

## ON THE GROWTH AND DERIVATIVES OF A POLYNOMIAL

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*Dedicated to Professor Gradimir V. Milovanović  
on the occasion of his 75<sup>th</sup> Anniversary*

This paper is devoted to Bernstein-type growth estimates for polynomials that do not vanish within a disk of positive radius. We derive several new sup-norm bounds for the growth of higher-order derivatives of such polynomials in the complex plane. These results encompass generalizations of the well-known inequality of Ankeny and Rivlin (Pacific J. Math., 5: 849-852, 1955), along with extensions of significant inequalities in approximation theory, notably those obtained by Jain (Turk. J. Math., 31: 89–94, 2007).

### 1. INTRODUCTION


The study of inequalities in various norms of polynomials in the complex plane is a well-established area of analysis, primarily due to its profound connections with geometric function theory. Beyond its theoretical significance, this field has a wide array of applications across science and engineering, and continues to be a source of active and ongoing research. Special classes of polynomials, such as Bernoulli, Euler, Bell, and Boubaker polynomials, as well as various families of orthogonal polynomials including Jacobi, Chebyshev, Laguerre, and Hermite, play a crucial role in the development of methods for solving differential and integral

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equations, including the construction of quadrature formulas (see, for instance, [3],[7],[8],[14]). These important polynomial classes and the extremal problems associated with them are comprehensively explored in the monograph by Milovanović et al. [16]. In addition to presenting foundational results along with their proofs, the book offers an in-depth treatment of recent developments in the study of extremal properties of polynomials and their derivatives across various metrics. It also includes a thorough analysis of inequalities related to trigonometric sums, algebraic polynomials, and their zeros. Among these, Bernstein-type inequalities are particularly notable, offering estimates for the growth of polynomials and their derivatives, and generalizing classical results in polynomial approximation theory (see, for example, [2],[4],[5],[12],[13],[15]-[19]).

Let  $f$  be an arbitrary entire function, and define  $M(f, r) := \max_{|z|=r} |f(z)|$ , the supremum norm of  $f$  on the circle of radius  $r > 0$ . By the Maximum Modulus Principle,  $M(f, r)$  is a non-decreasing function of  $r$ , and strictly increasing unless  $f$  is constant.

For a polynomial  $P(z)$  of degree  $n$ , it is known that

$$(1) \quad M(P, R) \leq R^n M(P, 1), \quad R \geq 1.$$

Inequality (1) follows directly from the Maximum Modulus Principle (see [16]). It is worth noting that equality in (1) holds if and only if all the zeros of  $P(z)$  are located at the origin. To obtain a sharper upper bound in (1), it is essential to impose certain restrictions on the location of the zeros of  $P(z)$ . It was shown by Ankeny and Rivlin [1] that if  $P(z) \neq 0$  in  $|z| < 1$ , then (1) can be replaced by

$$(2) \quad M(P, R) \leq \frac{R^n + 1}{2} M(P, 1), \quad R \geq 1.$$

Inequality (2) has been extended and generalized in various directions, with comprehensive overviews available in the books by Gardner et al. [6] and Milovanović et al. [16]. The problem of determining sharp upper bounds for  $\frac{M(P, R)}{M(P, 1)}$ , where  $R > 1$ , for polynomials with no zeros in  $|z| < k$  ( $k > 0$ ), was originally proposed by the late R. P. Boas. This problem has since attracted considerable attention; notable contributions include those by Govil and Rahman [9], and Govil, Qazi, and Rahman [10], among others. In 2003, Govil, Qazi, and Rahman [10] presented the following result as a partial extension of inequality (2) for polynomials that have no zeros in the disk  $|z| < k$ , where  $k \geq 1$ .

**Theorem A.** Let  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu \neq 0$  for  $|z| < k$  where  $k \geq 1$  and let  $\gamma = \frac{ka_1}{na_0}$ . Then for  $1 \leq R \leq k^2$ ,

$$M(P, R) \leq \left( \frac{R^2 + 2|\gamma|Rk + k^2}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1).$$

In recent years, there has been considerable interest in studying extremal problems involving polynomials under specific zero restrictions. Numerous papers

have been published on the subject, employing a range of methods to address various aspects. Most of these recent works deal with the extensions of the classical results of polynomial bounds in approximation theory. By examining polynomials that are zero-free in the disk  $|z| < k$ ,  $k \geq 1$ , and simultaneously considering their  $s^{\text{th}}$  derivatives for  $0 \leq s < n$ , Jain [11] derived the following generalization of inequality (2).

**Theorem B.** If  $P(z)$  is a polynomial of degree  $n$  with no zeros in  $|z| < k$ ,  $k \geq 1$ , then for  $0 \leq s < n$ ,

$$(3) \quad M(P^{(s)}, R) \leq \frac{1}{2} \left\{ \frac{d^s}{dR^s} (R^n + k^n) \right\} \left( \frac{2}{1+k} \right)^n M(P, 1), \text{ for } R \geq k.$$

Equality holds in (3) (with  $s = 0$  and  $k = 1$ ) for  $P(z) = z^n + 1$ .

Given the importance of the problem in approximation theory of determining the growth of analytic functions with respect to various norms, particularly polynomials, it is natural to investigate the behavior of higher-order derivatives of polynomials in the sup-norm, particularly under zero restrictions. The aim of this work is to establish several new bounds for polynomial derivatives, encompassing both generalizations and refinements of the previously mentioned inequalities, along with other related results.

## 2. MAIN RESULTS

For brevity, we use the following notations in our results:

$$B(R, k) := \frac{k|a_0| + Rk^n|a_n|}{|a_0| + k^n|a_n|}$$

and

$$m(P, k) := \min_{|z|=k} |P(z)|.$$

The primary objective of this paper is to demonstrate various refinements and generalizations of the previously mentioned inequalities. Specifically, we begin by proving the subsequent refinement and generalization of Theorem B.

**Theorem 1.** If  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n$  having no zeros in  $|z| < k$ ,  $k \geq 1$ , and let  $\gamma = \frac{ka_1}{na_0}$ , then for  $0 \leq s < n$  and  $R \geq k$ , we have

$$(4) \quad M(P^{(s)}, R) \leq \frac{B(R, k)}{R+k} \left\{ \frac{d^s}{dR^s} (R^n + k^n) \right\} \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1).$$

Equality holds in (4) (with  $k = 1$ ) for  $P(z) = z^n + 1$ .

It is easily seen that  $|\gamma| = \left| \frac{ka_1}{na_0} \right| \leq 1$ , and hence we can show in general

$$\left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} \leq \left( \frac{2}{1+k} \right)^n,$$

which is equivalent to showing

$$\frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \leq \left( \frac{2}{1+k} \right)^2,$$

that is,

$$(1 + |\gamma|)(1 + k)^2 \leq 2(1 + 2|\gamma|k + k^2),$$

which is equivalent to

$$(|\gamma| - 1)(k - 1)^2 \leq 0,$$

which clearly holds as  $|\gamma| \leq 1$ .

Thus from Theorem 1, we get the following result which is a refinement of Theorem B.

**Corollary 2.** *If  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n$  having no zeros in  $|z| < k$ ,  $k \geq 1$ , then for  $0 \leq s < n$  and  $R \geq k$ , we have*

$$(5) \quad M(P^{(s)}, R) \leq \frac{B(R, k)}{R + k} \left\{ \frac{d^s}{dR^s} (R^n + k^n) \right\} \left( \frac{2}{1+k} \right)^n M(P, 1).$$

*Equality holds in (5) (with  $k = 1$ ) for  $P(z) = z^n + 1$ .*

**Remark 3.** If  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu \neq 0$  in  $|z| < k$ ,  $k \geq 1$ , then  $k^n \left| \frac{a_n}{a_0} \right| \leq 1$ , which can also be taken as equivalent to

$$\frac{B(R, k)}{R + k} = \frac{k|a_0| + Rk^n|a_n|}{(R + k)(|a_0| + k^n|a_n|)} \leq \frac{1}{2} \text{ for } R \geq k.$$

Hence inequality (5) is an improvement of inequality (3). Infact, when  $P(z)$  has at least one zero in  $|z| > k$ ,  $k \geq 1$  and  $R > k$ , the bound obtained in (5) is always sharper than the bound obtained in (3) for every  $0 \leq s < n$ .

Instead of proving Theorem 1, we shall prove the following more refined result.

**Theorem 4.** If  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n \geq 2$  having no zeros in  $|z| < k$ ,  $k \geq 1$ , and let  $\gamma = \frac{ka_0}{na_1}$ , then for  $0 \leq s < n$  and  $R \geq k$ , we have

$$(6) \quad M(P^{(s)}, R) \leq \frac{B(R, k)}{R+k} \left\{ \frac{d^s}{dR^s} (R^n + k^n) \right\} \left[ \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1) - \frac{1}{k^n} \left\{ 1 - \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} \right\} m(P, k) \right].$$

Equality holds in (6) (with  $k = 1$ ) for  $P(z) = z^n + 1$ .

If we use Lemma 4 instead of Lemma 3 in the proof of Theorem 4, we get the following refinement of Theorem B.

**Corollary 5.** If  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n \geq 2$  having no zeros in  $|z| < k$ ,  $k \geq 1$ , then for  $0 \leq s < n$  and  $R \geq k$ , we have

$$(7) \quad M(P^{(s)}, R) \leq \frac{B(R, k)}{R+k} \left\{ \frac{d^s}{dR^s} (R^n + k^n) \right\} \left[ \left( \frac{2}{1+k} \right)^n M(P, 1) - \frac{1}{k^n} \left\{ 1 - \left( \frac{2}{1+k} \right)^n \right\} m(P, k) \right].$$

Equality holds in (7) (with  $k = 1$ ) for  $P(z) = z^n + 1$ .

**Remark 6.** It is worth noting that, for any polynomial of degree  $n \geq 2$  of the form  $P(z) = a_0 + a_1z + \dots + a_nz^n$ , having no zeros in  $|z| \leq k$  with  $k \geq 1$ , inequality (7) provides a sharper bound than (3) whenever  $R > k$  and  $0 \leq s < n$ . Moreover, for this same class of polynomials with  $R > k$  and  $k \geq 1$ , inequality (6) yields a significant improvement over (3) when  $|\gamma| = \left| \frac{ka_0}{na_1} \right| < 1$ .

For  $s = 0$  in Theorem 4, we get the following generalization of a result due to Dubinin [5] and related results.

**Corollary 7.** If  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$  is a polynomial of degree  $n \geq 2$  having no zeros in  $|z| < k$ ,  $k \geq 1$ , and let  $\gamma = \frac{ka_0}{na_1}$ , then for  $R \geq k$ , we have

$$(8) \quad M(P, R) \leq \frac{B(R, k)(R^n + k^n)}{R+k} \left[ \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1) - \frac{1}{k^n} \left\{ 1 - \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} \right\} m(P, k) \right].$$

Equality holds in (8) with  $k = 1$  for  $P(z) = z^n + 1$ .

**Remark 8.** For  $k = 1$ , (8) reduces to a result of Dubinin ([5], Corollary 1); see also Kumar ([12], Lemma 2.3).

**Remark 9.** As shown in Remark 3 that for  $R \geq k$ , we have

$$\frac{B(R, k)}{R + k} \leq \frac{1}{2},$$

the inequality (6) gives, in particular, the following known estimate (see [18], Theorem 3), for  $0 \leq s < n$ ,

$$M(P^{(s)}, R) \leq \frac{1}{2} \left\{ \frac{d^s}{dR^s} (R^n + k^n) \right\} \left[ \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1) - \frac{1}{k^n} \left\{ 1 - \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} \right\} m(P, k) \right].$$

### 3. AUXILIARY RESULTS

We require the following lemmas to prove our main results. The following two lemmas are due to Jain [11].

**Lemma 1.** Let  $P(z)$  be a polynomial of degree  $n$  having all its zeros in  $|z| \leq 1$ . If  $T(z)$  is a polynomial of degree at most  $n$  such that

$$|T(z)| \leq |P(z)|, \text{ for } |z| = 1,$$

then for  $0 \leq s < n$ ,

$$|T^{(s)}(z)| \leq |P^{(s)}(z)|, \text{ for } |z| \geq 1.$$

**Lemma 2.** If  $P(z)$  is a polynomial of degree at most  $n$ , then for  $0 \leq s < n$ ,

$$|P^{(s)}(z)| + |Q^{(s)}(z)| \leq \left\{ \left| \frac{d^s}{dz^s} (1) \right| + \left| \frac{d^s}{dz^s} (z^n) \right| \right\} M(P, 1), \text{ for } |z| \geq 1,$$

where  $Q(z) = z^n \overline{P(\frac{1}{z})}$ .

**Lemma 3.** Let  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$  be a polynomial of degree  $n \geq 2$  such that  $P(z) \neq 0$  for  $|z| < k$ ,  $k \geq 1$ , and let  $\gamma = \frac{ka_1}{na_0}$ . Then for  $1 \leq R \leq k^2$ , we have

$$M(P, R) \leq \left( \frac{R^2 + 2|\gamma|Rk + k^2}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1) - \frac{1}{k^n} \left\{ R^n - \left( \frac{R^2 + 2|\gamma|Rk + k^2}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} \right\} m(P, k).$$

*Proof of Lemma 3.* The above lemma is also proved in ([18], Theorem 1), however, for the sake of completeness, we present its brief outline. For this, first note that we can assume without loss of generality that  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$  has no zeros on  $|z| = k$ ; for otherwise the result follows by Theorem A. Assuming all zeros lie in  $|z| > k$ , it follows that  $m := m(P, k) > 0$ , and hence  $\frac{z^n}{P(z)}$  is analytic in  $|z| \leq k$ . By the Maximum Modulus Principle,

$$\left| \frac{z^n}{P(z)} \right| \leq \max_{|z|=k} \left| \frac{z^n}{P(z)} \right| = \frac{k^n}{\min_{|z|=k} |P(z)|},$$

which implies

$$|P(z)| \geq \frac{m|z|^n}{k^n} \quad \text{for } |z| \leq k.$$

Now, for any  $\alpha \in \mathbb{C}$  with  $|\alpha| < 1$ , Rouché’s theorem implies that the polynomial

$$P(z) + \frac{\alpha m z^n}{k^n}$$

has no zeros in  $|z| < k$ . Applying Theorem A to this polynomial of degree  $n \geq 2$ , we obtain

$$(9) \quad \left| P(z) + \frac{\alpha m z^n}{k^n} \right| \leq \left( \frac{R^2 + 2|\gamma|Rk + k^2}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} \max_{|z|=1} \left| P(z) + \frac{\alpha m z^n}{k^n} \right|,$$

for  $|z| = R$ ,  $1 \leq R \leq k^2$ .

Choosing the argument of  $\alpha$  such that

$$\left| P(z) + \frac{\alpha m z^n}{k^n} \right| = M(P, R) + \frac{|\alpha| m R^n}{k^n},$$

we derive

$$\begin{aligned} M(P, R) &\leq \left( \frac{R^2 + 2|\gamma|Rk + k^2}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1) \\ &\quad - \frac{|\alpha|}{k^n} \left\{ R^n - \left( \frac{R^2 + 2|\gamma|Rk + k^2}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} \right\} m. \end{aligned}$$

Finally, letting  $|\alpha| \rightarrow 1$ , the desired inequality follows, and the proof is complete.

**Lemma 4.** Let  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$  be a polynomial of degree  $n \geq 2$  such that  $P(z) \neq 0$  for  $|z| < k$ ,  $k \geq 1$ . Then for  $1 \leq R \leq k^2$ , we have

$$M(P, R) \leq \left( \frac{R+k}{1+k} \right)^n M(P, 1) - \frac{1}{k^n} \left\{ R^n - \left( \frac{R+k}{1+k} \right)^n \right\} m.$$

*Proof of Lemma 4.* Since  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu \neq 0$  in  $|z| < k$ ,  $k \geq 1$ , we have  $|\gamma| = \left| \frac{ka_1}{na_0} \right| \leq 1$ . Moreover, the function

$$x \mapsto \frac{R^2 + 2Rkx + k^2}{1 + 2kx + k^2}$$

is increasing on the interval  $[0, 1]$ . Using this in (9), we obtain

$$(10) \quad \left| P(z) + \frac{\alpha m z^n}{k^n} \right| \leq \left( \frac{R+k}{1+k} \right)^n \left\{ M(P, 1) + \frac{|\alpha|m}{k^n} \right\},$$

for  $|z| = R$ . Hence, following an argument analogous to the proof of Lemma 3 by choosing the argument of  $\alpha$  suitably, we get from (10) the following:

$$M(P, R) \leq \left( \frac{R+k}{1+k} \right)^n M(P, 1) - \frac{1}{k^n} \left\{ R^n - \left( \frac{R+k}{1+k} \right)^n \right\} m,$$

for  $1 \leq R \leq k^2$ . This completes the proof of Lemma 4.

**Lemma 5.** *If  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu \neq 0$  in  $|z| < 1$ , then for  $R \geq 1$ ,*

$$|P(Rz)| \leq \frac{|a_0| + R|a_n|}{|a_0|R + |a_n|} |Q(Rz)|$$

for  $|z| = 1$  and  $Q(z) = z^n \overline{P\left(\frac{1}{z}\right)}$ .

The above lemma is due to Kumar ([12], Lemma 2.3; inequality 2.4).

**Lemma 6.** *If  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu \neq 0$  in  $|z| < 1$  and  $Q(z) = z^n \overline{P\left(\frac{1}{z}\right)}$ , then for  $R \geq 1$  and  $0 \leq s < n$ , we have*

$$|P^{(s)}(Rz)| \leq \frac{|a_0| + R|a_n|}{|a_0|R + |a_n|} |Q^{(s)}(Rz)| \text{ for } |z| \geq 1.$$

*Proof of Lemma 6.* Since  $P(z) \neq 0$  in  $|z| < 1$ , the polynomial  $Q(z) = z^n \overline{P\left(\frac{1}{z}\right)}$  has all its zeros in  $|z| \leq 1$  and  $|P(z)| = |Q(z)|$  for  $|z| = 1$ .

For  $R = 1$ , the result follows by Lemma 1, henceforth, we assume that  $R > 1$ . Because  $Q(z)$  has all its zeros in  $|z| \leq 1$ , implies  $Q(Rz)$  has all its zeros in  $|z| \leq \frac{1}{R} < 1$  for every  $R > 1$ . By Lemma 5, it follows on applying Rouché's theorem that the polynomial

$$P(Rz) - \beta \left( \frac{|a_0| + R|a_n|}{|a_0|R + |a_n|} \right) Q(Rz)$$

has all its zeros in  $|z| \leq \frac{1}{R} < 1$  for every complex number  $\beta$  with  $|\beta| > 1$ . Therefore, by Gauss-Lucas theorem, the polynomial

$$(11) \quad G(z) = P^{(s)}(Rz) - \beta \left( \frac{|a_0| + R|a_n|}{|a_0|R + |a_n|} \right) Q^{(s)}(Rz), \quad 1 \leq s < n,$$

has all its zeros in  $|z| \leq \frac{1}{R} < 1$  for every  $\beta$  with  $|\beta| > 1$  and  $R > 1$ . This implies that for  $1 \leq s < n$ ,

$$(12) \quad |P^{(s)}(Rz)| \leq \frac{|a_0| + R|a_n|}{|a_0|R + |a_n|} |Q^{(s)}(Rz)| \text{ for } |z| \geq 1 \text{ and } R > 1.$$

Because, if (12) is not true, then there is a point  $z = z_0$  with  $|z_0| \geq 1$  such that

$$|P^{(s)}(Rz_0)| > \frac{|a_0| + R|a_n|}{|a_0|R + |a_n|} |Q^{(s)}(Rz_0)|, \quad 1 \leq s < n \text{ and } R > 1.$$

We choose

$$\beta = \frac{P^{(s)}(Rz_0)}{\left( \frac{|a_0| + R|a_n|}{|a_0|R + |a_n|} \right) Q^{(s)}(Rz_0)},$$

so that  $|\beta| > 1$ . With this choice of  $\beta$ , we get from (11) that

$$G(z_0) = 0, \text{ where } |z_0| \geq 1.$$

This is clearly a contradiction to the fact that all the zeros of  $G(z)$  lie in  $|z| < 1$ . Inequality (12) when combined with Lemma 5 gives the desired result, and this completes the proof of Lemma 6.

#### 4. PROOFS OF THE MAIN RESULTS

*Proof of Theorem 4.* Recall that  $P(z) = \sum_{\nu=0}^n a_\nu z^\nu \neq 0$  in  $|z| < k$ ,  $k \geq 1$ , therefore, the polynomial  $T(z) = P(kz) \neq 0$  in  $|z| < 1$ . Let  $H(z) = z^n \overline{T(\frac{1}{\bar{z}})}$ , therefore, on applying Lemma 6 to the polynomials  $T(z)$  and  $H(z)$ , we get for  $0 \leq s < n$  and  $r \geq 1$ ,

$$(13) \quad |T^{(s)}(re^{i\theta})| \leq \frac{|a_0| + rk^n|a_n|}{|a_0|r + k^n|a_n|} |H^{(s)}(re^{i\theta})|, \quad 0 \leq \theta < 2\pi.$$

Since,  $|T(z)| = |H(z)|$  for  $|z| = 1$ , by Lemma 2, we have for  $r \geq 1$  and  $0 \leq s < n$ ,

$$|T^{(s)}(re^{i\theta})| + |H^{(s)}(re^{i\theta})| \leq \left\{ \frac{d^s}{dr^s} (r^n + 1) \right\} M(T, 1), \quad 0 \leq \theta < 2\pi,$$

which on combining with (13) gives

$$|T^{(s)}(re^{i\theta})| \leq \frac{|a_0| + rk^n|a_n|}{(r+1)(|a_0| + k^n|a_n|)} \left\{ \frac{d^s}{dr^s}(r^n + 1) \right\} M(T, 1),$$

which is equivalent to

$$(14) \quad |P^{(s)}(kre^{i\theta})| \leq \frac{|a_0| + rk^n|a_n|}{k^s(r+1)(|a_0| + k^n|a_n|)} \left\{ \frac{d^s}{dr^s}(r^n + 1) \right\} M(P, k).$$

The above inequality (14) in conjunction with Lemma 3 gives

$$(15) \quad |P^{(s)}(kre^{i\theta})| \leq \frac{|a_0| + rk^n|a_n|}{k^s(r+1)(|a_0| + k^n|a_n|)} \left\{ \frac{d^s}{dr^s}(r^n + 1) \right\} \left[ \left( \frac{k^2 + 2|\gamma|k^2 + k^2}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1) - \frac{1}{k^n} \left\{ k^n - \left( \frac{k^2 + 2|\gamma|k^2 + k^2}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} m \right\} \right].$$

Now taking  $r = \frac{R}{k} \geq 1$  in (15), we get for  $0 \leq s < n$ ,

$$|P^{(s)}(Re^{i\theta})| \leq \frac{k|a_0| + Rk^n|a_n|}{(R+k)(|a_0| + k^n|a_n|)} \left\{ \frac{d^s}{dR^s}(R^n + k^n) \right\} \left[ \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} M(P, 1) - \frac{1}{k^n} \left\{ 1 - \left( \frac{2 + 2|\gamma|}{1 + 2|\gamma|k + k^2} \right)^{\frac{n}{2}} \right\} m \right],$$

which is equivalent to (6) and this completes the proof of Theorem 4.

*Proof of Theorem 1.* The proof of Theorem 1 proceeds along the same lines as that of Theorem 4, with Theorem A being used in place of Lemma 3. The details are therefore omitted.

## 5. CONCLUSION

The investigation of the growth of polynomials with restricted zeros, together with that of their derivatives in the uniform norm on circles of radius  $R > 0$ , occupies a central place in geometric function theory and approximation theory. This framework offers deep insight into the behavior, smoothness, and asymptotic properties of polynomials under varying radii. Moreover, such studies are intrinsically connected to applications in conformal mapping, potential theory, and the broader analysis of geometric structures within complex analysis.

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