, \text{ for all } l \geq 0,$$

where $\tilde{A} = \tilde{A}_{n+1}$ is the following square matrix of order $n+1$:

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 2 & 0 \end{pmatrix} \tag{35}$$

Theorem 7. *The characteristic polynomial of the matrix $\tilde{A}$ defined by* (35) *is the Vieta-Lucas polynomial:*

$$v_{n+1}(\lambda) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \frac{n+1}{n-k+1} \binom{n-k+1}{k} \lambda^{n-2k+1}.$$

Proof. The characteristic polynomial of the matrix $\tilde{A} = \tilde{A}_{n+1}$ is:

$$\tilde{P}_{n+1}(\lambda) = \det(\lambda I_{n+1} - \tilde{A}) = \begin{vmatrix} \lambda & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & \lambda & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \lambda & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & -1 \\ 0 & 0 & 0 & 0 & \dots & -2 & \lambda \end{vmatrix}$$

Hence,

$$\begin{aligned}\tilde{P}_{n+1}(\lambda) &= \lambda V_{n+1}(\lambda) - 2V_n(\lambda)\\ &= V_{n+2}(\lambda) - V_n(\lambda) = v_{n+1}(\lambda), \text{ for all } n = 1, 2, \ldots\end{aligned}$$

and the proof is complete. □

By Cayley-Hamilton Theorem we can write:

$$\tilde{A}^l = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} \frac{n+1}{n-k+1} \binom{n-k+1}{k} \tilde{A}^{l-2k}, \text{ for all } l \geq n+1.$$

Since $\tilde{X}^{(l)} = \tilde{A}^l \tilde{U}$, for every $l \geq 0$, the next corollary follows by multiplying with $\tilde{U}$ the relation above.

Corollary 2. *Let $w = 2n$ be an even number and $l \geq 1$. If $x_{i,l}$, $i = 0, \ldots, w$ count the Dyck-type lattice paths of length l, confined in the strip $0 \leq y \leq w$, starting on the y-axis and ending up at the point (l, i), then the following recurrence relation holds for every $i = 0, \ldots, w$ and $l \geq n+1$:*

$$x_{i,l} = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} \frac{n+1}{n-k+1} \binom{n-k+1}{k} x_{i,l-2k} \tag{36}$$

The total number $N(w, l)$ of Dyck-type lattice paths of length l, confined in the strip $0 \leq y \leq w$ satisfies, for all $l \geq n+1$, the same recurrence relation

$$N(w, l) = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k-1} \frac{n+1}{n-k+1} \binom{n-k+1}{k} N(w, l-2k).$$

Thus, for every $i = 0 \ldots w$, the terms of the sequence $\{x_{i,l}\}_{l \geq 0}$ can be computed, for $l \geq \lfloor \frac{w}{2} \rfloor$ by a recurrence relation of order $\lfloor \frac{w}{2} \rfloor + 1$, which is the relation (34) if $w = 2n - 1$, and the relation (36) if $w = 2n$, respectively. From (8) and (9), the initial values, $x_{i,l}$, $l = 0, 1, \ldots \lfloor \frac{w}{2} \rfloor$ are given by the formula:

$$x_{i,l} = \begin{cases} \sum_{k=\lfloor \frac{l-i}{2} \rfloor}^{\lfloor \frac{l+i}{2} \rfloor} \binom{l}{k}, & \text{if } i \leq \lfloor \frac{w}{2} \rfloor \\ \sum_{k=\lfloor \frac{l-w+i}{2} \rfloor}^{\lfloor \frac{l+w-i}{2} \rfloor} \binom{l}{k}, & \text{if } i > \lfloor \frac{w}{2} \rfloor \end{cases}$$

(see Remark 3 and Remark 4).

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