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NEW RESULTS ON THE ADJUGATE MATRIX APPLIED TO GRAPHS WITH A ROTATED EDGE

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The adjugate matrix adj(G, x) of a simple graph G having adjacency matrix $\mathbf{A}$ is the adjugate of $x\mathbf{I} - \mathbf{A}$. In this paper, we prove two new results on adj(G, x), one of which is a more general version of Jacobi's theorem that is in terms of the characteristic polynomials of G and those of the vertex-deleted subgraphs $G - u$, $G - v$ and $G - u - v$. These two results on adj(G, x) are used to present several necessary and sufficient conditions for G to keep its original characteristic polynomial after performing an edge rotation.

1. INTRODUCTION

Throughout this work, G is a simple graph on n vertices and $\mathbf{A}$ denotes its adjacency matrix. The characteristic polynomial of G, denoted by $\phi(G, x)$, is the determinant of $x\mathbf{I} - \mathbf{A}$, where $\mathbf{I}$ is the $n \times n$ identity matrix. The roots of $\phi(G, x)$ are the eigenvalues of G. The spectrum of G is the multiset of the eigenvalues of G, with possible repetitions. An eigenvalue is simple if its multiplicity in $\phi(G, x)$ is one. We remark that the characteristic polynomial of the graph whose vertex and edge sets are both empty is 1.

If u and v are adjacent vertices, then we may write $u \sim v$ to convey this fact. If $u \sim v$ in the graph G, then the graph $G - uv$ is G with the edge uv removed. In the case that u and v are not adjacent vertices, then we may write $u \not\sim v$ and then $G + uv$ would be the graph that results after introducing the edge uv to G.

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If u is a vertex of G, then the vertex-deleted graph $G-u$ is the induced graph on $n-1$ vertices obtained after deleting the vertex u and all edges incident to it. This notation can be extended; for instance, $G-u-v$ is the graph $(G-u)-v$. Indeed, the notation $G-V$ represents the graph $G-v_1-\cdots-v_k$, where $V=\{v_1,\ldots,v_k\}$ is a subset of the vertex set of G.

The degree of vertex v, denoted by $\deg(v)$, is the number of vertices adjacent to v. If the vertices of a graph G have the same degree, then G is regular. The set containing the vertices of G that are adjacent to v is denoted by $N_G(v)$; the vertices in this set are called the neighbours of v. The cardinality of set S is denoted by $\#S$. Thus we have, for example, $\#(N_G(v))=\deg(v)$.

A walk of length k in G starting from vertex u and ending at vertex v is a sequence of k edges $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_{k+1}$ of G such that $u=v_1$ and $v=v_{k+1}$. Such a walk is a path whenever the $k+1$ vertices $v_1,\ldots,v_{k+1}$ are distinct. Note that the (unique) path from any vertex to itself must have length zero.

An isomorphism from a graph G to a graph H is a bijection σ from the vertex set of G to the vertex set of H such that for any two vertices u and v in G, $u\sim v$ in G if and only if $\sigma(u)\sim\sigma(v)$ in H. If such an isomorphism σ exists, then G and H are isomorphic graphs.

For any square matrix $\mathbf{M}$, $\mathbf{M}_{jk}$ denotes the entry in its j^{th} row and its k^{th} column. The adjugate of $\mathbf{M}$ is the transpose of its matrix of cofactors. The adjugate of G, denoted by $\operatorname{adj}(G,x)$, is the adjugate of $x\mathbf{I}-\mathbf{A}$. Each entry of $\operatorname{adj}(G,x)$ is a polynomial in the variable x with integer coefficients.

From the definition of an adjugate matrix, it is clear that the diagonal entries of $\operatorname{adj}(G,x)$ are $\phi(G-v_1,x),\ldots,\phi(G-v_n,x)$, where $v_1,\ldots,v_n$ are the vertices of G. The off-diagonal entries of $\operatorname{adj}(G,x)$, on the other hand, may be found using a very important result arising from a corollary of Jacobi's theorem (Theorem 1) that relates $\operatorname{adj}(G,x)_{uv}$ with $\phi(G,x)$, $\phi(G-u,x)$, $\phi(G-v,x)$ and $\phi(G-u-v,x)$.

In this paper, we first deduce two new results on the entry $\operatorname{adj}(G-v)_{uw}$. The second of these results generalizes the aforementioned Theorem 1. We then use these new results to investigate conditions for G to keep its original spectrum after performing a slight alteration to it. This alteration is described in the next paragraph.

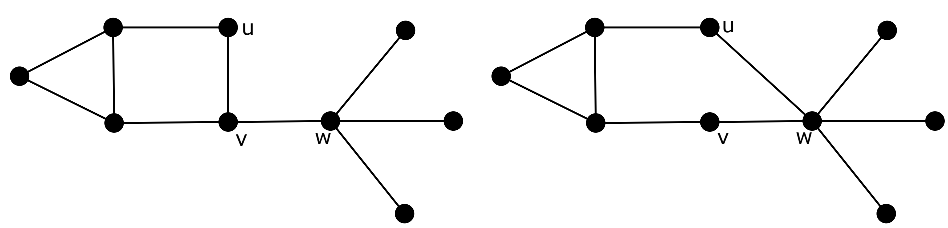

Figure 1: The edge uv in graph G (left) is rotated to become edge uw in graph H (right).

We start from the graph G on at least three vertices and choose three vertices u, v and w from it such that $u \sim v$ and $u \not\sim w$. Then we remove edge uv and introduce edge uw, to produce the graph $H = G - uv + uw$. During this process, the edge uv that was originally incident to v in G has seemingly 'rotated' about vertex u to become incident to w in H, otherwise the remaining vertices and edges in G and H are identical [**2**]; see Figure 1. We are thus calling this process of transforming G to H "rotating edge uv to edge uw". This paper, thus, investigates conditions for G to keep its original spectrum after performing such an edge rotation, aided by our new results on the adjugate matrix.

The paper is organized as follows. The next section contains preliminary results on the adjugate matrix that are useful for the rest of the paper. Section 3 contains our two new results on the adjugate matrix, the second of which is the generalization of Jacobi's theorem we mentioned earlier in this introduction. From Section 4 onwards, our paper switches its focus to the problem of finding conditions for G, having vertices u, v and w such that $u \sim v$ but $u \not\sim w$, to keep its original spectrum after rotating edge uv to edge uw. Section 4 itself applies the two new results of Section 3 to present several necessary and sufficient conditions for this to happen, the main results being Theorem 9, Theorem 10 and Theorem 11. Theorem 12 expresses the exact characteristic polynomial of G in terms of only entries from $\operatorname{adj}(G, x)$ in the case when it is kept intact after rotating uv to uw. Sections 5 and 6 contains only necessary conditions for G to keep its spectrum after performing an edge rotation, most of which are in terms of relatively easily-accessible graph parameters like vertex degrees and walks of graphs. The last section illustrates examples of graphs that keep their original spectrum after performing an edge rotation, found using our main results proved in the previous sections.

2. PRELIMINARY RESULTS ON THE ADJUGATE MATRIX

In this section, we first remind the reader of results on the adjugate matrix that shall be useful in subsequent sections.

We have already mentioned Theorem 1 below in the introduction. This result has been used successfully in the context of determining the conductance or insulation between atoms in hydrocarbons [**7, 13**], and originates from a result by Jacobi [**9**, p. 26] relating the determinant of a block matrix taken from the adjugate of a matrix $\mathbf{M}$ with determinants of block matrices taken from $\mathbf{M}$ itself.

Theorem 1. *If G is a graph having two distinct vertices u and v, then*

$$\left(\operatorname{adj}(G, x)_{uv}\right)^2 = \phi(G - u, x)\,\phi(G - v, x) - \phi(G, x)\,\phi(G - u - v, x).$$

There are also other relationships between off-diagonal entries of $\operatorname{adj}(G, x)$, the characteristic polynomial of G, and those of subgraphs of G. A case in point is the following theorem.

Theorem 2. [**4**, Theorem 5.2] Let u and v be two distinct vertices of G.

a) If $u \sim v$, then $-2\,\mathrm{adj}(G,x)_{uv} = \phi(G,x) - \phi(G-uv,x) - \phi(G-u-v,x)$;

b) If $u \not\sim v$, then $\quad 2\,\mathrm{adj}(G,x)_{uv} = \phi(G,x) - \phi(G+uv,x) - \phi(G-u-v,x)$.

The next result reveals what the coefficient of x^{n-3} of any off-diagonal entry of $\mathrm{adj}(G,x)$ is equal to in terms of walks of G.

Theorem 3. [**4**, Theorem 2.3] *The coefficient of* x^{n-3} *in* $\mathrm{adj}(G,x)_{uv}$, *where* $u \neq v$, *is equal to the number of walks of length two starting from* u *and ending at* v *in* G.

Moreover, each entry in $\mathrm{adj}(G,x)$, for instance that in the u^{th} row and v^{th} column, can be determined from all the paths linking vertices u and v. The following theorem, called the fundamental theorem of adjugate matrices in [**6**], elaborates further.

Theorem 4 (Fundamental Theorem of Adjugate Matrices [**6**])**.** *Let* G *be a graph and* u *and* v *be any two (not necessarily distinct) vertices of* G. *Let* $\mathcal{P}_{uv}$ *be the set of all distinct paths starting from* u *and ending at* v *in* G, *and let* $V(H)$ *be the set of vertices in graph* H. *Then*

$$\mathrm{adj}(G,x)_{uv} = \sum_{P \in \mathcal{P}_{uv}} \phi(G - V(P)).$$

We remark that if $u = v$, then Theorem 4 yields $\mathrm{adj}(G,x)_{uu} = \phi(G-u,x)$ for any vertex u, a fact that was already mentioned in the introduction.

3. NEW RESULTS ON THE ADJUGATE MATRIX

We now reveal two new results on the adjugate matrix.

The first result relates entries of $\mathrm{adj}(G+uv,x)$, $\mathrm{adj}(G,x)$ and $\mathrm{adj}(G-v,x)$, assuming that $u \not\sim v$ in G.

Theorem 5. *Let* G *be a graph and let* u *and* v *be two non-adjacent, distinct vertices of* G. *If* w *is any other vertex of* G, *then*

$$\mathrm{adj}(G+uv,x)_{vw} = \mathrm{adj}(G,x)_{vw} + \mathrm{adj}(G-v,x)_{uw}.$$

Proof. Consider any path P starting from v and ending at w in $G+uv$, having the edge sequence $vv_1, v_1v_2, \ldots, v_kw$. We consider two cases: $v_1 \neq u$ and $v_1 = u$.

Case 1: When $v_1 \neq u$, P is a path in G as well.

Case 2: When $v_1 = u$, the path $P - v$, which is the path P without the initial edge vu in its edge sequence, is a path in $G - v$ starting from u and ending at w.

Since each path from v to w in $G+uv$ must satisfy exactly one of the above two cases, the result follows by applying Theorem 4. □

Theorem 5 may be seen as updating the entry in the v^{th} row and w^{th} column of $\operatorname{adj}(G,x)$ when the edge uv is introduced to G, simply by adding $\operatorname{adj}(G-v)_{uw}$ to it.

This adjugate entry $\operatorname{adj}(G-v)_{uw}$ also features in our second new result, Theorem 7, which is a generalization of Theorem 1. We shall describe how Theorem 7 generalizes Theorem 1 after its proof.

Before revealing Theorem 7, we need the following lemma first.

Lemma 6. [**5**] *Let $\mathbf{R}(H)_w$ be the w^{th} column of the inverse of $x\mathbf{I}-\mathbf{A}_H$, where $\mathbf{A}_H$ is the adjacency matrix associated with the graph H. Consider any graph G and let r_{jk} be the j^{th} entry of $\mathbf{R}(G)_k$. Moreover, let $\kappa_{uv}=(1-r_{uv})^2-r_{uu}r_{vv}$ and let $\varkappa_{uv}=(1+r_{uv})^2-r_{uu}r_{vv}$.*

a) *If $u\not\sim v$ in G, then*

$$\begin{aligned}\mathbf{R}(G+uv)_w = \mathbf{R}(G)_w &+ \frac{1}{\kappa_{uv}}\big((1-r_{uv})r_{vw}+r_{vv}r_{uw}\big)\mathbf{R}(G)_u \\ &+ \frac{1}{\kappa_{uv}}\big((1-r_{uv})r_{uw}+r_{uu}r_{vw}\big)\mathbf{R}(G)_v. \end{aligned} \tag{1}$$

b) *If $u\sim v$ in G, then*

$$\begin{aligned}\mathbf{R}(G-uv)_w = \mathbf{R}(G)_w &- \frac{1}{\varkappa_{uv}}\big((1+r_{uv})r_{vw}-r_{vv}r_{uw}\big)\mathbf{R}(G)_u \\ &- \frac{1}{\varkappa_{uv}}\big((1+r_{uv})r_{uw}-r_{uu}r_{vw}\big)\mathbf{R}(G)_v. \end{aligned} \tag{2}$$

We are now sufficiently prepared to present Theorem 7 and its proof.

Theorem 7. *If G is a graph having three distinct vertices u, v and w, then* $\operatorname{adj}(G,x)_{uv}\ \operatorname{adj}(G,x)_{vw}=\phi(G-v,x)\ \operatorname{adj}(G,x)_{uw}-\phi(G,x)\ \operatorname{adj}(G-v,x)_{uw}$.

Proof. We first suppose that $u\not\sim v$ in G. By Lemma 6, (1) is true. We now focus on κ_{uv} as defined in the same lemma and note that if we shorten the notation $\operatorname{adj}(G,x)_{uv}$ to a_{uv}, then:

$$\begin{aligned}\kappa_{uv} &= \left(1-\frac{a_{uv}}{\phi(G,x)}\right)^2-\frac{a_{uu}\,a_{vv}}{(\phi(G,x))^2}.\\ \kappa_{uv} &= \frac{(\phi(G,x)-a_{uv})^2-\phi(G-u,x)\,\phi(G-v,x)}{(\phi(G,x))^2}.\\ (\phi(G,x))^2\,\kappa_{uv} &= (\phi(G,x))^2-2a_{uv}\,\phi(G,x)+{a_{uv}}^2-\phi(G-u,x)\,\phi(G-v,x).\\ \phi(G,x)\,\kappa_{uv} &= \phi(G,x)-2a_{uv}-\phi(G-u-v,x),\ \text{by Theorem 1.}\\ \phi(G,x)\,\kappa_{uv} &= \phi(G+uv,x),\ \text{by Theorem 2 b).} \end{aligned} \tag{3}$$

Let the w^{th} row of $\operatorname{adj}(G,x)$ be $\mathbf{a}_w$ and the w^{th} row of $\operatorname{adj}(G+uv,x)$ be $\mathbf{b}_w$. Clearly $\phi(G,x)\,\mathbf{R}(G)_w = \mathbf{a}_w$ and $\phi(G+uv,x)\,\mathbf{R}(G+uv)_w = \mathbf{b}_w$. By putting (3) into (1) and simplifying, we obtain:

$$\begin{aligned}(\phi(G,x))^2\,\mathbf{b}_w &= \phi(G,x)\,\phi(G+uv,x)\,\mathbf{a}_w\\ &\quad+\big((\phi(G,x)-a_{uv})a_{vw}+a_{vv}a_{uw}\big)\mathbf{a}_u\\ &\quad+\big((\phi(G,x)-a_{uv})a_{uw}+a_{uu}a_{vw}\big)\mathbf{a}_v.\end{aligned}\tag{4}$$

Now consider the v^{th} entry of the vectors on both sides of (4). We get:

$$\begin{aligned}(\phi(G,x))^2\ \operatorname{adj}(G+uv)_{vw} &= \phi(G,x)\,\phi(G+uv,x)\,a_{vw}\\ &\quad+\big((\phi(G,x)-a_{uv})a_{vw}+a_{vv}a_{uw}\big)a_{uv}\\ &\quad+\big((\phi(G,x)-a_{uv})a_{uw}+a_{uu}a_{vw}\big)a_{vv}.\\ =\phi(G,x)\,\phi(G+uv,x)\,a_{vw} &+\phi(G,x)\,a_{uv}\,a_{vw}-{a_{uv}}^2\,a_{vw}+a_{vv}\,a_{uw}\,a_{uv}\\ &+\phi(G,x)\,a_{uw}\,a_{vv}-a_{uv}\,a_{uw}\,a_{vv}+a_{uu}\,a_{vw}\,a_{vv}.\\ =\phi(G,x)\,\phi(G+uv,x)\,a_{vw} &+\phi(G,x)\big(a_{uv}\,a_{vw}+a_{uw}\,a_{vv}\big)\\ &+a_{vw}\big(a_{uu}\,a_{vv}-{a_{uv}}^2\big).\end{aligned}$$

We now apply Theorem 1 again to the last expression in brackets, getting:

$$\begin{aligned}\phi(G,x)\ \operatorname{adj}(G+uv)_{vw} &= \big(\phi(G+uv,x)+\phi(G-u-v,x)+a_{uv}\big)a_{vw}\\ &\quad+a_{uw}\,a_{vv}.\end{aligned}\tag{5}$$

The expression in brackets in (5) is equal to $\phi(G,x)-a_{uv}$ by Theorem 2 b). In addition, we apply Theorem 5 to the left hand side of (5) to obtain:

$$\phi(G,x)\big(a_{vw}+\operatorname{adj}(G-v)_{uw}\big)=\big(\phi(G,x)-a_{uv}\big)a_{vw}+\phi(G-v,x)\,a_{uw}$$

which simplifies to

$$\phi(G,x)\operatorname{adj}(G-v)_{uw}=-\operatorname{adj}(G,x)_{uv}\ \operatorname{adj}(G,x)_{vw}+\phi(G-v,x)\ \operatorname{adj}(G,x)_{uw}$$

proving the result for the case $u\not\sim v$.

Now suppose $u\sim v$ in G. By Lemma 6, (2) is true. By using a similar argument that previously yielded (3) and using Theorem 2 a) instead, we discover that

$$\phi(G,x)\,\varkappa_{uv}=\phi(G-uv,x)\tag{6}$$

where $\varkappa_{uv}$ is as defined in Lemma 6. If we let $\mathbf{c}_w$ be the w^{th} row of the matrix $\operatorname{adj}(G-uv,x)$, we get the following very similar equation to (4) by substituting (6) in (2):

$$\begin{aligned}(\phi(G,x))^2\,\mathbf{c}_w &= \phi(G,x)\,\phi(G-uv,x)\,\mathbf{a}_w\\ &\quad-\big((\phi(G,x)+a_{uv})a_{vw}-a_{vv}a_{uw}\big)\mathbf{a}_u\\ &\quad-\big((\phi(G,x)+a_{uv})a_{uw}-a_{uu}a_{vw}\big)\mathbf{a}_v.\end{aligned}\tag{7}$$

By again considering the v^{th} entry of the vectors on both sides of (7) and simplifying as before by using Theorem 1, we obtain the following expression akin to (5):

$$\begin{aligned}\phi(G,x)\ \operatorname{adj}(G-uv)_{vw} &= \big(\phi(G-uv,x)+\phi(G-u-v,x)-a_{uv}\big)a_{vw}\\ &\quad - a_{uw}\,a_{vv}.\end{aligned} \tag{8}$$

By using Theorem 2 a), the expression in brackets in (8) is $\phi(G,x)+a_{uv}$. Moreover, if we suppose that $G = G^\star + uv$, then $G^\star = G - uv$. Rewriting Theorem 5 with G replaced by $G^\star$, the equation

$$\operatorname{adj}(G,x)_{vw} = \operatorname{adj}(G-uv,x)_{vw} + \operatorname{adj}(G-v,x)_{uw}$$

is yielded, noting that $G - v$ and $G^\star - v$ are isomorphic. Plugging in this equation in (8), we get:

$$\phi(G,x)\left(a_{vw} - \operatorname{adj}(G-v,x)_{uw}\right) = \big(\phi(G,x)+a_{uv}\big)a_{vw} - a_{uw}\,a_{vv}$$

which becomes

$$-\phi(G,x)\ \operatorname{adj}(G-v,x)_{uw} = \operatorname{adj}(G,x)_{uv}\ \operatorname{adj}(G,x)_{vw} - \phi(G-v,x)\ \operatorname{adj}(G,x)_{uw}.$$

This proves the result for the case $u \sim v$. □

Theorem 7 generalizes Theorem 1 for the following reason. If we substitute $u = w$ in Theorem 7, we get:

$$\operatorname{adj}(G,x)_{uv}\ \operatorname{adj}(G,x)_{vu} = \phi(G-v,x)\ \operatorname{adj}(G,x)_{uu} - \phi(G,x)\ \operatorname{adj}(G-v,x)_{uu}.$$

But $\operatorname{adj}(G,x)_{uu} = \phi(G-u,x)$ and $\operatorname{adj}(G-v,x)_{uu} = \phi(G-v-u,x)$. Thus, the above statement is equivalent to that of Theorem 1. This means that Theorem 7 is also true if vertices u and w are the same; in that case, the theorem would be a restatement of Theorem 1.

4. NECESSARY AND SUFFICIENT CONDITIONS FOR G TO KEEP ITS ORIGINAL SPECTRUM AFTER PERFORMING AN EDGE ROTATION

Henceforth, we consider any graph G on at least three vertices u, v and w such that $u \sim v$ and $u \not\sim w$. We investigate the following problem: which conditions cause G to keep its original spectrum when its edge uv is rotated to edge uw? Our newly-established theorems, Theorem 5 and Theorem 7, shall be used to answer this question.

Clearly we need to focus on the equality $\phi(G-uv+uw,x) = \phi(G,x)$, so we first attempt to rewrite the left hand side of this relationship in terms of substructures related to G that we can work with better.

Note that uw is neither an edge in G nor in $G-uv$. By applying Theorem 2 b) and then Theorem 2 a) on $\phi(G-uv+uw,x)$ and on $\phi(G-uv,x)$ respectively, we get:

$$\begin{aligned}
\phi(G-uv+uw,x) &= \phi(G-uv,x) - 2\,\mathrm{adj}(G-uv,x)_{uw} \\
&\quad - \phi((G-uv)-u-w,x). \\
\phi(G-uv+uw,x) &= \big(\phi(G,x) - \phi(G-u-v,x) + 2\,\mathrm{adj}(G,x)_{uv}\big) \\
&\quad - 2\,\mathrm{adj}(G-uv,x)_{uw} - \phi(G-u-w,x). \\
\phi(G-uv+uw,x) - \phi(G,x) &= 2\,\mathrm{adj}(G,x)_{uv} - 2\,\mathrm{adj}(G-uv,x)_{uw} \\
&\quad - \phi(G-u-v,x) - \phi(G-u-w,x). \qquad (9)
\end{aligned}$$

By observing that $\phi(G-uv+uw,x)=\phi(G,x)$ if and only if the polynomial on the right hand side of (9) is zero, we deduce the following theorem.

Theorem 8. *Let G be a graph and let u, v and w be vertices of G such that $u\sim v$ and $u \not\sim w$. Then G keeps its original spectrum after uv is rotated to uw if and only if*

$$\phi(G-u-v,x) + \phi(G-u-w,x) = 2\big(\mathrm{adj}(G,x)_{uv} - \mathrm{adj}(G-uv,x)_{uw}\big).$$

By using Theorem 5 in the result of Theorem 8, we obtain another necessary and sufficient condition for G to keep its original spectrum after rotating uv to uw, but this time in terms of the entry $\mathrm{adj}(G-u,x)_{vw}$.

Theorem 9. *G keeps its spectrum after rotating uv to uw if and only if*

$$\begin{aligned}
&\phi(G-u-v,x) + \phi(G-u-w,x) \\
&\qquad = 2\big(\mathrm{adj}(G,x)_{uv} - \mathrm{adj}(G,x)_{uw} + \mathrm{adj}(G-u,x)_{vw}\big).
\end{aligned}$$

Proof. Since $u\sim v$ in G, we rewrite Theorem 5 in the following manner:

$$\mathrm{adj}((G-uv)+uv,x)_{uw} = \mathrm{adj}((G-uv),x)_{uw} + \mathrm{adj}((G-uv)-u,x)_{vw}$$

which may be rewritten as

$$\mathrm{adj}(G-uv,x)_{uw} = \mathrm{adj}(G,x)_{uw} - \mathrm{adj}(G-u,x)_{vw}$$

since $(G-uv)-u$ and $G-u$ are isomorphic. Substituting this expression for $\mathrm{adj}(G-uv,x)_{uw}$ in Theorem 8, the result is proved. □

4.1 Condition Depending Only on $G-u$

It may be easier to find conditions for G to keep its original spectrum after rotating uv to uw that only require knowledge of $G-u$. This subsection focuses on this approach. We slightly shorten the notation $G-u$ to G_u throughout the proof of the next result.

Theorem 10. *G keeps its spectrum after rotating uv to uw if and only if*

$$\phi(G_u - v, x) - \phi(G_u - w, x) = 2 \sum_{t \in N_G(u) \setminus \{v\}} \big[\operatorname{adj}(G_u, x)_{tw} - \operatorname{adj}(G_u, x)_{tv} \big].$$

Proof. Let $\mathbf{A}_G$ and $\mathbf{A}_H$ be the adjacency matrices of G and of $H = G - uv + uw$ respectively. Since $\operatorname{adj}(G, x)$ is an adjugate matrix, $(x\mathbf{I} - \mathbf{A}_G) \operatorname{adj}(G, x) = \phi(G, x)\,\mathbf{I}$. That is

$$[(x\mathbf{I} - \mathbf{A}_G) \operatorname{adj}(G, x)]_{ii} = \phi(G, x) \quad \text{for some } 1 \le i \le n.$$

Similarly for H, we have:

$$[(x\mathbf{I} - \mathbf{A}_H) \operatorname{adj}(H, x)]_{jj} = \phi(H, x) \quad \text{for some } 1 \le j \le n.$$

Thus, $\phi(G, x) = \phi(H, x)$ if and only if

$$[(x\mathbf{I} - \mathbf{A}_G) \operatorname{adj}(G, x)]_{ii} = [(x\mathbf{I} - \mathbf{A}_H) \operatorname{adj}(H, x)]_{jj} \quad \text{for some } 1 \le i, j \le n.$$

If we consider the entry that corresponds to vertex u, then

$$x\,\phi(G_u, x) - \sum_{t \in N_G(u)} \operatorname{adj}(G, x)_{ut} = x\,\phi(H_u, x) - \sum_{t \in N_H(u)} \operatorname{adj}(H, x)_{ut}.$$

Since G_u and H_u are isomorphic graphs, $\phi(G, x) = \phi(H, x)$ if and only if

$$\sum_{t \in N_G(u)} \operatorname{adj}(G, x)_{ut} = \sum_{t \in N_H(u)} \operatorname{adj}(H, x)_{ut}.$$

Note that the sets $N_G(u) \setminus \{v\}$ and $N_H(u) \setminus \{w\}$ are equal. If we denote the elements of $N_G(u) \setminus \{v\}$ by $u_1, \ldots, u_k$, then by Theorem 4,

$$\sum \operatorname{adj}(G, x)_{ut} = \sum \operatorname{adj}(G_u, x)_{tv} + \sum \operatorname{adj}(G_u, x)_{tu_1} + \cdots + \sum \operatorname{adj}(G_u, x)_{tu_k},$$

where each summation spans all vertices $t \in N_G(u)$. Similarly,

$$\sum \operatorname{adj}(H, x)_{ut} = \sum \operatorname{adj}(G_u, x)_{tw} + \sum \operatorname{adj}(G_u, x)_{tu_1} + \cdots + \sum \operatorname{adj}(G_u, x)_{tu_k},$$

where each summation spans all vertices $t \in N_H(u)$.

For all p, the expressions $\sum_{t \in N_G(u)} \operatorname{adj}(G_u, x)_{tu_p}$ and $\sum_{t \in N_H(u)} \operatorname{adj}(G_u, x)_{tu_p}$ differ only by $\operatorname{adj}(G_u, x)_{vu_p}$ in the first sum and by $\operatorname{adj}(G_u, x)_{wu_p}$ in the other. Thus

$$\left(\sum_{t \in N_G(u)} \operatorname{adj}(G_u, x)_{tv} \right) + \operatorname{adj}(G_u, x)_{vu_1} + \cdots + \operatorname{adj}(G_u, x)_{vu_k}$$
$$= \left(\sum_{t \in N_H(u)} \operatorname{adj}(G_u, x)_{tw} \right) + \operatorname{adj}(G_u, x)_{wu_1} + \cdots + \operatorname{adj}(G_u, x)_{wu_k}.$$

Now

$$\operatorname{adj}(G_u, x)_{vu_1} + \cdots + \operatorname{adj}(G_u, x)_{vu_k} = \sum_{t \in N_G(u)\setminus\{v\}} \operatorname{adj}(G_u, x)_{tv}$$

and

$$\operatorname{adj}(G_u, x)_{wu_1} + \cdots + \operatorname{adj}(G_u, x)_{wu_k} = \sum_{t \in N_G(u)\setminus\{w\}} \operatorname{adj}(G_u, x)_{tw}.$$

Noting that $\operatorname{adj}(G_u, x)_{vv} = \phi(G_u - v)$ and $\operatorname{adj}(G_u, x)_{ww} = \phi(G_u - w)$, we deduce that $\phi(G, x) = \phi(H, x)$ if and only if

$$\phi(G_u - v, x) + 2 \sum_{t \in N_G(u)\setminus\{v\}} \operatorname{adj}(G_u, x)_{tv} = \phi(G_u - w, x) + 2 \sum_{t \in N_H(u)\setminus\{w\}} \operatorname{adj}(G_u, x)_{tw},$$

which proves the result. □

4.2 Conditions on the Characteristic Polynomial of G

We now apply our generalized version of Jacobi's theorem, Theorem 7, to one of our earlier theorems, obtaining another necessary and sufficient condition for G to keep its original spectrum after rotating uv to uw, but this time involving only the characteristic polynomial of G and entries from the adjugate of G.

Theorem 11. *The graph G keeps its original spectrum when uv is rotated to uw if and only if*

$$\begin{aligned}\phi(G-u, x)\big(\phi(G-v, x) + \phi(G-w, x) - 2\operatorname{adj}(G, x)_{vw}\big)\\ = \big(\operatorname{adj}(G, x)_{uv} - \operatorname{adj}(G, x)_{uw}\big)\big(2\,\phi(G, x) + \operatorname{adj}(G, x)_{uv} - \operatorname{adj}(G, x)_{uw}\big).\end{aligned}$$

Proof. We start from the result of Theorem 9, with both sides multiplied by $\phi(G, x)$:

$$\begin{aligned}(10)\quad \phi(G, x)\big(\phi(G-u-v, x) + \phi(G-u-w, x)\big)\\ = 2\,\phi(G, x)\big(\operatorname{adj}(G, x)_{uv} - \operatorname{adj}(G, x)_{uw} + \operatorname{adj}(G-u, x)_{vw}\big).\end{aligned}$$

By Theorem 1, the left hand side of (10) can be rewritten as

$$\phi(G-u, x)\,\phi(G-v, x) - \operatorname{adj}(G, x)_{uv}{}^2 + \phi(G-u, x)\,\phi(G-w, x) - \operatorname{adj}(G, x)_{uw}{}^2.$$

We apply Theorem 7 (with u and v swapped) to the right hand side of (10), so that it becomes

$$\begin{aligned}2\,\phi(G, x)\operatorname{adj}(G, x)_{uv} - 2\,\phi(G, x)\operatorname{adj}(G, x)_{uw}\\ + 2\big(\phi(G-u, x)\operatorname{adj}(G, x)_{vw} - \operatorname{adj}(G, x)_{uv}\operatorname{adj}(G, x)_{uw}\big).\end{aligned}$$

By grouping terms involving $\phi(G-u, x)$ and those involving $\phi(G, x)$ and noticing that

$$\begin{aligned}\operatorname{adj}(G, x)_{uv}{}^2 - 2\operatorname{adj}(G, x)_{uv}\operatorname{adj}(G, x)_{uw} + \operatorname{adj}(G, x)_{uw}{}^2\\ = \big(\operatorname{adj}(G, x)_{uv} - \operatorname{adj}(G, x)_{uw}\big)^2\end{aligned}$$

we arrive at the relation

$$\phi(G-u,x)\big(\phi(G-v,x)+\phi(G-w,x)-2\,\mathrm{adj}(G,x)_{vw}\big) \\ = 2\,\phi(G,x)\big(\mathrm{adj}(G,x)_{uv}-\mathrm{adj}(G,x)_{uw}\big)+\big(\mathrm{adj}(G,x)_{uv}-\mathrm{adj}(G,x)_{uw}\big)^2,$$

proving the result. □

Note that since, in Theorem 11, we are assuming that $u \sim v$ and $u \not\sim w$, then by Theorem 4, the degree of the polynomial $\mathrm{adj}(G,x)_{uv}$ would be $n-2$, while that of $\mathrm{adj}(G,x)_{uw}$ would be less than $n-2$. Thus, $\mathrm{adj}(G,x)_{uv}-\mathrm{adj}(G,x)_{uw}$ is a (monic) polynomial of degree $n-2$. Henceforth, we refer to this polynomial as $a_u^{vw}(x)$. The fact that $a_u^{vw}(x)$ is nonzero allows us to rewrite the formula of Theorem 11 with $\phi(G,x)$ as its subject, and with its right hand side consisting solely of entries of $\mathrm{adj}(G,x)$.

Theorem 12. *The graph G keeps its original characteristic polynomial $\phi(G,x)$ after rotating uv to uw if and only if $\phi(G,x)$ is equal to*

$$\frac{\phi(G-u,x)\big(\phi(G-v,x)+\phi(G-w,x)-2\,\mathrm{adj}(G,x)_{vw}\big)-\big(a_u^{vw}(x)\big)^2}{2\,a_u^{vw}(x)},$$

where $a_u^{vw}(x)=\mathrm{adj}(G,x)_{uv}-\mathrm{adj}(G,x)_{uw}$.

5. NECESSARY CONDITIONS FOR G TO KEEP ITS ORIGINAL SPECTRUM AFTER ROTATING uv TO uw

We now give necessary conditions for G to keep its spectrum after the rotation of edge uv to edge uw is made.

Since the right hand side of the statement of Theorem 8 is a polynomial with even coefficients, the following corollary is deduced immediately.

Corollary 13. *A necessary condition for G to keep its original spectrum after uv is rotated to uw is that all the coefficients of $\phi(G-u-v,x)+\phi(G-u-w,x)$ are even numbers. Equivalently, for all k, the coefficient of x^k in $\phi(G-u-v,x)$ and $\phi(G-u-w,x)$ have the same parity.*

The subsequent necessary conditions are in terms of properties of the graph G. The next one uses Sachs' coefficient theorem [**12**], which relates each coefficient of the characteristic polynomial of a graph G with substructures of G called elementary subgraphs. In particular, we have these well-known consequences about the coefficients of x^{n-2} and x^{n-3} in $\phi(G,x)$.

Lemma 14. *Let $\phi(G,x)=\sum_{j=1}^{n} c_j x^j$. If m is the number of edges of G and t is the number of triangles (circuits of size three) in G, then $c_{n-2}=-m$ and $c_{n-3}=-2t$.*

We then have:

Corollary 15. *If G keeps its original spectrum after uv is rotated to uw, then* $\deg(v)$ *and* $\deg(w)$ *have different parities in G.*

Proof. The graph $G-u-v$ has $n-2$ vertices and $m-\deg(u)-\deg(v)+1$ edges, where m denotes the number of edges of G. By Lemma 14, the coefficient of x^{n-4} in $\phi(G-u-v,x)$ is thus equal to $-(m-\deg(u)-\deg(v)+1)$. Similarly, the coefficient of x^{n-4} in $\phi(G-u-w,x)$ is equal to $-(m-\deg(u)-\deg(w))$. From Corollary 13, these two numbers must have the same parity, and the result follows. □

Of course, if $\deg(v)$ and $\deg(w)$ have different parities in G, then they must also have different parities in the resulting graph after the rotation of uv to uw is performed.

We now consider the number of walks of length two from u to v in G.

Corollary 16. *If G keeps its original spectrum after uv is rotated to uw, then the number of walks of length two from u to v in G is equal to the number of walks of length two from u to w in the graph resulting after the rotation of uv to uw.*

Proof. We refer to the relation in Theorem 8. The coefficient of x^{n-3} in the polynomial $\phi(G-u-v,x)+\phi(G-u-w,x)$, which is the left hand side of Theorem 8, is zero. Hence this coefficient is also zero on the right hand side, which is $2\big(\mathrm{adj}(G,x)_{uv}-\mathrm{adj}(G-uv,x)_{uw}\big)$. By applying Theorem 3 to the polynomial $\mathrm{adj}(G,x)_{uv}-\mathrm{adj}(G-uv,x)_{uw}$, we deduce that the number of walks of length two from u to v in G is equal to the number of walks of length two from u to w in $G-uv$. If we introduce edge uw to the graph $G-uv$, then the number of walks of length two from u to w in the graph is unaffected. This proves the result. □

Let $W_{uw}^{(2)}$ and $\bar{W}_{uw}^{(2)}$ be the number of walks of length 2 in G and in $G-uv+uw$ respectively starting from u and ending at w. It can be seen that

$$\bar{W}_{uw}^{(2)}=\begin{cases} W_{uw}^{(2)}, & \text{if } v\not\sim w \text{ in } G;\\ W_{uw}^{(2)}-1, & \text{if } v\sim w \text{ in } G.\end{cases} \tag{11}$$

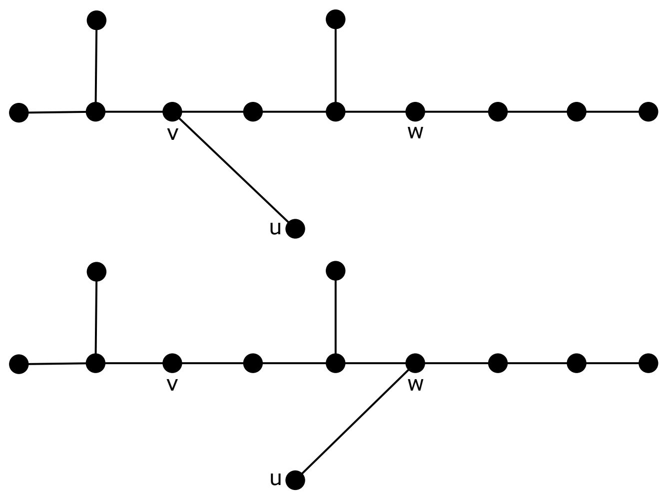

Figure 2: The graph G (top), having vertex u of degree one, keeps its original spectrum after rotating edge uv to edge uw (bottom).

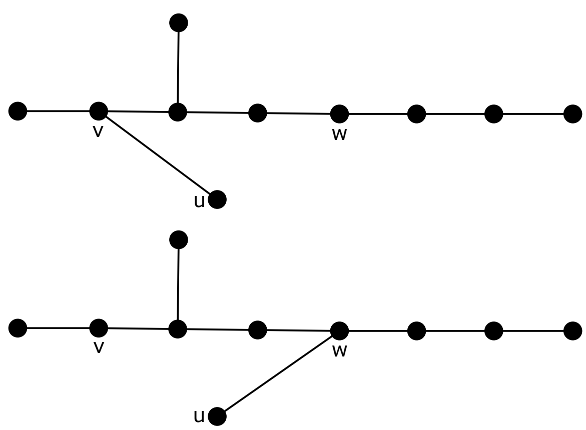

Figure 3: Another graph G (top), having vertex u of degree one, also keeps its original spectrum after rotating edge uv to edge uw (bottom).

The next corollary is an alternative way of writing Corollary 16 after applying (11) to its result.

Corollary 17. *Suppose the graph G keeps its original spectrum after its edge uv is rotated to edge uw. Then:*

a) *If $v \not\sim w$ in G, then $\#\big(N_G(u) \cap N_G(v)\big) = \#\big(N_G(u) \cap N_G(w)\big)$. That is, if in*

G, $v \not\sim w$, then the number of common neighbours of u and v is equal to the number of common neighbours of u and w.

b) *If $v \sim w$ in G, then $\#\big(N_G(u) \cap N_G(v)\big) = \#\big(N_G(u) \cap N_G(w)\big) - 1$. That is, if in G, $v \sim w$, then the number of common neighbours of u and v is one less than the number of common neighbours of u and w.*

Observe that, by Lemma 14, the coefficient of x^{n-3} of $\phi(G,x)$ is equal to -2 times the number of triangles in G. Corollary 17 above is necessary in order to leave the number of triangles in G unchanged after the edge uv is rotated to uw.

The right hand side of the formula in the statement of Theorem 11 is $a_u^{vw}(x)\big(a_u^{vw}(x) + 2\,\phi(G,x)\big)$, where $a_u^{vw}(x) = \operatorname{adj}(G,x)_{uv} - \operatorname{adj}(G,x)_{uw}$. Since $\phi(G-u,x)$ is a factor of the left hand side of the same formula, we have the following rather strict criterion that is required for G to keep its original characteristic polynomial after rotating uv to uw.

Theorem 18. *Let $a_u^{vw}(x)$ be the polynomial* $\operatorname{adj}(G,x)_{uv} - \operatorname{adj}(G,x)_{uw}$. *If G keeps its original spectrum after rotating uv to uw, then* **each** *of the $n-1$ eigenvalues λ of $G-u$ must either satisfy $a_u^{vw}(\lambda) = 0$ or $a_u^{vw}(\lambda) = -2\,\phi(G,\lambda)$.*

Proof. Immediate by substituting λ on both sides of Theorem 11. □

6. RELATION BETWEEN THE ADJUGATE MATRIX AND EIGENVECTOR ENTRIES

We now briefly relate entries of the adjugate matrix with eigenvector entries. This connection lets us establish a further result similar to that of Theorem 18. In the subsequent work, any eigenvector associated with the eigenvalue λ of G (that is, any nonzero vector $\mathbf{x}$ satisfying $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$) is assumed to be normalized.

In [**11**] (see also [**8**]), it was proved that if λ is a simple eigenvalue of a graph G with associated eigenvector $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^{\mathsf{T}}$, then

$$x_u{}^2 = \frac{\phi(G-u,\lambda)}{\phi'(G,\lambda)}, \tag{12}$$

where $\phi'(G,x)$ is the derivative of $\phi(G,x)$ with respect to x.

This result can be generalized as follows. Let G have n simple eigenvalues. It is well-known (see [**1, 3**]) that the generating function for the number of walks in G from vertex u to vertex v is

$$H(x)_{uv} = \sum_{i=1}^{n} \frac{(\mathbf{e}_u^{\mathsf{T}}\mathbf{x}_i)(\mathbf{e}_v^{\mathsf{T}}\mathbf{x}_i)}{1-\lambda_i x} = \mathbf{e}_u^{\mathsf{T}}(\mathbf{I} - x\mathbf{A})^{-1}\mathbf{e}_v,$$

where $\lambda_1, \ldots, \lambda_n$ are the n eigenvalues of G, $\mathbf{x}_i$ is the eigenvector associated with λ_i for all i and $\mathbf{e}_j$ is the j^{th} column of $\mathbf{I}$. By substituting x^{-1} for x in the above relationship, we get

$$x^{-1}H(x^{-1})_{uv} = \sum_{i=1}^{n} \frac{(\mathbf{e}_u^\mathsf{T}\mathbf{x}_i)(\mathbf{e}_v^\mathsf{T}\mathbf{x}_i)}{x - \lambda_i} = \frac{\mathbf{e}_u^\mathsf{T}\,\mathrm{adj}(G,x)\mathbf{e}_v}{\phi(G,x)}.$$

By focusing on the second equality and taking a common denominator for the summation, we obtain the following useful result.

Theorem 19. *If G is a graph having the n simple eigenvalues $\lambda_1, \ldots, \lambda_n$ and, for all i, $\mathbf{x}_i$ is the eigenvector associated with λ_i, then*

$$\mathrm{adj}(G,x)_{uv} = \sum_{i=1}^{n} \left((\mathbf{e}_u^\mathsf{T}\mathbf{x}_i)(\mathbf{e}_v^\mathsf{T}\mathbf{x}_i) \prod_{j \neq i} (x - \lambda_j) \right).$$

Now let $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\mathsf{T}$ be the eigenvector associated with the simple eigenvalue λ_k. By substituting $x = \lambda_k$ in Theorem 19, we have

$$\mathrm{adj}(G,\lambda_k)_{uv} = x_u\, x_v \prod_{\substack{j=1 \\ j \neq k}}^{n} (\lambda_k - \lambda_j).$$

But $\phi'(G,x) = \sum_{i=1}^{n} \left(\prod_{j \neq i} (x - \lambda_j) \right)$, so that $\phi'(G,\lambda_k) = \prod_{\substack{j=1 \\ j \neq k}}^{n} (\lambda_k - \lambda_j)$. Hence we obtain the following corollary.

Corollary 20. *If λ is a simple eigenvalue of G with associated eigenvector $(x_1, x_2, \ldots, x_n)^\mathsf{T}$, then $x_u\, x_v = \dfrac{\mathrm{adj}(G,\lambda)_{uv}}{\phi'(G,\lambda)}$.*

Thus, the above corollary generalizes (12), since (12) is Corollary 20 for the particular case $u = v$.

Now consider a connected graph G, so that its largest eigenvalue λ_1 is simple and its associated eigenvector $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\mathsf{T}$ has strictly positive entries. In [**3**, p. 230], there is the following result that is related to our work.

Theorem 21. [**3**] *Let G be a connected graph and let u, v, w be vertices in G such that $u \sim v$ and $u \not\sim w$. Moreover, let λ_1 be the largest eigenvalue of G with associated eigenvector $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\mathsf{T}$, and let H be the graph obtained from G after rotating the edge uv to the edge uw. If $x_w \geq x_v$, then the largest eigenvalue of H is larger than λ_1.*

Thus, if $x_w \geq x_v$, then G does not keep its spectrum when edge uv is rotated to edge uw, no matter which vertex u, incident to v but not incident to w, is chosen to rotate edge uv about to obtain edge uw. This is related to adjugate entries of G as follows.

By Corollary 20, $x_u(x_v - x_w) = \dfrac{\mathrm{adj}(G,\lambda_1)_{uv} - \mathrm{adj}(G,\lambda_1)_{uw}}{\phi'(G,\lambda_1)}$. Note that in the previous section, we had abbreviated the numerator of the previous expression to $a_u^{vw}(\lambda_1)$. Moreover, $\phi'(G,\lambda_1)$ is a positive number, since by the shape of the graph of the polynomial $\phi(G,x)$ (which has a positive leading coefficient), the slope at its largest root must be positive. Thus $x_v - x_w \leq 0$ if and only if $a_u^{vw}(\lambda_1) \leq 0$ for all vertices u incident to v and not incident to w. Thus, using Theorem 21 and adjugate entries, we deduce the following result.

Theorem 22. *Let G be a connected graph with largest eigenvalue λ_1 and let u, v, w be vertices in G such that $u \sim v$ but $u \not\sim w$. Suppose $a_u^{vw}(\lambda_1)$ is not a positive number, where $a_u^{vw}(x)$ is as in Theorem 18. Then G does not keep its spectrum after rotating uv to uw. Moreover, if u' is any other vertex in G satisfying $u' \sim v$ and $u' \not\sim w$, then G does not keep its spectrum after rotating $u'v$ to $u'w$.*

7. EXAMPLES

By applying our necessary and sufficient criteria in Section 4, we made several computer searches, using the Mathematica software package, to find hundreds of examples of graphs that maintain their original spectrum when one of their edges is rotated to another edge. Some of these examples are depicted in Figures 2, 3, 4 and 5.

In our search, we found some examples where the graph with the rotated edge is isomorphic to the original graph. Of course, in such instances, the original graph and the one with the rotated edge have the same spectrum as well. We thus focused on examples where the graph obtained after an edge is rotated is not isomorphic to the original.

During our search, we noticed that when the degree of the pivoting vertex u is one, the necessary and sufficient criterion for G to maintain the same spectrum after rotating uv to uw is very simple — we only require $\phi(G-u-v,x)$ to be equal to $\phi(G-u-w,x)$. This is, in fact, a corollary of Theorem 10, and is well-known in the literature [**14**]. Indeed, the examples in Figures 2 and 3 are quite well-known in the literature as well; see, for example, [**10, 14**].

Figures 4 and 5 depict six examples of cases where the pivoting vertex through which we are rotating an edge has degree two. All of these six examples keep their original spectrum after performing the edge rotation, but the resulting graph is not isomorphic to the original one. We found 101 such examples having nine vertices (five of which are shown in Figures 4 and 5) where the graph with the rotated edge is not isomorphic to the original graph, but has the same spectrum as the original graph. The first example is particularly intriguing in that it is originally disconnected, but becomes connected after rotating an edge, and yet the graph still keeps its original spectrum after the edge rotation is performed.

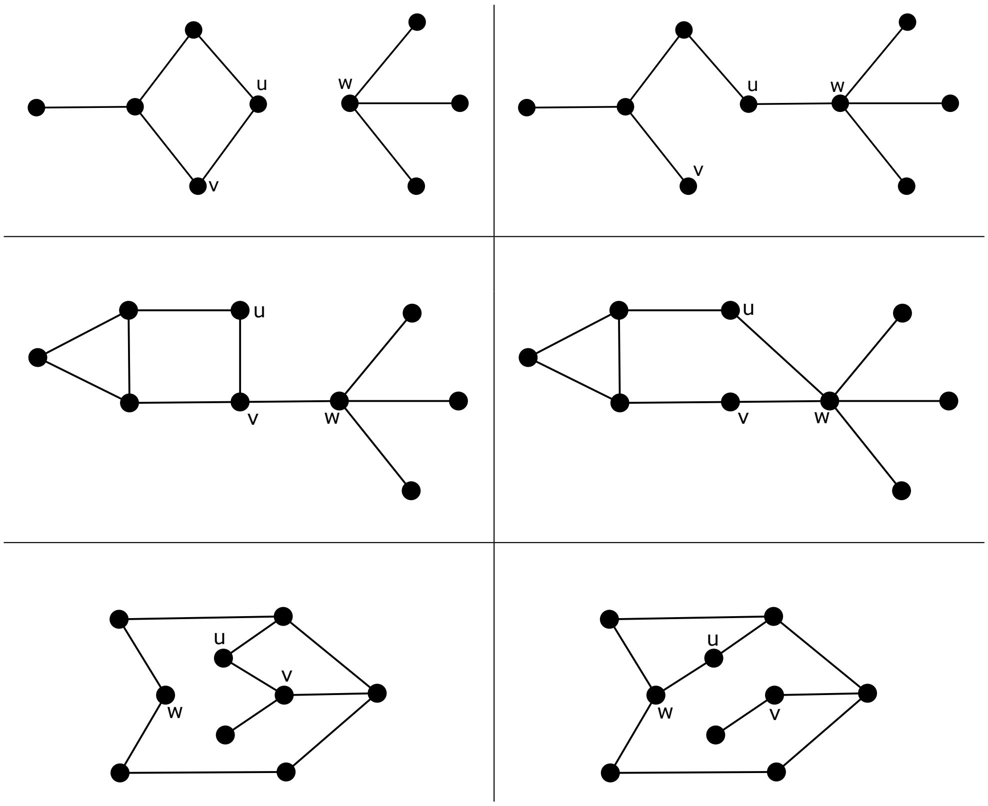

Figure 4: Three examples of graphs with $\deg(u) = 2$ where any graph on the right is obtained by starting from the graph on its left and rotating the edge uv to uw. In each case, the resulting graph is not isomorphic but keeps the same spectrum as the original.

To illustrate results from Section 5, we consider the examples of Figures 4 and 5. The degrees of v and w have different parities, while the number of walks of length two from u to v is equal to those from u to w, except when $v \sim w$, in which case it's one less. These statements are true both for the original graph and for the one that has had its edge rotated (swapping v and w in the rotated edge case).

We illustrate Theorem 18 by considering the last example of Figure 5. The eigenvalues of $G - u$ are -2.414, -1.778, -1.618, -0.169, 0.414, 0.618, 0.804 and 4.143. For the graph on the left, $a_u^{vw}(x)$ is equal to $x^8 - 15x^6 - 16x^5 + 26x^4 + 18x^3 - 21x^2 + 2x + 1$. The numbers -1.778, -1.618, -0.169, 0.618, 0.804 and 4.143, which are six of the roots of $\phi(G - u, x)$, are also roots of $a_u^{vw}(x)$. For the remaining two eigenvalues of $G - u$, -2.414 and 0.414, we have the relations $a_u^{vw}(-2.414) = -102.912$ and $a_u^{vw}(0.414) = -1.088$. Both graphs have characteristic polynomial $\phi(G, x) = x^9 - 17x^7 - 16x^6 + 49x^5 + 38x^4 - 52x^3 - 14x^2 + 17x - 2$, and indeed $-2\,\phi(G, -2.414) = -102.912$ and $-2\,\phi(G, 0.414) = -1.088$.

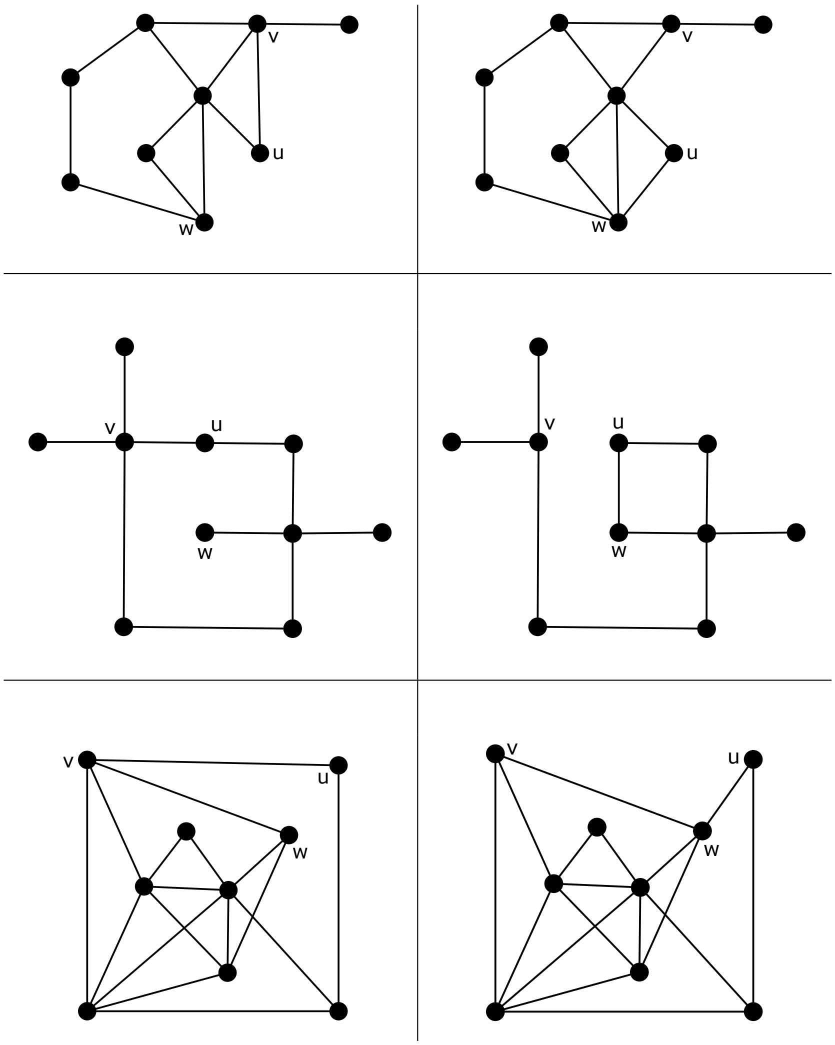

Figure 5: Another three examples of graphs with $\deg(u) = 2$ where any graph on the right is obtained by starting from the graph on its left and rotating the edge uv to uw. In each case, the resulting graph is not isomorphic but keeps the same spectrum as the original.

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