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EXPANSIONS OF CERTAIN INVERSE FACTORIAL SERIES VIA SERIES AND SEQUENCE TRANSFORMATIONS

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In this study, factorial series involving generalizations of the harmonic numbers are investigated. New expressions for the Dirichlet series (also called Euler sums) associated with hyperharmonic and skew-harmonic numbers are obtained. In addition, the relationships between the inverse factorial series of a given sequence and the inverse factorial series of the binomial, Stirling and Lah transformations of that sequence are investigated. Furthermore, closed-form formulas are derived for inverse factorial series whose coefficients are given by the p-Stirling numbers.

1. INTRODUCTION

Inverse factorial series, also known as factorial series of the first type are defined by [**?**]

$$f(z)=\sum_{n=0}^{\infty}\frac{a_n n!}{z(z+1)(z+2)\cdots(z+n)}. \tag{1}$$

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The relation between the ordinary generating function of the sequence (a_n) and $f(z)$ can be obtained as follows. Considering (**??**) one can write

$$f(z)=\sum_{n=0}^{\infty}a_n\int_0^1 t^{z-1}(1-t)^n\,dt.$$

The series

$$\phi(t)=\sum_{n=0}^{\infty}a_n(1-t)^n \tag{2}$$

is called the *generating function* of the inverse factorial series (see [**?**]). With this notation it is possible to write the following *Laplace type integral*

$$f(z)=\int_0^1 t^{z-1}\phi(t)\,dt. \tag{3}$$

On the other hand the solution of this integral equation gives $\phi(t)$ as

$$\phi(t)=\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty}t^{-z}f(z)\,dz,$$

where l is any number greater than the abscissa of absolute convergence of the factorial series for $f(z)$. That is, for a given function $f(z)$, the generating function $\phi(t)$ of the inverse factorial series can be determined (see [**?, ?**], for further explanation). This transition between $f(z)$ and $\phi(t)$ will be used throughout this paper.

One of the most important application areas of the factorial series is the difference equations, because the role of power series in differential equations is played by factorial series in difference equations (see [**?**], for more detail). More precisely, applying the *difference operator* Δ, defined by

$$\Delta h(z)=h(z+1)-h(z)$$

to the function

$$F_n(z)=\frac{n!}{z(z+1)(z+2)\cdots(z+n)}$$

gives

$$\Delta F_n(z)=-F_{n+1}(z).$$

In this sense, the function $\frac{n!}{z^{n+1}}$ from differential calculus has its finite difference analogue in $F_n(z)$ (see [**?**]). From this point of view, the role of power series solutions in the theory of differential equations is taken over by factorial series in the field of linear difference equations. Note that applying difference operator n-times to a suitable function $h(z)$ we get (e.g. [**?**])

$$\Delta^k h(z)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}h(z+i). \tag{4}$$

The factorial series was first seen in the works of Newton (see [**?**]), Stirling (see [**?**]) and later Schlömilch [**?**, **?**]. These series are important in asymptotic analysis (see [**?**, **?**], for more detail). One of the first example of inverse factorial series was given by [**?**] as

$$\log\left(1+\frac{1}{z}\right)=\frac{1}{z}-\frac{\alpha_1}{z\,(z+1)}+\frac{\alpha_2}{z\,(z+1)\,(z+2)}-\cdots, \tag{5}$$

where $\operatorname{Re}(z)>0$ and

$$\alpha_n=\int_0^1 t\,(1-t)\,(2-t)\cdots(n-1-t)\,dt.$$

Finally, we briefly review the general literature. Major contributions to the theory were made by Bendixson [**?**], Nielsen [**?**, **?**, **?**, **?**], Landau [**?**] and Nörlund [**?**, **?**, **?**]. The general theory and important properties of inverse factorial series were given by Milne-Thomson [**?**], Nörlund [**?**, **?**, **?**] and Watson [**?**]. Also, contemporary studies on various properties and applications of these series are ongoing (e.g. Boyadzhiev [**?**]; Karp and Prilepkina [**?**]; Karp [**?**]; Weniger [**?**]).

Motivated by these developments, in this paper we study several aspects of inverse factorial series connected with harmonic-type sequences, sequence transformations, and generalized Stirling numbers. More specifically, we first derive inverse factorial expansions corresponding to the asymptotic series of hyperharmonic and skew-hyperharmonic numbers, together with their special cases, and express the corresponding coefficients in terms of the Cauchy numbers. We then obtain new series representations for Euler sums associated with hyperharmonic, harmonic, and skew-harmonic numbers. Next, we investigate how inverse factorial series behave under the binomial, Stirling, and Lah transformations of a sequence, and as an application we derive a closed-form expression for a particular inverse factorial series in terms of the lower incomplete gamma function. Finally, we establish closed-form inverse factorial expansions involving the p-Stirling numbers of the first kind, thereby extending Boyadzhiev's formula in (**??**), and we also derive an identity expressing these numbers in terms of the ordinary Stirling numbers of the first kind.

The paper is organized as follows. In Section 2, we recall the necessary preliminaries on inverse factorial series, harmonic-type numbers, Euler sums, and Stirling numbers. Section 3.1 is devoted to inverse factorial expansions and Euler-sum representations involving generalized harmonic numbers. Section 3.2 studies the effect of binomial, Stirling, and Lah transformations on inverse factorial series. Section 3.3 concerns inverse factorial series involving p-Stirling numbers and their consequences.

2. PRELIMINARIES

Before presenting our results, in this section we briefly recall the concept that we need from the literature.

The well-known Euler's *gamma function* is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad \mathrm{Re}(z) > 0. \tag{6}$$

The quantity

$$\frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1)\cdots(z+n-1)$$

is known as *the rising factorial* of z (e.g. [**?**]). If we write the rising factorial in powers of z as

$$z(z+1)\cdots(z+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z^k,$$

the resulting coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ are called the *unsigned Stirling numbers of the first kind* (we call them *Stirling numbers of the first kind* for short) (see [**?**, **?**]). We note two different generating functions for the numbers $\begin{bmatrix} n \\ k \end{bmatrix}$, the first one is (see [**?**, **?**])

$$\frac{1}{z^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{z(z+1)\cdots(z+n)}, \tag{7}$$

which is closer to the concept of the inverse factorial series. The second one is the more familiar generating function given by (see [**?**, **?**, **?**])

$$\frac{(-\ln(1-z))^k}{k!} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n!}, \qquad |z| < 1. \tag{8}$$

On the other hand, the *Stirling numbers of the second kind*, denoted $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, is defined by (see [**?**, **?**, **?**])

$$\frac{(e^z - 1)^k}{k!} = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{z^n}{n!}.$$

Boyadzhiev [**?**] demonstrated the relationship between Stirling numbers of the first kind and the inverse factorial series. We recall the p-Stirling numbers of the first kind since they arise naturally in the extension of equation (**??**), and hence in the corresponding generalization of Boyadzhiev's result. *p-Stirling numbers of the first kind*, denoted by $\begin{bmatrix} n+p \\ k+p \end{bmatrix}_p$ (for simplicity we will use the notation $\begin{bmatrix} n \\ k \end{bmatrix}_p$) and defined as

$$\sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_p \frac{z^n}{n!} = \frac{(-\ln(1-z))^k}{(1-z)^p \, k!},$$

where $p \geq 0$ and $n \geq k > 0$. It is obvious that $\begin{bmatrix} n \\ k \end{bmatrix}_0 = \begin{bmatrix} n \\ k \end{bmatrix}$. As has been noted in the literature, Merris's p-Stirling numbers [**?**] are essentially equivalent to Broder's r-Stirling numbers [**?**], since both represent the same underlying generalization of the classical Stirling numbers, differing primarily in notation, indexing conventions, and point of view. More broadly, such generalizations of the Stirling numbers have been considered by several authors at different times, and some of the earliest well-known developments in this direction go back in essence to Carlitz's work from 1980 (see [**?, ?**]).

Let us note two special equations for the Stirling numbers of the first kind, $\begin{bmatrix} n+1 \\ 1 \end{bmatrix} = n!$ and $\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = n!H_n$ (see [**?**]) where H_n is *the n-th harmonic number* defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad H_0 := 0.$$

It can be easily seen that the generating function of these numbers is in the following form:

$$-\frac{\ln(1-t)}{1-t} = \sum_{n=0}^{\infty} H_n t^n.$$

Harmonic numbers and their variations appear in many areas of mathematics. In our study, we frequently encounter series containing these numbers. The alternating variation of the harmonic numbers is called the *skew-harmonic numbers* (see [**?**]), denoted $\widetilde{H}_n$ and defined by

$$\widetilde{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}, \quad \widetilde{H}_0 := 0.$$

Obviously, generating function of the skew-harmonic numbers is

$$\frac{\log(1+t)}{1-t} = \sum_{n=0}^{\infty} \widetilde{H}_n t^n. \tag{9}$$

One of the most important mathematical concepts related to the harmonic numbers and their generalizations is *the multiple zeta function*, in the special case, *Euler sums*. The following Dirichlet series of the harmonic numbers

$$\zeta_H(m) = \sum_{n=1}^{\infty} \frac{H_n}{n^m}, \quad m = 2, 3, 4, \ldots$$

is called *the Euler sum*. Euler showed that this series can be evaluated in terms of *Riemann zeta values* $\zeta(m) = \sum_{n=1}^{\infty} \frac{1}{n^m}$.

In the light of (**??**), Boyadzhiev [**?**] showed that the following equality holds for the Euler sums.

Theorem 1. *([?]) Let $k \geq 1$. Then*

$$\zeta_H(k+1) = \sum_{m=1}^{\infty} \frac{H_m}{m^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\psi'(n)}{n!},$$

where $\psi'(z) = \frac{d}{dz}\psi(z) = \frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)}$.

In Section 3.1, we used this theorem to obtain a series representation of the Euler sums involving generalized harmonic numbers (see Theorem **??**). There are many interesting generalizations of the harmonic numbers. We recall two of them that we need. For a positive integer n and an integer m *the n-th generalized harmonic number of order m*, denoted $H_n^{(m)}$, is defined by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

For $r \in \mathbb{N} = \{1, 2, 3, \ldots\}$; *the n-th hyperharmonic number of order r* denoted $h_n^{(r)}$ is defined by [?]

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}, \qquad h_n^{(0)} = \frac{1}{n}.$$

One can obtain the generating function for the hyperharmonic numbers as [?, ?]

$$\sum_{n=0}^{\infty} h_n^{(r)} z^n = -\frac{\ln(1-z)}{(1-z)^r}. \tag{10}$$

Motivated by this generating function, Kargın and Can [?] defined the *skew-hyperharmonic numbers*, denoted by $\widetilde{h}_n^{(r)}$, as follows:

$$\sum_{n=0}^{\infty} \widetilde{h}_n^{(r)} z^n = \frac{\ln(1+z)}{(1-z)^r},$$

where $\widetilde{h}_n^{(1)} = \widetilde{H}_n$. In Section 3.1, we study the connection between the asymptotic expansions and the inverse factorial expansions of hyperharmonic and skew-hyperharmonic numbers, as well as their special cases. We also examine the corresponding Dirichlet series (Euler sums) associated with hyperharmonic, harmonic, and skew-harmonic numbers from the viewpoint of inverse factorial series. In particular, we consider the Euler sums of hyperharmonic numbers, denoted by $\zeta_{h^{(r)}}(k)$ and defined as [?, ?]

$$\zeta_{h^{(r)}}(k) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^k},$$

where $k > r$. Obviously this is a generalization of the Euler sums of harmonic numbers $\zeta_H(k)$.

Boyadzhiev gave the following result, relating the asymptotic series and the inverse factorial series, by the help of (**??**).

Theorem 2. *([?]) For a given sequence* (a_n) *we have*

$$\sum_{k=0}^{\infty}\frac{a_k}{z^{k+1}}=\sum_{n=0}^{\infty}\frac{1}{z\left(z+1\right)\cdots\left(z+n\right)}\left\{\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}a_k\right\},$$

where z *is assumed not to belong to* $\{0,-1,-2,\ldots\}$.

By splitting the integral in (**??**) at a point $x\geq 0$, we obtain two incomplete gamma functions (e.g. [**?**]). We will deal with the lower part of them, namely *the lower incomplete gamma function*, denoted by $\gamma\left(z,x\right)$, is defined as

$$\gamma\left(z,x\right)=\int_0^x t^{z-1}e^{-t}dt \quad x>0, \qquad \mathrm{Re}\left(z\right)>0, \tag{11}$$

which has the following inverse factorial expansion [**?**]

$$\gamma\left(z,x\right)=x^z e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{z\left(z+1\right)\cdots\left(z+n\right)}. \tag{12}$$

3. RESULTS

3.1 Factorial series for generalized harmonic numbers

In this section, we study asymptotic series and inverse factorial expansions involving harmonic numbers and their generalizations. We also consider the corresponding Euler sums for harmonic numbers, skew-harmonic numbers, and related generalized sequences, and show how several known results in the literature arise as special cases.

Firstly, in the light of (**??**) we give the inverse factorial series corresponding to the asymptotic series of harmonic, hyperharmonic, skew-harmonic and skew-hyperharmonic numbers. We express the coefficients of the inverse factorial series in terms of the Cauchy numbers, which are more familiar than the coefficients α_n in (**??**). Recall that the Cauchy numbers c_n are defined by [**?**]

$$c_n=\int_0^1 t\left(t-1\right)\left(t-2\right)\cdots\left(t-n+1\right)dt.$$

Hence we have the relation

$$\alpha_n=\left(-1\right)^{n-1}c_n$$

with $\alpha_1=c_1:=-1$.

Proposition 3. *Let* r *be a positive integer, and let* z *satisfy* $|z|>1$ *with* z *not a negative integer. Then*

$$\sum_{n=0}^{\infty}\left(-1\right)^n\frac{h_n^{(r)}}{z^n}=\frac{z^r}{\left(z+1\right)^r}\left(-\frac{1}{z}+\sum_{n=1}^{\infty}\frac{c_n}{z\left(z+1\right)\left(z+2\right)\cdots\left(z+n\right)}\right)$$

and

$$\sum_{n=0}^{\infty}\frac{\widetilde{h}_n^{(r)}}{z^n}=\frac{(-1)^r z^r}{(1-z)^r}\left(\frac{1}{z}-\sum_{n=1}^{\infty}\frac{c_n}{z(z+1)(z+2)\cdots(z+n)}\right).$$

Proof. Manipulation of (**??**) gives

$$\sum_{n=0}^{\infty}(-1)^n\frac{h_n^{(r)}}{z^n}=-\frac{\ln\left(1+\frac{1}{z}\right)}{\left(1+\frac{1}{z}\right)^r}=-\frac{z^r}{(1+z)^r}\ln\left(1+\frac{1}{z}\right).$$

Considering (**??**) the RHS becomes

$$-\left(\frac{z}{z+1}\right)^r\ln\left(1+\frac{1}{z}\right)=-\frac{z^r}{(z+1)^r}\left(\frac{1}{z}-\frac{\alpha_1}{z(z+1)}+\frac{\alpha_2}{z(z+1)(z+2)}-\cdots\right).$$

Hence after some arrangements we get the first equation. Result for the skew-hyperharmonic numbers can be obtained in a similar way. □

The special case $r=1$ gives the following relation between the asymptotic series of the harmonic and skew-harmonic numbers and the inverse factorial series of the Cauchy numbers.

Corollary 4. *Let z satisfy $|z|>1$ with z not a negative integer, then we have*

$$\sum_{n=0}^{\infty}(-1)^n\frac{H_n}{z^n}=\frac{1}{z+1}\left(-1+\sum_{n=1}^{\infty}\frac{c_n}{(z+1)(z+2)\cdots(z+n)}\right)$$

and

$$\sum_{n=0}^{\infty}\frac{\widetilde{H}_n}{z^n}=\frac{1}{z-1}\left(1-\sum_{n=1}^{\infty}\frac{c_n}{(z+1)(z+2)\cdots(z+n)}\right).$$

Now we give an extension of Theorem **??** for the hyperharmonic numbers. First, we need a lemma which is similar to the result [**?**, Prop. 5]. A slightly different version of (**??**) is given in [**?**, Lemma 4] and various similar results are given in [**?**].

Lemma 5. *Let $n+1>r>0$. We have*

$$\frac{1}{n!\,(n+1-r)^2}=\sum_{m=0}^{\infty}\frac{h_m^{(r)}}{(m+1)(m+2)\cdots(m+n+1)}. \tag{13}$$

Proof. It is easy to see that

$$\frac{1}{n!}\int_0^1 t^n(1-t)^{m-1}\,dt=\frac{1}{m(m+1)\cdots(m+n)}.$$

For $n+1>r>0$, we can write

$$\frac{1}{n!}\int_0^1 t^n \sum_{m=2}^{\infty} h_{m-1}^{(r)} (1-t)^{m-1}\, dt = \sum_{m=2}^{\infty} \frac{h_{m-1}^{(r)}}{m(m+1)\cdots(m+n)}.$$

We may use (**??**) to conclude that

$$-\frac{1}{n!}\int_0^1 t^{n-r} \ln t dt = \sum_{m=1}^{\infty} \frac{h_{m-1}^{(r)}}{m(m+1)\cdots(m+n)},$$

which gives desired result after straightforward computation of LHS. □

The next theorem gives a new representation for the Euler sums of hyperharmonic numbers $\zeta_{h^{(r)}}(k)$. The reader can find various results of this kind in the works [**?**, **?**].

Theorem 6. *Let $k+1>r>0$. Then*

$$\zeta_{h^{(r)}}(k+1) = \zeta(k+2) + \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{H_n^{(2)} - H_{n-r}^{(2)}}{n!}.$$

Proof. Considering that $h_m^{(r)} = h_{m-1}^{(r)} + h_m^{(r-1)}$ we can write

$$\zeta_{h^{(r)}}(k+1) - \zeta_{h^{(r-1)}}(k+1) = \sum_{m=1}^{\infty} \frac{h_{m-1}^{(r)}}{m^{k+1}}.$$

With the aid of (**??**), RHS can be written in the form

$$\sum_{m=1}^{\infty} \frac{h_{m-1}^{(r)}}{m^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{m=1}^{\infty} \frac{h_{m-1}^{(r)}}{m(m+1)\cdots(m+n)},$$

which combines with Lemma **??** to give

$$\zeta_{h^{(r)}}(k+1) - \zeta_{h^{(r-1)}}(k+1) = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!\,(r-n-1)^2}.$$

For convenience, let us use the notation $\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(r-n-1)^2} = T(k,r)$. Thus we get the following iteration steps.

$$\begin{aligned} \zeta_{h^{(r)}}(k+1) &= T(k,r) + \zeta_{h^{(r-1)}}(k+1) \\ &= T(k,r) + T(k,r-1) + \zeta_{h^{(r-2)}}(k+1) \\ &= T(k,r) + T(k,r-1) + \cdots + T(k,1) + \zeta_{h^{(0)}}(k+1). \end{aligned}$$

Then the above equation can be written as

$$\begin{aligned}\zeta_{h^{(r)}}(k+1) &= \zeta(k+2) + \sum_{p=1}^{r} T(k,p) \\ &= \zeta(k+2) + \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!} \sum_{p=1}^{r} \frac{1}{(n+1-p)^2}.\end{aligned}$$

We complete the proof using the equation

$$\sum_{p=1}^{r} \frac{1}{(n+1-p)^2} = H_n^{(2)} - H_{n-r}^{(2)}.$$

□

The special case $r = 1$, gives a different proof for the equality of $\zeta_H(k)$ which is given by Boyadzhiev [**?**, Eq. (3.2)].

Corollary 7. *For every $k > 0$ we have the identity*

$$\zeta_H(k+1) = \zeta(k+2) + \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!n^2}.$$

We give a closed form evaluation formula for inverse factorial type series of skew-harmonic numbers.

Proposition 8. *We have*

$$\sum_{m=1}^{\infty} \frac{\widetilde{H}_{m-1}}{m(m+1)\cdots(m+n)} = \frac{2^n}{n!n}\left(\ln 2 - H_n + \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{2^k k}\right).$$

Proof. Evaluation of the well-known integral gives

$$\frac{1}{n!}\int_0^1 t^n \widetilde{H}_{m-1}(1-t)^{m-1}\,dt = \frac{\widetilde{H}_{m-1}}{m(m+1)\cdots(m+n)}.$$

Considering (**??**), the above equation can equally well be written as

$$-\frac{1}{n!}\int_0^1 t^{n-1}\ln(2-t)\,dt = \sum_{m=1}^{\infty}\frac{\widetilde{H}_{m-1}}{m(m+1)\cdots(m+n)}.$$

Here integration by parts and some arrangements give

$$\begin{aligned}\int_0^1 t^{n-1}\ln(2-t)\,dt &= -\frac{1}{n}\int_0^1 \frac{t^n}{t-2}dt \\ &= \frac{-1}{n}\int_0^1 \frac{\sum_{k=0}^{n}\binom{n}{k}(t-2)^k 2^{n-k}}{t-2}dt.\end{aligned}$$

If we separate the case $k=0$ and calculate the integrals, then we obtain that

$$\begin{aligned}&\frac{1}{n}\int_0^1\frac{2^n}{2-t}dt-\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}2^{n-k}\int_0^1(t-2)^{k-1}\,dt\\&=\frac{2^n}{n}\ln 2-\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{2^{n-k}}{k}(-1)^k-\frac{1}{n}2^n\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}.\end{aligned}$$

But we already know that $\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}=H_n$ (see [**?**]) which completes the proof. □

Now we give a representation for the Euler sums of the skew-harmonic numbers, denoted by $\zeta_{\widetilde{H}}(k)$ and defined as

$$\zeta_{\widetilde{H}}(k)=\sum_{m=1}^{\infty}\frac{\widetilde{H}_m}{m^k}.$$

For this we first give a lemma.

Lemma 9. *We have*

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^2(m+1)\cdots(m+n)}=\frac{1}{n!}\sum_{k=1}^{n}\frac{1}{k}\left(\binom{n}{k}\widetilde{H}_k-2^k\ln 2\right).$$

Proof. Setting $f_n=(-1)^{n-1}$ and $p=2$ in the formulas (see [**?**, p. 951])

$$\sum_{n=1}^{\infty}\frac{f_n}{n^p\binom{n+l}{l}}=\sum_{a=1}^{l}(-1)^{a-1}\binom{l}{a}a\sum_{n=1}^{\infty}\frac{f_n}{n^p(n+a)}$$

and

$$\sum_{n=1}^{\infty}\frac{f_n}{n^p(n+a)}=\sum_{m=1}^{p-1}\frac{(-1)^{m-1}}{a^m}\sum_{n=1}^{\infty}\frac{f_n}{n^{p+1-m}}+\frac{(-1)^{p-1}}{a^{p-1}}\sum_{n=1}^{\infty}\frac{f_n}{n(n+a)}$$

give

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2(n+1)(n+2)\cdots(n+l)}=\frac{1}{l!}\sum_{a=1}^{l}(-1)^{a-1}\binom{l}{a}a\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2(n+a)}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2(n+a)}=\frac{1}{a}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+a)}\right)$$

respectively. Also note that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\zeta(2)}{2}$$

and [?, p. 656]

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+a)}=\frac{1}{a}\left[(-1)^a-1\right]\ln 2+\frac{(-1)^a}{a}\sum_{k=1}^{a}\frac{(-1)^k}{k}.$$

Taking these into account, the following equation is obtained after some adjustments.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2(n+1)(n+2)\cdots(n+l)}=-\frac{\ln 2}{l!}\sum_{a=1}^{l}\binom{l}{a}\frac{1}{a}-\frac{\ln 2}{l!}H_l+\frac{1}{l!}\sum_{a=1}^{l}\binom{l}{a}\frac{\widetilde{H}_a}{a}.$$

Considering the binomial equation (see [?, Eq. (9.28)])

$$\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}=\sum_{k=1}^{n}\frac{2^k}{k}-H_n$$

in the expression on the RHS and making some adjustments, the desired result is obtained. □

In the next proposition we give an alternative representation for the Euler sums of the skew-harmonic numbers.

Proposition 10. *We have*

$$\zeta_{\widetilde{H}}(k+1)$$
$$=\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}\frac{1}{n!n}\left\{2^nH_n+\widetilde{H}_n+\sum_{k=1}^{n}\binom{n}{k}\frac{2^{n-k}(-1)^{k-1}}{k}+\sum_{k=1}^{n-1}\frac{n}{k}\left(\binom{n}{k}\widetilde{H}_k-2^k\ln 2\right)\right\}.$$

Proof. In the light of (??) we write

$$\sum_{m=1}^{\infty}\frac{\widetilde{H}_m}{m^{k+1}}=\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}\sum_{m=1}^{\infty}\frac{\widetilde{H}_m}{m(m+1)\cdots(m+n)}.$$

Using $\widetilde{H}_n=\widetilde{H}_{n-1}+\frac{(-1)^{n-1}}{n}$ this becomes

$$\zeta_{\widetilde{H}}(k+1)=\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}\left\{\sum_{m=1}^{\infty}\frac{\widetilde{H}_{m-1}}{m(m+1)\cdots(m+n)}+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^2(m+1)\cdots(m+n)}\right\}.$$

Now utilizing Proposition ?? and Lemma ?? we prove the stated result. □

3.2 Sequence Transformations and Inverse Factorial Series

In this section, we study the relationships between the inverse factorial series associated with certain sequences and those associated with their binomial, Stirling, and Lah transforms. We begin by recalling these transformations.

Binomial transform (see [**?**, **?**, **?**]): Given a sequence (a_n) its binomial transform (b_n) is the new sequence defined by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$$

with inversion

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \tag{14}$$

Stirling transform (see [**?**, **?**, **?**]): Given a sequence (a_n) its Stirling transform (b_n) is the new sequence defined by

$$b_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} a_k$$

with inversion

$$a_n = \sum_{k=0}^{n} \left[{n \atop k} \right] (-1)^{n-k} b_k.$$

*Lah transform (see [**?**]):* Given a sequence (a_n) its Lah transform (b_n) is the new sequence defined by

$$b_n = \sum_{k=0}^{n} L(n,k) a_k$$

with inversion

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} L(n,k) b_k,$$

where $L(n,k)$ are the Lah numbers and have the following generating function [**?**]:

$$\frac{1}{k!} \left(\frac{x}{1-x} \right)^k = \sum_{n=0}^{\infty} \frac{L(n,k) x^n}{n!}. \tag{15}$$

In the following theorems, (a_n) will denote a sequence and (b_n) its sequence transform (e.g. Binomial, Stirling and Lah). We write their inverse factorial series as

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z(z+1)\cdots(z+n)} \text{ and } \widetilde{f}(z) = \sum_{n=0}^{\infty} \frac{b_n}{z(z+1)\cdots(z+n)}$$

and their generating functions are

$$\phi(t) = \sum_{n=0}^{\infty} a_n \frac{(1-t)^n}{n!} \text{ and } \widetilde{\phi}(t) = \sum_{n=0}^{\infty} b_n \frac{(1-t)^n}{n!},$$

respectively.

Theorem 11. *Let (b_n) be the binomial transform of a given sequence (a_n). Then we have*

a)

$$\widetilde{\phi}(t) = e^{1-t}\phi(t).$$

b)

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \widetilde{f}(z+k).$$

Proof. a) It is an immediate result considering (**??**) and Cauchy product as

$$\begin{aligned}\phi(t) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k \frac{(1-t)^n}{n!} \\ &= e^{t-1}\widetilde{\phi}(t).\end{aligned}$$

b) In the light of $\phi(t) = e^{t-1}\widetilde{\phi}(t)$, we have

$$f(z) = \int_0^1 t^{z-1}\phi(t)\,dt = \int_0^1 t^{z-1}e^{t-1}\widetilde{\phi}(t)\,dt.$$

Our aim is to relate the functions $\widetilde{f}(z)$ and $f(z)$. By definition,

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 t^{z-1}(1-t)^n \widetilde{\phi}(t)\,dt.$$

By the binomial theorem we write

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^1 t^{z+k-1}\widetilde{\phi}(t)\,dt.$$

Now (**??**) together with $\widetilde{f}(z) = \int_0^1 t^{z-1}\widetilde{\phi}(t)\,dt$ give the desired result. □

Example 12. *One can obtain interesting results considering Theorem* **??** *with known binomial transforms. For instance let us consider the following binomial transforms (see [**?**])*

$$\begin{aligned}\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} H_k &= \frac{1}{n}, \\ \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{1}{k} &= H_n.\end{aligned}$$

Setting

$$a_n = \frac{(-1)^{n+1}}{n} \qquad \text{and} \qquad b_n = H_n,$$

we then obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{1}{z(z+1)\cdots(z+n)}=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\binom{m+n}{m}(-1)^{m}H_{n}}{z(z+1)\cdots(z+m+n)},$$

*where we have used (see [**?**, Eq. 8.34])*

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^{k}}{(z+k)(z+k+1)\cdots(z+k+n)}=\frac{(m+n)!}{n!z(z+1)\cdots(z+m+n)}.$$

The proof of the next theorem is similar to that of Theorem **??** by considering Cauchy product and the generating function (**??**).

Theorem 13. *a) Let (b_n) be the Stirling transform of a given sequence (a_n). Then we have*

$$\widetilde{\phi}(t)=\phi\left(2-e^{1-t}\right).$$

b) Let the sequences (a_n) and (b_n) are given in the form

$$a_n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}b_k.$$

Then we have

$$\widetilde{\phi}(t)=\phi\left(e^{t-1}\right).$$

Example 14. *Consider the following equation*

$$B_n=\sum_{k=0}^{n}S(n,k)\frac{(-1)^{k}k!}{k+1},$$

where B_n is the n-th Bernoulli number. Setting $a_n=\frac{(-1)^n n!}{n+1}$ and $b_n=B_n$ in the Stirling transform and also by remembering the generating function of Bernoulli numbers, we have

$$\widetilde{\phi}(t)=\frac{1-t}{e^{1-t}-1}.$$

Since $\widetilde{\phi}(t)=\phi\left(2-e^{1-t}\right)$ we get

$$\phi(t)=\frac{\ln(2-t)}{1-t},$$

which is obviously generating function of the sequence $a_n=\frac{(-1)^n n!}{n+1}$.

Now we give an inverse factorial expansion related to the lower incomplete gamma function $\gamma(z,1)$.

Corollary 15. *The inverse factorial series* $\sum_{n=0}^{\infty}\frac{n}{z(z+1)\cdots(z+n)}$ *has the following closed form expression*

$$\sum_{n=0}^{\infty}\frac{n}{z\left(z+1\right)\cdots\left(z+n\right)}=e(1-z)\gamma\left(z,1\right)+1.$$

Proof. Considering the inverse factorial series [**?**, page 28]

$$\frac{1}{(z-1)^2}=\sum_{n=0}^{\infty}\frac{n!H_n}{z\left(z+1\right)\cdots\left(z+n\right)}$$

we get the generating function for this series as

$$\phi\left(t\right)=\sum_{n=0}^{\infty}H_n\left(1-t\right)^n=\frac{-\ln\left(t\right)}{t}$$

which permits us to write

$$\phi\left(e^{t-1}\right)=\frac{1-t}{e^{t-1}}.$$

In the light of $\phi\left(e^{t-1}\right)=\widetilde{\phi}\left(t\right)$ we write

$$\widetilde{f}(z)=\int_0^1 t^{z-1}\widetilde{\phi}\left(t\right)dt=\int_0^1 t^{z-1}\phi\left(e^{t-1}\right)dt.$$

Hence

$$\widetilde{f}(z)=\int_0^1 t^{z-1}\left(\frac{1-t}{e^{t-1}}\right)dt=e\left(\int_0^1 t^{z-1}e^{-t}dt-\int_0^1 t^{z}e^{-t}dt\right).$$

We may use (**??**) and (**??**) to conclude that

$$e(\gamma\left(z,1\right)-\gamma\left(z+1,1\right))=\sum_{n=0}^{\infty}\frac{n}{z\left(z+1\right)\cdots\left(z+n\right)}.$$

This result can be further simplified if we use integration by parts as:

$$\gamma\left(z+1,1\right)=-e^{-1}+z\int_0^1 t^{z-1}e^{-t}dt=-e^{-1}+z\gamma\left(z,1\right).$$

This proves the stated result. □

The proof of the next theorem is similar to that of Theorem **??** by considering Cauchy product and the generating function of Lah numbers given by (**??**).

Theorem 16. *Let* (b_n) *be the Lah transform of a given sequence* (a_n)*. Hence we have* $\phi\left(2-\frac{1}{t}\right)=\widetilde{\phi}\left(t\right)$*.*

3.3 Inverse factorial series involving p-Stirling numbers

We now turn to inverse factorial series involving the p-Stirling numbers. The following theorem gives a closed-form expression in this setting and extends equation (**??**).

Theorem 17. *For $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $p \geq k \geq 0$ we have*

$$\frac{1}{(z-p)^{k+1}} = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_p \frac{1}{z(z+1)\cdots(z+n)}, \qquad \operatorname{Re} z > p$$

or equally

$$\frac{1}{z^{k+1}} = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_p \frac{1}{(z+p)(z+p+1)\cdots(z+p+n)}, \quad \operatorname{Re} z > 0. \tag{16}$$

Proof. Let us consider the series

$$f(z) = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_p \frac{1}{z(z+1)\cdots(z+n)}.$$

Here we want to describe the function f. For the sequence $a_n = \frac{1}{n!}\begin{bmatrix} n \\ k \end{bmatrix}_p$, equation (**??**) gives

$$\phi(t) = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_p \frac{(1-t)^n}{n!} = \frac{(-\ln t)^k}{t^p k!}.$$

When we use this in (**??**) we get

$$\begin{aligned} f(z) &= \int_0^1 t^{z-1} \frac{(-\ln t)^k}{t^p k!} dt \\ &= \frac{1}{(z-p)^{k+1}}, \end{aligned}$$

which completes the proof. □

The special case $k = 0$ gives the following well-known result of Nörlund [**?**].

Corollary 18. *For $p \in \mathbb{N}$ and $\operatorname{Re} z > p$ we have*

$$\frac{1}{z-p} = \sum_{n=0}^{\infty} \frac{p(p+1)\cdots(p+n-1)}{z(z+1)\cdots(z+n)}. \tag{17}$$

The next corollary depends on the equation (see [**?**])

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p w^k = (p+w)(p+w+1)\cdots(p+w+n-1),$$

where $w \in \mathbb{C}$, $p \in \mathbb{N}$.

Corollary 19. *For* $\operatorname{Re} z > p$ *and* $w \in \mathbb{C}$ *we have*

$$\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{(p+w)(p+w+1)\cdots(p+w+n-1)}{(z+p)(z+p+1)\cdots(z+p+n)}. \tag{18}$$

Remark 20. *Let* (a_n) *be a sequence and*

$$f(z) = \sum_{n=0}^{\infty} \frac{n! a_n}{z(z+1)\cdots(z+n)}.$$

Then we have

$$f(z) = \sum_{n=0}^{\infty} \frac{n! b_n}{(z+p)(z+p+1)\cdots(z+p+n)},$$

where

$$b_n = a_n + \binom{p}{1} a_{n-1} + \binom{p+1}{2} a_{n-2} + \cdots + \binom{p+n-1}{n} a_0. \tag{19}$$

This transformation is called $(z, z+p)$ *transform of inverse factorial series (see [?], for more detail). Thanks to* $(z, z+p)$ *transform, we express p-Stirling numbers of the first kind as a combination of Stirling numbers of the first kind (see [?] for similar formulas).*

Corollary 21.

$$\begin{bmatrix} n \\ k \end{bmatrix}_p \frac{1}{n!} = \sum_{r=k}^{n} \binom{p+n-r-1}{p-1} \begin{bmatrix} r \\ k \end{bmatrix} \frac{1}{r!}.$$

Proof. Considering (**??**) and (**??**) in the light of (**??**) we get the desired result. □

Here we rediscover a slightly different form of the Vandermonde convolution (see for example [**?**, Eq. (5.6)]) by the help of $(z, z+p)$ transform.

Corollary 22. *We have*

$$\binom{x+p+n-1}{n} = \sum_{k=0}^{n} \binom{x+k-1}{k} \binom{p+n-k-1}{n-k}.$$

Proof. Comparing (**??**) with (**??**) gives $a_n = \binom{x+n-1}{n}$ and $b_n = \binom{x+p+n-1}{n}$. Substitution these in (**??**) gives the desired result. □

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