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ADVANCES IN MITRINOVIĆ–ADAMOVIĆ-TYPE INEQUALITIES

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In this paper, we provide several new Mitrinović–Adamović-type inequalities and pose some open problems.

1. INTRODUCTION AND MOTIVATION

Mitrinović and Adamović [**8**] proved the inequality

$$\cos x < \left(\frac{\sin x}{x}\right)^3$$

for $x \in \left(0, \frac{\pi}{2}\right)$, and showed that the exponent 3 is the largest possible. Based on the previous inequality, many authors gave new inequalities: C.-P. Chen and B. Malešević [**2**], T. Lutovac, B. Malešević, M. Rašajski [**4**], B. Malešević, T. Lutovac, M. Rašajski, C. Mortici [**6**], C. Mortici [**9**], J.-L. Sun and C.-P. Chen [**11**], S. Wu and A. Baricz [**15**], Z.-H. Yang and Y.-M. Chu [**16**], L. Zhu [**17**].

In this paper, our aim is to find some real functions $u(x)$ and $v(x)$ such that

$$\left(\frac{\sin x}{x}\right)^3 > u(x) > \cos x > v(x)$$

for $x \in (0, a)$, where $a > 0$.

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In the following two theorems, we give some concrete choices of such functions $u(x)$ and $v(x)$.

Theorem 1. *For* $x \in \left(0, \frac{2}{3}\right)$, *it holds that*

$$\left(\frac{\sin x}{x}\right)^3 > \left(\frac{\sin x}{x}\right)^2 \left(\frac{\arcsin x}{x}\right)^{-1} > \cos x > \frac{\sin x}{x}\left(\frac{\arcsin x}{x}\right)^{-2}. \tag{1}$$

Furthermore, the first and the last inequalities in (1) *also hold for all* $x \in (0,1)$.

Theorem 2. *For* $x \in (0,1)$, *it holds that*

$$\left(\frac{\sin x}{x}\right)^3 > \left(\frac{\sin x}{x}\right)^{\frac{27}{7}} \left(\frac{\arctan x}{x}\right)^{-\frac{3}{7}} > \cos x > \left(\frac{\sin x}{x}\right)^{\frac{9}{5}}\left(\frac{\arcsin x}{x}\right)^{-\frac{6}{5}}. \tag{2}$$

Let us note that due to the choice of the functions $u(x)$ and $v(x)$, the inequalities (1) and (2) are defined on the interval $(0,1)$.

In Sections 3 and 4 we will prove the previous theorems. In Section 5 we will pose some open problems.

2. PRELIMINARIES

In our proofs, we will use the Bernoulli's numbers B_j and Euler's numbers E_j. They are defined by the following generating functions:

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}, \quad |t| < 2\pi,$$

respective

$$\frac{2e^t}{e^{2t}+1} = \sum_{j=0}^{\infty} E_j \frac{t^j}{j!}, \quad |t| < \pi.$$

The first few Bernoulli's numbers are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, while $B_{2k+1} = 0$, for all positive integers k.

The first few Euler's numbers are $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, while $E_{2k+1} = 0$, for all positive integers k.

The following asymptotic expansion holds:

$$E_n \sim (-1)^n \frac{4}{\pi}\left(\frac{2}{\pi}\right)^{2n} (2n)!, \quad n \to \infty.$$

By using the Stirling's formula, we get:

$$|E_{2n}| > 8\sqrt{\frac{n}{\pi}}\left(\frac{4n}{\pi e}\right)^{2n} > 1$$

(see, e.g., [**10**]).

Bernoulli's numbers have many applications in asymptotic analysis. We use the formula

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{2^{2n-2}((n-1)!)^2}{(2n-1)!} x^{2n-1}, \quad |x| < 1.$$

The following inequality holds:

$$\frac{2\,(2n)!}{(2\pi)^{2n}} < (-1)^{n+1} B_{2n} < \frac{2\,(2n)!}{(2\pi)^{2n}} \cdot \frac{1}{1-2^{1-2n}}, \quad n \geq 1.$$

The Bernoulli's numbers are related to the Riemann-zeta function

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}, \quad k \geq 1. \tag{3}$$

We also use the formulas

$$\ln \frac{\sin x}{x} = -\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k\pi^{2k}} x^{2k}, \quad |x| < \pi, \tag{4}$$

and

$$\ln \cos x = -\sum_{k=1}^{\infty} \frac{\left(4^k - 1\right)\zeta(2k)}{k\pi^{2k}} x^{2k}. \tag{5}$$

For proofs and further properties see, e.g., [**1, 12, 14**].

3. A PROOF OF THEOREM 1

We will prove Theorem 1 by proving the following inequalities:

$$\left(\frac{\sin x}{x}\right)^3 > \left(\frac{\sin x}{x}\right)^2 \left(\frac{\arcsin x}{x}\right)^{-1}, \quad x \in (0,1), \tag{6}$$

$$\left(\frac{\sin x}{x}\right)^2 \left(\frac{\arcsin x}{x}\right)^{-1} > \cos x, \quad x \in \left(0, \frac{2}{3}\right), \tag{7}$$

and

$$\cos x > \frac{\sin x}{x} \left(\frac{\arcsin x}{x}\right)^{-2}, \quad x \in (0,1). \tag{8}$$

3.1 A proof of inequality (6)

The inequality (6) is equivalent to the inequality

$$\arcsin x > \frac{x^2}{\sin x}, \quad x \in (0,1),$$

i.e., to the inequality

$$(\arcsin x)^2 > \frac{x^4}{\sin^2 x}, \quad x \in (0,1).$$

Based on the inequality

$$(\arcsin x)^2 > x \tan x, \quad x \in (0,1),$$

(which is given by (10) and will be proved in Subsection 3.3) and the Mitrinović--Adamović inequality, it holds that

$$(\arcsin x)^2 > x \tan x = x\frac{\sin x}{\cos x} > x\frac{\sin x}{\left(\frac{\sin x}{x}\right)^3} = \frac{x^4}{\sin^2 x}$$

for $x \in (0,1)$. The proof is completed.

3.2 A proof of inequality (7)

The inequality (7) is equivalent to the inequality

$$x \arcsin x < \tan x \sin x, \quad x \in \left(0, \frac{2}{3}\right).$$

Since

$$\tan x \sin x = \frac{\sin^2 x}{\cos x} = \frac{1-\cos^2 x}{\cos x} = \sec x - \cos x,$$

by using the power series expansions, we get

$$\tan x \sin x = x^2 + \frac{1}{6}x^4 + \frac{31}{360}x^6 + \frac{173}{5040}x^8 + \frac{25261}{1814400}x^{10} + \sum_{n\geq 6} (-1)^n \left(E_{2n} - 1\right) \frac{x^{2n}}{(2n)!}.$$

We will prove the inequality (7) by proving that

$$x \arcsin x < x^2 + \frac{1}{6}x^4 + \frac{31}{360}x^6 + \frac{173}{5040}x^8 + \frac{25261}{1814400}x^{10} < \tan x \sin x \tag{9}$$

for $x \in \left(0, \frac{2}{3}\right)$.

A proof of the left side of the inequality (9). Let us observe the function

$$f\left(x\right) = \arcsin x - x - \frac{1}{6}x^3 - \frac{31}{360}x^5 - \frac{173}{5040}x^7 - \frac{25261}{1814400}x^9$$

for $x \in \left(0, \frac{2}{3}\right)$. By substitution $x = \sin t$, for $t \in \left(0, \arcsin\frac{2}{3}\right) = (0, 0.72972...)$, we obtain the function

$$g(t) = f(\sin t) = t - \sin t - \frac{1}{6}\sin^3 t - \frac{31}{360}\sin^5 t - \frac{173}{5040}\sin^7 t - \frac{25261}{1814400}\sin^9 t.$$

It holds that $g(t) < 0$ for $t \in (0, 0.73)$ based on the method for proving mixed trigonometric polynomial (MTP) inequalities from [**5, 7**]. Thus,

$$f(x) < 0$$

for $x \in \left(0, \frac{2}{3}\right)$.

A proof of the right side of the inequality (9). This inequality is equivalent to the MTP inequality

$$\sin^2 x - \left(x^2 + \frac{1}{6}x^4 + \frac{31}{360}x^6 + \frac{173}{5040}x^8 + \frac{25261}{1814400}x^{10}\right)\cos x > 0, \quad x \in \left(0, \frac{2}{3}\right),$$

which could be proved using the method from [**5, 7**].

The proof is completed.

3.3 A proof of inequality (8)

The inequality (8) is equivalent to the inequality

$$(\arcsin x)^2 > x\tan x, \quad x \in (0,1)\,. \tag{10}$$

Let us observe the function

$$f(x) = (\arcsin x)^2 - x\tan x$$

for $x \in (0,1)$. We have that

$$f'(x) = -\frac{b(x)}{\sqrt{-x^2+1}},$$

where

$$b(x) = \left(x\tan^2 x + x + \tan x\right)\sqrt{-x^2+1} - 2\,\arcsin x.$$

It holds that

$$b'(x) = \frac{b_1(x)}{\cos^3 x\sqrt{-x^2+1}},$$

where

$$b_1(x) = -x\sin x\cos^2 x - 2\,x^3\sin x - 2\,\cos^3 x - 3\,x^2\cos x + 2\,x\sin x + 2\,\cos x.$$

Based on the method for proving MTP inequalities from [**5, 7**], it holds that

$$b_1(x) < 0$$

for $x \in (0,1)$. Thus, the function $b(x)$ is decreasing for $x \in (0,1)$. Since $b(0) = 0$, it holds that

$$b(x) < 0$$

for $x \in (0,1)$. Therefore, the function $f(x)$ is increasing for $x \in (0,1)$. Since $f(0) = 0$, it holds that

$$f(x) > 0$$

for $x \in (0,1)$, which completes the proof.

4. A PROOF OF THEOREM 2

We will prove Theorem 2 by proving the following inequalities:

$$\left(\frac{\sin x}{x}\right)^3 > \left(\frac{\sin x}{x}\right)^{\frac{27}{7}} \left(\frac{\arctan x}{x}\right)^{-\frac{3}{7}}, \quad x \in (0,1)\,, \tag{11}$$

$$\left(\frac{\sin x}{x}\right)^{\frac{27}{7}} \left(\frac{\arctan x}{x}\right)^{-\frac{3}{7}} > \cos x, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{12}$$

and

$$\cos x > \left(\frac{\sin x}{x}\right)^{\frac{9}{5}} \left(\frac{\arcsin x}{x}\right)^{-\frac{6}{5}}, \quad x \in (0,1)\,. \tag{13}$$

4.1 A proof of inequality (11)

The inequality (11) is equivalent to the inequality

$$\left(\frac{\sin x}{x}\right)^{-\frac{6}{7}} > \left(\frac{\arctan x}{x}\right)^{-\frac{3}{7}}, \quad x \in (0,1)\,,$$

i.e., to the inequality

$$\arctan x > \frac{\sin^2 x}{x}, \quad x \in (0,1)\,.$$

Let us observe the function

$$f(x) = \arctan x - \frac{\sin^2 x}{x}$$

for $x\in(0,1)$. We have that

$$f'(x)=\frac{b(x)}{(x^2+1)\,x^2},$$

where

$$b=\left(-x^2-1\right)\cos^2 x-2\,x\left(x^2+1\right)\sin x\cos x+2\,x^2+1.$$

It holds that

$$b(x)>0$$

for $x\in(0,1)$ based on the method for proving MTP inequalities from [**5, 7**]. Thus, the function $f(x)$ is increasing for $x\in(0,1)$. Since $f(0+)=0$, it holds that

$$f(x)>0$$

for $x\in(0,1)$, which completes the proof.

4.2 A proof of inequality (12)

The inequality (12) is equivalent to the inequality

$$x^8\arctan x<(\sin x)^9\,(\cos x)^{-\frac{7}{3}}\,,\quad x\in\left(0,\frac{\pi}{2}\right).$$

Let us notice that

$$x^8\arctan x=x^9-\frac{1}{3}x^{11}+\frac{1}{5}x^{13}-\frac{1}{7}x^{15}+O\left(x^{17}\right)$$

and

$$(\sin x)^9\,(\cos x)^{-\frac{7}{3}}=x^9-\frac{1}{3}x^{11}+\frac{1}{5}x^{13}-\frac{16}{2835}x^{15}+O\left(x^{17}\right).$$

Therefore, we will prove the inequality (12) by proving that

$$(14)\qquad x^8\arctan x<x^9-\frac{1}{3}x^{11}+\frac{1}{5}x^{13}-\frac{16}{2835}x^{15}<(\sin x)^9\,(\cos x)^{-\frac{7}{3}}$$

for $x\in\left(0,\frac{\pi}{2}\right)$.

A proof of the left side of the inequality (14). Let us observe the function

$$g(x)=\arctan x-x+\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{16}{2835}x^7$$

for $x\in\left(0,\frac{\pi}{2}\right)$. We have that

$$g'(x)=\frac{x^6(16x^2-389)}{405(x^2+1)}<0$$

for $x \in \left(0, \frac{\pi}{2}\right)$. Thus, the function $g(x)$ is strictly decreasing for $x \in \left(0, \frac{\pi}{2}\right)$. Since $g(0) = 0$, it holds that

$$g(x) < 0$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

A proof of the right side of the inequality (14). This inequality is equivalent to the inequality

$$\left(\frac{\sin x}{x}\right)^9 (\cos x)^{-\frac{7}{3}} > 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{16}{2835}x^6, \quad x \in \left(0, \frac{\pi}{2}\right),$$

By taking the logarithm, it suffices to prove that

$$9 \ln \frac{\sin x}{x} - \frac{7}{3} \ln \cos x > \ln \left(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{16}{2835}x^6\right), \quad x \in \left(0, \frac{\pi}{2}\right).$$

Let us notice that

$$9 \ln \frac{\sin x}{x} - \frac{7}{3} \ln \cos x = -\frac{1}{3}x^2 + \frac{13}{90}x^4 + \frac{46}{945}x^6 + \frac{293}{18900}x^8 + O\left(x^{10}\right)$$

and

$$\ln \left(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{16}{2835}x^6\right) = -\frac{1}{3}x^2 + \frac{13}{90}x^4 + \frac{46}{945}x^6 - \frac{467}{170100}x^8 + O\left(x^{10}\right).$$

Therefore, we will prove the right side of the inequality (14) by proving that

$$(15) \quad 9 \ln \frac{\sin x}{x} - \frac{7}{3} \ln \cos x > -\frac{1}{3}x^2 + \frac{13}{90}x^4 + \frac{46}{945}x^6 > \ln\left(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{16}{2835}x^6\right)$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

A proof of the left side of the inequality (15). From (3)-(5), we get

$$\begin{aligned} 9 \ln \frac{\sin x}{x} - \frac{7}{3} \ln \cos x &= \sum_{k=1}^{\infty} \frac{-9 + \frac{7}{3}\left(4^k - 1\right)}{k\pi^{2k}} \zeta(2k) x^{2k} \\ &= -\frac{2}{\pi^2}\zeta(2)x^2 + \frac{26}{2\pi^4}\zeta(4)x^4 + \frac{138}{3\pi^6}\zeta(6)x^6 \\ &\quad + \sum_{k \geq 4} \frac{7 \cdot 4^k - 34}{3k\pi^{2k}} \zeta(2k) x^{2k}. \end{aligned}$$

Since

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945},$$

we obtain that

$$9 \ln \frac{\sin x}{x} - \frac{7}{3} \ln \cos x = -\frac{x^2}{3} + \frac{13}{90}x^4 + \frac{46}{945}x^6 + \sum_{k\geq 4} \frac{7 \cdot 4^k - 34}{3k\pi^{2k}} \zeta(2k) x^{2k}$$
$$> -\frac{x^2}{3} + \frac{13}{90}x^4 + \frac{46}{945}x^6$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

A proof of the right side of the inequality (15). Let us observe the function

$$h(x) = \ln \left(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{16}{2835}x^6\right) + \frac{1}{3}x^2 - \frac{13}{90}x^4 - \frac{46}{945}x^6$$

for $x \in \left(0, \frac{\pi}{2}\right)$. It holds that

$$h'(x) = \frac{A(x)}{B(x)},$$

where

$$A(x) = 2x^7 \left(-736x^4 + 24626x^2 + 9807\right)$$

and

$$B(x) = 315(16x^6 - 567x^4 + 945x^2 - 2835).$$

By an application of the Sturm's theorem (see [**3, 13**]), we obtain that $A(x) > 0$ and $B(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$. Thus,

$$h'(x) < 0$$

for $x \in \left(0, \frac{\pi}{2}\right)$ and consequently, the function $h(x)$ is strictly decreasing for $x \in \left(0, \frac{\pi}{2}\right)$. Since $h(0) = 0$, it holds that

$$h(x) < 0$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

The proof is completed.

Let us note that the inequalities (14) could be reduced to the corresponding MTP inequalities that could be proved using the method from [**5, 7**].

4.3 A proof of inequality (13)

The inequality (13) is equivalent to the inequality

$$(\sin x)^3 (\cos x)^{-\frac{5}{3}} < x (\arcsin x)^2, \quad x \in (0, 1).$$

Let us notice that

$$x\left(\arcsin x\right)^2 = x^3 + \frac{1}{3}x^5 + \frac{8}{45}x^7 + \frac{4}{35}x^9 + \frac{128}{1575}x^{11} + O\left(x^{13}\right)$$

and

$$\left(\sin x\right)^3 \left(\cos x\right)^{-\frac{5}{3}} = x^3 + \frac{1}{3}x^5 + \frac{8}{45}x^7 + \frac{47}{567}x^9 + \frac{1613}{42525}x^{11} + O\left(x^{13}\right).$$

We will prove the inequality (13) by proving that

$$(16) \qquad \left(\sin x\right)^3 \left(\cos x\right)^{-\frac{5}{3}} < x^3 + \frac{1}{3}x^5 + \frac{8}{45}x^7 + \frac{4}{35}x^9 + \frac{128}{1575}x^{11} < x\left(\arcsin x\right)^2$$

for $x \in (0,1)$.

A proof of the right side of the inequality (16). Let us observe the function

$$h(x) = \left(\arcsin x\right)^2 - x^2 - \frac{1}{3}x^4 - \frac{8}{45}x^6 - \frac{4}{35}x^8 - \frac{128}{1575}x^{10}$$

for $x \in (0,1)$. We have that

$$\begin{aligned} h'(x) &= \frac{2\arcsin x}{\sqrt{1-x^2}} - 2x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{32}{35}x^7 - \frac{256}{315}x^9 \\ &= 2\sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!}x^{2n+1} - 2x - \frac{4}{3}x^3 - \frac{16}{15}x^5 - \frac{32}{35}x^7 - \frac{256}{315}x^9 \\ &= 2\sum_{n\ge 5} \frac{2^{2n}(n!)^2}{(2n+1)!}x^{2n+1} > 0 \end{aligned}$$

for $x \in (0,1)$. Thus, the function $h(x)$ is strictly increasing for $x \in (0,1)$. Since $h(0) = 0$, it holds that

$$h(x) > 0$$

for $x \in (0,1)$.

A proof of the left side of the inequality (16). This inequality is equivalent to the inequality

$$3\ln\frac{\sin x}{x} - \frac{5}{3}\ln\cos x < \ln\left(1 + \frac{1}{3}x^2 + \frac{8}{45}x^4 + \frac{4}{35}x^6 + \frac{128}{1575}x^8\right), \quad x \in (0,1).$$

Let us observe the function

$$f(x) = 3\ln\frac{\sin x}{x} - \frac{5}{3}\ln\cos x - \ln\left(1 + \frac{1}{3}x^2 + \frac{8}{45}x^4 + \frac{4}{35}x^6 + \frac{128}{1575}x^8\right)$$

for $x \in (0,1)$. Based on the Lemmas 1.1 and 1.2 from [**7**], it holds that

$$\frac{\sin x}{x} < 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4$$

and

$$\cos x > 1 - \frac{1}{4}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$$

for $x \in (0,1)$. Therefore, one upward polynomial approximation of the function $f(x)$ on the interval $(0,1)$ is

$$\begin{aligned} g(x) = 3\ln\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4\right) - \frac{5}{3}\ln\left(1 - \frac{1}{4}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) \\ - \ln\left(1 + \frac{1}{3}x^2 + \frac{8}{45}x^4 + \frac{4}{35}x^6 + \frac{128}{1575}x^8\right). \end{aligned}$$

It holds that

$$g'(x) = \frac{P(x)}{Q(x)},$$

where

$$\begin{aligned} P(x) = 2x^5\big(- 384x^{12} + 20120x^{10} - 436160x^8 + \\ + 5220880x^6 - 27687575x^4 + 13631400x^2 + 12573000\big) \end{aligned}$$

and

$$Q(x)=(x^4-20x^2+120)(x^6-30x^4+360x^2-720)(128x^8+180x^6+280x^4+252x^2+1575)$$

By an application of the Sturm's theorem (see [**3, 13**]), we obtain that $P(x) > 0$ and $Q(x) < 0$ for $x \in (0,1)$. Thus,

$$g'(x) < 0$$

for $x \in (0,1)$ and consequently, the function $g(x)$ is strictly decreasing for $x \in (0,1)$. Since $g(0) = 0$, it holds that

$$g(x) < 0$$

for $x \in (0,1)$. Considering that $f(x) < g(x)$ for $x \in (0,1)$, it holds that

$$f(x) < 0$$

for $x \in (0,1)$.

The proof is completed.

Let us note that the inequalities (16) could be reduced to the corresponding MTP inequalities that could be proved using the method from [**5, 7**].

5. OPEN PROBLEMS

We propose the following open problems:

(1) what are the best positive constants α and β for which

$$\cos x > \left(\frac{\sin x}{x}\right)^{\alpha}\left(\frac{\arcsin x}{x}\right)^{-\beta}$$

holds for $x \in (0,1)$.

(2) what are the best positive constants γ and δ for which

$$\cos x < \left(\frac{\sin x}{x}\right)^{\gamma}\left(\frac{\arctan x}{x}\right)^{-\delta}$$

holds for $x \in \left(0, \frac{\pi}{2}\right)$.

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