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# APPLICATION OF NEW FIXED POINT THEOREMS ON PROXIMITY SPACES TO FRACTIONAL BOUNDARY VALUE PROBLEM

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We present two new fixed point theorems on proximity spaces using $p$-distance in this paper and compare them with the earlier results. Then, the existence and uniqueness of a fractional boundary value problem (FBVP) with Riemann-Liouville fractional derivatives have been established using one of these fixed point theorems.

## 1. INTRODUCTION AND PRELIMINARIES

The study of fractional calculus, originated by Leibniz in the end of 17th century has emerged as one of the primary areas of mathematics and it has deep applications in diverse and widespread fields of science such as ecological systems, fluid mechanics, electrochemistry, optics, image processing, biological population models and signal processing etc. [**21**]. Therefore, finding the solutions of boundary value problems for nonlinear fractional differential equations has become vital for many mathematicians over the years (see, [**1, 2, 6, 22**]). In 2015, Hollon et al. [**7**] considered the following fractional boundary value problem:

$$(1)\qquad \begin{cases} D^{\alpha}_{0+}\xi(t) + a(t)f(\xi(t)) = 0,\ t \in (0,1) \\ \xi(0) = D^{\beta}_{0+}\xi(1) = 0, \end{cases}$$

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where $\alpha \in (1,2]$, $\beta \in [0,1]$, $D_{0+}^{\gamma}$ is Riemann-Liouville derivative of order $\gamma$, $a : [0,1] \longrightarrow [0,+\infty)$ is an integrable function on $[0,1]$ and and $f : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous. By using fixed point theorem of Krasnosel'skii [**10**], they provided only the existence of fractional boundary value problem (FBVP) (1) if certain conditions on the $f$ and $a$ are met. This motivates us to study the existence and also uniqueness of the solution of FBVP (1) under some certain conditions on the functions $a$ and $f$ by considering the one of our new fixed point results on proximity spaces.

In 1908, Riesz [**17**] introduced the notions of proximity spaces at the mathematical congress in Rome. Further, in 1934 Efremovich [**4**] revived this theory under the name of infinitesimal space. He introduced proximity relation as $M$ is near to $N$ for any subsets $M$ and $N$ of $X$, and using this nearness of sets, he defined closure of a set which eventually laid a foundation to define a topology (completely regular) in proximity space. He also proved that every completely regular topological spaces can be converted into a proximity spaces using Urysohn's function. In 1964, Smirnov [**19, 20**] further improved proximity spaces and showed that a topology which admits a proximity relation is compatible with the given topology. He was also the first to determine the relationship between uniformities and proximities. As a result, the uniform properties of metric spaces and the continuity properties of topology are generalized in proximity spaces. During the past few years, different mathematical aspects of proximity spaces has been studied by several mathematicians [**11, 12, 14, 18**].

Let $X$ be a nonempty set and $\delta$ be a relation on the power set $P(X)$. Then, the pair $(X, \delta)$ is called a proximity space if the followings hold: for all $M, N, L \in P(X)$

(P1) $M\delta N$ implies $N\delta M$;

(P2) $M\delta N$ implies $M, N \neq \emptyset$;

(P3) $M\delta(N \cup L)$ iff $M\delta N$ or $M\delta L$;

(P4) $M \cap N \neq \emptyset$ implies $M\delta N$;

(P5) For all $D \subseteq X$, $M\delta D$ or $N\delta(X - D)$ implies $M\delta N$.

For all $\xi \in X$ and $M \subseteq X$, we use the notation $\xi\delta M$ and $M\delta\xi$ instead of $\{\xi\}\delta M$ and $M\delta\{\xi\}$, respectively. A proximity space $(X, \delta)$ is said to be separated if $\xi\delta\zeta$ implies $\xi = \zeta$ for all $\xi, \zeta \in X$. Every proximity relation $\delta$ on $X$ induces a topology $\tau_\delta$ via Kuratowski closure operator which can be defined by $cl(M) = \{\xi : \xi\delta M\}$ for all $\xi \in X$ and $M \subseteq X$, and this induced topology $\tau_\delta$ is always completely regular. If $\delta$ is a proximity on $X$ and $(X, \tau)$ is a topological space such that $\tau_\delta = \tau$, then $\tau$ and $\delta$ are compatible. It is well known that every completely regular topology on $X$ has a compatible proximity. Moreover, if $\xi \in X$ and a sequence $\{\xi_n\}$ converges to $\xi$ with respect to $\tau_\delta$, then we have $\xi\delta\{\xi_n\}$.

**Example 1.** *If $(X, \rho)$ is a metric space, then the relation $\delta$ defined by*

$$M\delta N \Leftrightarrow \rho(M,N) = \inf\{\rho(a,b) : a \in M, b \in N\} = 0$$

*is a proximity on $X$. Moreover, $\tau_\rho$ and $\delta$ are compatible, where $\tau_\rho$ is the induced topology by $\rho$.*

**Example 2.** *If $(X, \tau)$ is a $T_4$ topological space, then the relation $\delta$ defined by*

$$M \delta N \Leftrightarrow \overline{M} \cap \overline{N} \neq \emptyset$$

*is a proximity on $X$, where $\overline{M}$ is the closure of $M$ with respect to $\tau$. In addition, $\tau$ and $\delta$ are compatible.*

**Example 3.** *If $(X, \tau)$ is a completely regular topological space, then the relation $\delta$ defined by $M \delta N$ iff there exists a continuous mapping $f : X \to [0, 1]$ such that $f(M) = 0$, $f(N) = 1$, is a proximity on $X$. Furthermore, $\tau$ and $\delta$ are compatible.*

In 2021, Kostić [**9**] introduced the concept of $p$-distance inspired by [**8**] in proximity spaces and by using this concept, he succeeded in extending the metric fixed point theory into proximity space. He initially, proved the Banach fixed point theorem on proximity space via $p$-distance, then Qasim et al. [**15**] obtained some nonlinear versions of it.

**Definition 4** ([**9**])**.** *Let $(X, \delta)$ be a proximity space and let $p : X \times X \longrightarrow [0, +\infty)$ be a mapping. If $p$ satisfies*

- *w1) $p(\xi, M) = 0$ and $p(\xi, N) = 0$ implies $M \delta N$, for any $\xi \in X$ and $M, N \in P(X)$, then $p$ is called a $p$-distance on $X$, where*

$$p(\xi, M) = \inf \{p(\xi, \zeta) : \zeta \in M\}.$$

*Further, a $p$-distance on $(X, \delta)$ is said to be $p_0$-distance if the followings hold:*

- *w2) $p(\xi, \eta) \leq p(\xi, \zeta) + p(\zeta, \eta)$ for all $\xi, \zeta, \eta \in X$,*
- *w3) $p$ is lower semicontinuous (lsc) in both variable with respect to $\tau_\delta$, i.e., for all $\xi, \zeta \in X$ we have*

$$p(\xi, \zeta) \leq \liminf_{\xi' \to \xi} p(\xi', \zeta) = \sup_{V \in \mathcal{U}_\xi} \inf_{\xi' \in V} p(\xi', \zeta)$$

*and*

$$p(\zeta, \xi) \leq \liminf_{\xi' \to \xi} p(\zeta, \xi') = \sup_{V \in \mathcal{U}_\xi} \inf_{\xi' \in V} p(\zeta, \xi'),$$

*where $\mathcal{U}_\xi$ is a neighborhoods base of $\xi \in X$.*

**Remark 5.** *Clearly, if $p$ is lsc in both variable w.r.t. $\tau_\delta$, then we have $p(\xi, \zeta) \leq \liminf_{n \to +\infty} p(\xi_n, \zeta)$ and $p(\zeta, \xi) \leq \liminf_{n \to +\infty} p(\zeta, \xi_n)$ for any sequence $\{\xi_n\}$ converging to $\xi$ w.r.t. $\tau_\delta$.*

**Example 6.** *Let* $X = C[0,2] = \{f \mid f : [0,2] \to \mathbb{R}$ *is continuous*$\}$, *endowed with the metric* $\rho$ *defined by*

$$\rho(f,g) = \int_0^2 |f(t) - g(t)|\,dt$$

*and the proximity* $\delta$ *induced by the metric* $\rho$ *as follows: for all* $M, N \subseteq X$,

$$M\delta N \Leftrightarrow \rho(M,N) = \inf\{\rho(a,b) : a \in M, b \in N\} = 0.$$

*Define the mapping* $p : X \times X \longrightarrow [0,+\infty)$ *by*

$$p(f,g) = \int_0^2 |g(t)|\,dt,$$

*then* $p$ *is* $p_0$*-distance on* $X$.

For further examples of $p_0$-distance on proximity space (see [**9, 15**]).

**Lemma 7** ([**9**])**.** *Let* $p$ *be a* $p_0$ *distance on a proximity space* $(X,\delta)$. *Then, the followings hold:*

*(i) If* $(X,\delta)$ *is separated, then* $p(\xi,\zeta) = 0$ *and* $p(\xi,\eta) = 0$ *implies* $\zeta = \eta$.

*(ii) If* $p(\zeta,\xi) = 0$ *and* $p(\zeta,\xi_n) \to 0$ *as* $n \to +\infty$, *then* $\{\xi_n\}$ *has a subsequence converging to* $\xi$ *with respect to* $\tau_\delta$.

**Theorem 8** ([**9**])**.** *Let* $p$ *be a* $p_0$ *distance on a separated proximity space* $(X,\delta)$. *Consider the mapping* $T : X \longrightarrow X$ *satisfying the followings:*

*(i) there exists* $k \in (0,1)$ *such that*

$$p(T\xi, T\zeta) \le kp(\xi,\zeta)$$

*for all* $\xi, \zeta \in X$;

*(ii) for all* $\xi \in X$, *any Picard sequence* $\{T^n\xi\}$ *has a convergent subsequence w.r.t* $\tau_\delta$.

*Then there exists unique* $\eta \in X$ *such that* $\eta = T\eta$ *and* $p(\eta,\eta) = 0$.

In 2022, Qasim et al. [**15**] proved the proximity space version of both Matkowski [**13**] and Boyd-Wong [**3**] fixed point theorems which are stated as follows:

**Theorem 9** ([**15**])**.** *Let* $p$ *be a* $p_0$ *distance on a separated proximity space* $(X,\delta)$ *and* $T : X \longrightarrow X$ *be a mapping such that* *(i) for all* $\xi, \zeta \in X$,

$$p(T\xi, T\zeta) \leq \phi(p(\xi, \zeta)),$$

*where* $\phi : [0, +\infty) \to [0, +\infty)$ *is a non-decreasing function such that* $\lim_{n \to +\infty} \phi^n(t) = 0$ *for all* $t \geq 0$,

*(ii) for all* $\xi \in X$, *any Picard sequence* $\{T^n \xi\}$ *has a convergent subsequence w.r.t* $\tau_\delta$.

*Then there exists unique* $\eta \in X$ *such that* $\eta = T\eta$ *and* $p(\eta, \eta) = 0$.

**Theorem 10** ([**15**])**.** *Let* $p$ *be a* $p_0$ *distance on a separated proximity space* $(X, \delta)$ *and* $T : X \longrightarrow X$ *be a mapping such that*

*(i) for all* $\xi, \zeta \in X$

$$p(T\xi, T\zeta) \leq \phi(p(\xi, \zeta)),$$

*where* $\phi : [0, +\infty) \to [0, +\infty)$ *is upper semicontinuous from the right such that* $\phi(0) = 0$, $\phi(t) < t$ *for all* $t > 0$ *and right continuous at* $0$,

*(ii) for all* $\xi \in X$, *any Picard sequence* $\{T^n \xi\}$ *has a convergent subsequence w.r.t* $\tau_\delta$.

*Then there exists unique* $\eta \in X$ *such that* $\eta = T\eta$ *and* $p(\eta, \eta) = 0$.

The paper is arranged as under: We obtain the proximity space equivalents of the well-known Rakotch [**16**] and Geraghty [**5**] fixed-point theorems in section 2 and compare them with the earlier results. In section 3, we present both existence and uniqueness of solutions for the FBVP stated in [**7**], taking into account Geraghty fixed-point theorem for proximity space.

## 2. FIXED POINT RESULTS

Here, we present our main theorems.

**Theorem 11.** *Let* $p$ *be a* $p_0$ *distance on a separated proximity space* $(X, \delta)$ *and* $T : X \longrightarrow X$ *be a mapping sucg that*

*(i) there exists a function* $L : [0, +\infty) \longrightarrow [0, 1)$ *satisfying*

$$p(T\xi, T\zeta) \leq L(p(\xi, \zeta))p(\xi, \zeta) \tag{2}$$

*for all* $\xi, \zeta \in X$ *with*

$$\sup\{L(r) : 0 < p \leq r \leq q\} < 1;$$

*(ii) for all* $\xi \in X$, *any Picard sequence* $\{T^n \xi\}$ *has a convergent subsequence w.r.t* $\tau_\delta$.

*Then, there exists unique $\eta \in X$ such that $\eta = T\eta$ and $p(\eta,\eta) = 0$.*

*Proof.* Let $\xi_0 \in X$ be any arbitrary point. Consider the corresponding Picard sequence $\{\xi_n\}$ constructed by

$$\xi_n = T^n \xi_0 = T\xi_{n-1}.$$

For simplicity, let $p_n = p(\xi_n, \xi_{n+1})$ for all $n \in \mathbb{N}$. In this case we have

$$\begin{aligned} p_n &= p(\xi_n, \xi_{n+1}) \\ &= p(T\xi_{n-1}, T\xi_n) \\ &\leq L(p(\xi_{n-1}, \xi_n))p(\xi_{n-1}, \xi_n) \\ &\leq p(\xi_{n-1}, \xi_n) = p_{n-1}. \end{aligned}$$

Thus, $\{p_n\}$ is a monotone non-increasing sequence, and there exists $\sigma \geq 0$ such that

$$\lim_{n\to+\infty} p_n = \sigma. \tag{3}$$

Note that for all $n \in \mathbb{N}$,

$$\sigma \leq p_n \leq p_{n-1} \leq \cdots \leq p_0.$$

If $\sigma > 0$, and take $\sup\{L(r) : 0 < \sigma \leq r \leq p_0\} = \lambda$, then we have $L(p_n) \leq \lambda$ for all $n \in \mathbb{N}$, and consequently,

$$\begin{aligned} 0 &< \sigma \leq p_n \\ &\leq L(p_{n-1})p_{n-1} \\ &\leq \lambda p_{n-1} \\ &\vdots \\ &\leq \lambda^n p_0 \to 0 \end{aligned}$$

as $n \to +\infty$, a contradiction. Therefore, $\sigma = 0$. Now, let $\epsilon > 0$ and take $\sup\{L(r) : \frac{\epsilon}{2} \leq r \leq \epsilon\} = \sigma(\epsilon) < 1$. Since $\lim\limits_{n\to+\infty} p_n = 0$ and $1 - \sigma(\epsilon) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $p_n < \frac{1-\sigma(\epsilon)}{2}\epsilon$ for all $n \geq n_0$.

We claim that

$$p(\xi_n, \xi_m) < \epsilon \tag{4}$$

for all $m, n \in \mathbb{N}$ with $m > n > n_0$. We will use the principle of Mathematical Induction to proof our claim. First,

$$p(\xi_n, \xi_{n+1}) = p_n < \frac{1-\sigma(\epsilon)}{2}\epsilon < \epsilon,$$

hence (4) is true for $n+1$. Assume that inequality (4) holds for $m$. Now, if $p(\xi_n,\xi_m) < \frac{\epsilon}{2}$, then by (w2),

$$\begin{aligned} p(\xi_n,\xi_{m+1}) &\leq p(\xi_n,\xi_m) + p(\xi_m,\xi_{m+1}) \\ &< \frac{\epsilon}{2} + \frac{1-\sigma(\epsilon)}{2}\epsilon \\ &< \epsilon. \end{aligned}$$

Now, if $\frac{\epsilon}{2} \leq p(\xi_n,\xi_m) < \epsilon$, then

$$\begin{aligned} p(\xi_{n+1},\xi_{m+1}) &\leq L(p(\xi_n,\xi_m))p(\xi_n,\xi_m) \\ &\leq \sigma(\epsilon)p(\xi_n,\xi_m) \\ &\leq \sigma(\epsilon)\epsilon. \end{aligned}$$

By (w2),

$$\begin{aligned} p(\xi_n,\xi_{m+1}) &\leq p(\xi_n,\xi_{n+1}) + p(\xi_{n+1},\xi_{m+1}) \\ &< \frac{1-\sigma(\epsilon)}{2}\epsilon + \sigma(\epsilon)\epsilon \\ &< \epsilon. \end{aligned}$$

Thus, (4) is true for $m+1$. Similarly, we can obtain

$$\lim_{n\to+\infty} p(\xi_{n+1},\xi_n) = 0.$$

Also, we can obtain that for every $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$p(\xi_m,\xi_n) < \epsilon \tag{5}$$

for $m > n \geq n_1$.

By (ii), there is a subsequence $\{\xi_{n_k}\}$ of sequence $\{\xi_n\}$, which is convergent w.r.t. $\tau_\delta$ to some $\eta \in X$, and by (4), we have

$$p(\xi_{n_k},\eta) \leq \lim_{l\to+\infty}\inf p(\xi_{n_k},\xi_{n_l}) \leq \epsilon \tag{6}$$

for all $n_k \geq n_0$. Then, we have

$$p(\eta,\eta) \leq \lim_{k\to+\infty}\inf p(\xi_{n_k},\xi) \leq \epsilon$$

and consequently, $p(\eta,\eta) = 0$. Similarly by (5), we have

$$p(\eta,\xi_{n_k}) \leq \lim_{l\to+\infty}\inf p(\xi_{n_l},\xi_{n_k}) \leq \epsilon \tag{7}$$

for all $n_k \geq n_1$.

Finally, by (6) and (7), for all $n_i > \max\{n_0, n_1\}$, we have

$$\begin{aligned} p(\eta, T\eta) &\leq p(\eta, \xi_{n_i}) + p(\xi_{n_i}, T\eta) \\ &= p(\eta, \xi_{n_i}) + p(T\xi_{n_i-1}, T\eta) \\ &\leq \epsilon + L(p(\xi_{n_i-1}, \eta))p(\xi_{n_i-1}, \eta) \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Hence, by Lemma 7 (i), we have $\eta = T\eta$. For the uniqueness, let $\zeta \in X$ be a different fixed point of $T$. Then, we have $p(\eta, \zeta) > 0$ since $\eta \neq \zeta$ and $p(\eta, \eta) = 0$.

By (i), we have

$$0 < p(\eta, \zeta) = p(T\eta, T\zeta) \leq L(p(\eta, \zeta))p(\eta, \zeta) < p(\eta, \zeta),$$

a contradiction. Therefore, the fixed point of $T$ is unique. □

**Remark 12.** *In above theorem, if we take $\phi(t) = L(t)t$, where $L : [0, +\infty) \longrightarrow [0, 1)$ is a non-increasing function, then for any $t > 0$, $\phi(t) < t$ and*

$$\lim_{t \to t_0^+} \sup \phi(t) = \lim_{t \to t_0^+} \sup L(t)t = \lim_{t \to t_0^+} \sup L(t_0)t = L(t_0)t_0 = \phi(t_0).$$

*Thus, $\phi$ is upper semicontinuous from right, and also,*

$$p(T\xi, T\zeta) \leq \phi(p(\xi, \zeta)).$$

*Therefore, above theorem reduces to Theorem 10.*

Let $\Omega : [0, +\infty) \to [0, 1)$ be a function such that for any $t_n > 0$, $t_n \to 0$ whenever $\Omega(t_n) \to 1$ as $n \to +\infty$. We represent the set of all functions $\Omega$ by $\mathcal{G}$. Some of the examples of the functions belonging to $\mathcal{G}$ are $\Omega_1(t) = \frac{1}{t+1}$,

$$\Omega_2(t) = \begin{cases} \frac{\ln(1+t)}{t} & , \quad t > 0 \\ 0 & , \quad t = 0 \end{cases}$$

and $\Omega_3(t) = K$, where $0 \leq K < 1$.

In 1973, Geraghty [**5**] extended the famous Banach fixed point theorem for complete metric space and obtained some new interesting results. Considering the importance of Geraghty fixed point theorem for metric space, we come up with the proximity space version of it.

**Theorem 13.** *Let $p$ be a $p_0$ distance on a separated proximity space $(X, \delta)$ and $T : X \to X$ be a mapping such that*

*(i) there exists $\Omega \in \mathcal{G}$, satisfying*

$$p(T\xi, T\zeta) \leq \Omega(p(\xi, \zeta))p(\xi, \zeta)$$

*for all $\xi, \zeta \in X$;*

*(ii) for all $\xi \in X$, any Picard sequence $\{T^n\xi\}$ has a convergent subsequence w.r.t. $\tau_\delta$.*

*Then, there exists unique $\eta \in X$ such that $\eta = T\eta$ and $p(\eta, \eta) = 0$.*

*Proof.* Let $\xi_0 \in X$ be any arbitrary point. Consider the corresponding Picard sequence $\{\xi_n\}$ constructed by

$$\xi_n = T^n\xi_0 = T\xi_{n-1}.$$

For simplicity, suppose $p_n = p(\xi_n, \xi_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $p_{n_0} = p(\xi_{n_0}, \xi_{n_0+1}) = 0$, then we have

$$\begin{aligned} p(\xi_{n_0}, \xi_{n_0+2}) &\leq p(\xi_{n_0}, \xi_{n_0+1}) + p(\xi_{n_0+1}, \xi_{n_0+2}) \\ &= p(\xi_{n_0+1}, \xi_{n_0+2}) \\ &\leq \Omega(p(\xi_{n_0}, \xi_{n_0+1}))p(\xi_{n_0}, \xi_{n_0+1}) \\ &= 0. \end{aligned}$$

Therefore, by Lemma 7 (i), $\xi_{n_0+1}$ is a fixed point of $T$. Now, assume $p_n > 0$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} p_n &= p(T\xi_{n-1}, T\xi_n) \\ &\leq \Omega(p_{n-1})p_{n-1} \\ &< p_{n-1}. \end{aligned}$$

Thus, $\{p_n\}$ is a monotone non-increasing sequence and it is bounded below, and there exists $\sigma \geq 0$ such that

$$\lim_{n\to+\infty} p_n = \sigma. \tag{8}$$

Suppose $\sigma > 0$. It follows that

$$\frac{p_n}{p_{n-1}} \leq \Omega(p_{n-1}).$$

Taking limit $n \to +\infty$, we get $\lim_{n\to+\infty} \Omega(p_{n-1}) = 1$. Since $\Omega \in \mathcal{G}$, it follows that $\lim_{n\to+\infty} p_{n-1} = 0$, a contradiction. Thus, $\sigma = 0$. Hence, we have

$$\lim_{n\to+\infty} p_n = \lim_{n\to+\infty} p(\xi_n, \xi_{n+1}) = 0.$$

Now, if we call $\overline{p}_n = p(\xi_{n+1}, \xi_n)$, then in a same way, we can obtain

$$\lim_{n\to+\infty} \overline{p}_n = \lim_{n\to+\infty} p(\xi_{n+1}, \xi_n) = 0.$$

Now, we claim that

$$\lim_{m,n\to+\infty} \sup p(\xi_n, \xi_m) = 0.$$

Assume the contrary, $\lim_{m,n\to+\infty}\sup p(\xi_n,\xi_m) > 0$. By (w2), we have, for all $m,n\in\mathbb{N}$,

$$\begin{aligned} p(\xi_n,\xi_m) &\leq p(\xi_n,\xi_{n+1}) + p(\xi_{n+1},\xi_{m+1}) + p(\xi_{m+1},\xi_m) \\ &\leq p_n + \Omega(p(\xi_n,\xi_m))p(\xi_n,\xi_m) + \overline{p}_m. \end{aligned}$$

Thus,

$$[1-\Omega(p(\xi_n,\xi_m))]p(\xi_n,\xi_m) \leq [p_n + \overline{p}_m] \to 0$$

as $m,n\to+\infty$ and it follows that

$$\lim_{m,n\to+\infty}\sup \Omega(p(\xi_n,\xi_m)) = 1.$$

Since $\Omega\in\mathcal{G}$, we have

$$\lim_{m,n\to+\infty}\sup p(\xi_n,\xi_m) = 0,$$

a contradiction. Thus, $\lim_{m,n\to+\infty}\sup p(\xi_n,\xi_m) = 0$. Therefore, for every $\epsilon$, there exists $N\in\mathbb{N}$ such that

$$p(\xi_n,\xi_m) < \epsilon \tag{9}$$

for all $m,n > N$.

By assumption (ii), there is a subsequence $\{\xi_{n_k}\}$ of sequence $\{\xi_n\}$, which is convergent w.r.t. $\tau_\delta$ to some $\eta\in X$, and by inequality (4), we have

$$p(\eta,\xi_{n_k}) \leq \lim_{l\to+\infty}\inf p(\xi_{n_l},\xi_{n_k}) \leq \epsilon \tag{10}$$

and

$$p(\xi_{n_k},\eta) \leq \epsilon \tag{11}$$

for all $n_k > N$. Then, we have

$$p(\eta,\eta) \leq \lim_{k\to+\infty}\inf p(\xi_{n_k},\eta) \leq \epsilon$$

and consequently, $p(\eta,\eta) = 0$.

On other hand, by inequalities (10) and (11), we have, for $n_k > N$

$$\begin{aligned} p(\eta,T\eta) &\leq p(\eta,\xi_{n_k+1}) + p(\xi_{n_k+1},T\eta) \\ &= p(\eta,\xi_{n_k+1}) + p(T\xi_{n_k},T\eta) \\ &< p(\eta,\xi_{n_k+1}) + \Omega(p(\xi_{n_k},\eta))p(\xi_{n_k},\eta) \\ &< p(\eta,\xi_{n_k+1}) + p(\xi_{n_k},\eta) \\ &< \epsilon + \epsilon \\ &= 2\epsilon. \end{aligned}$$

Therefore, by Lemma 7 (i), we have $\eta = T\eta$. For the uniqueness, let $\zeta \in X$ be a different fixed point of $T$. Then, we get $p(\eta, \zeta) > 0$ since $\eta \neq \zeta$ and $p(\eta, \eta) = 0$.

By (ii), we have

$$0 < p(\eta, \zeta) = p(T\eta, T\zeta) \leq \Omega(p(\eta, \zeta))p(\eta, \zeta) < p(\eta, \zeta),$$

a contradiction. Thus, the fixed point of $T$ is unique. □

Now, we provide both illustrative and comperative example.

**Example 14.** *Let $X = [0, +\infty)$, equipped with the lower limit topology $\tau_l$ and the proximity $\delta$ induced by $\tau_l$ is given as follows: for all $M, N \subseteq X$,*

$$M\delta N \Leftrightarrow \overline{M} \cap \overline{N} \neq \emptyset.$$

*Clearly, $(X, \delta)$ is a separated proximity space and also $\tau_l = \tau_\delta$. Define the mapping $p : X \times X \longrightarrow [0, +\infty)$ by $p(\xi, \zeta) = \zeta$ for all $\xi, \zeta \in X$, then it is a $p_0$ distance on $X$. Consider the self mapping $T : X \longrightarrow X$ by $T\xi = \frac{\xi}{1+\xi}$ and the function $\Omega \in \mathcal{G}$ by*

$$\Omega(t) = \begin{cases} \frac{\ln(1+t)}{t} & , \quad t > 0 \\ 0 & , \quad t = 0 \end{cases}.$$

*Then, for all $\xi \in X$, we have*

$$p(T\xi, T0) = T0 = 0 \leq \Omega(p(\xi, 0))p(\xi, 0),$$

*and for all $\xi \in X$ and $\zeta > 0$, we get*

$$\begin{aligned} p(T\xi, T\zeta) &= T\zeta = \frac{\zeta}{1+\zeta} \\ &\leq \ln(1+\zeta) \\ &= \frac{\ln(1+\zeta)}{\zeta}\zeta \\ &= \Omega(p(\xi, \zeta))p(\xi, \zeta). \end{aligned}$$

*Also for all $\xi \in X$, $T^n\xi = \frac{\xi}{1+n\xi}$ is convergent w.r.t. $\tau_\delta$. As a result, all conditions of Theorem 13 are satisfied. Therefore, $T$ has a unique fixed point. However, since*

$$\sup_{\zeta \in X} \frac{p(T\xi, T\zeta)}{p(\xi, \zeta)} = 1,$$

*then there isn't any $k \in (0, 1)$ satisfying*

$$p(T\xi, T\zeta) \leq kp(\xi, \zeta),$$

*for all $\xi, \zeta \in X$. Hence, we can't apply Theorem 8 to this example.*

## 3. APPLICATION

In this part, we propose a novel application in which we demonstrate the existence and uniqueness of the solution to a FBVP using the Theorem 13: Here, for an integrable function $a : [0,1] \to \mathbb{R}$ and for a continuous function $f : \mathbb{R} \to \mathbb{R}$, we consider the FBVP given as

$$\begin{cases} D^{\alpha}_{0+}\xi(t) + a(t)f(\xi(t)) = 0, \ t \in (0,1) \\ \xi(0) = D^{\beta}_{0+}\xi(1) = 0, \end{cases} \tag{12}$$

where $\alpha \in (1,2]$, $\beta \in [0,1]$ and $D^{\gamma}_{0+}$ is Riemann-Liouville derivative of order $\gamma$. It is well known that the operator $D^{\gamma}_{0+}$ is defined as, for positive integer $n$ and $\gamma \in (n-1, n]$,

$$D^{\gamma}_{0+}\xi(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\gamma-1}\xi(s)ds$$

for a function $\xi : [0,1] \to \mathbb{R}$, provided the right hand side exists. It is demonstrated in [**7**] that (12) is equivalent to the following integral equation:

$$\xi(t) = \int_0^1 G(t,s)a(s)f(\xi(s))ds, \ 0 \le t \le 1, \tag{13}$$

where $G(t,s)$ is associated Green's function defined by

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} & , \quad 0 \le s \le t \le 1 \\ \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} & , \quad 0 \le t \le s \le 1 \end{cases}.$$

Define an operator $T : C[0,1] \to C[0,1]$ by

$$T\xi(t) = \int_0^1 G(t,s)a(s)f(\xi(s))ds.$$

Hence, $\eta$ is a solution of (13) whenever it is a fixed point of $T$, and so equivalently it is a solution of the FBVP (12).

Let $(X, \delta)$ be the proximity space, where $X = C[0,1]$ and $\delta$ is induced by the supremum metric $\rho_\infty$

$$\rho_\infty(\xi, \zeta) = \sup\{|\xi(t) - \zeta(t)| : t \in [0,1]\}.$$

In this case $(X, \delta)$ is separated. Consider the following $p_0$-distances $p_\gamma$ on $X$ defined by

$$p_\gamma(\xi, \zeta) = \sup\{e^{-\gamma t} |\xi(t) - \zeta(t)| : t \in [0,1]\},$$

where $\gamma > 0$ is a constant.

Now consider the following assumptions:

(A1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$|f(t) - f(s)| \leq h(t,s)\,|t-s|$$

for all $t, s \in \mathbb{R}$,

(A2) There exists $\Omega \in \mathcal{G}$ such that

$$\sup_{t\in[0,1]} h(\xi(t), \zeta(t)) \leq \Omega(p_\gamma(\xi, \zeta))$$

for all $\xi, \zeta \in C[0,1]$,

(A3) $a : [0,1] \to \mathbb{R}$ is continuous.

**Theorem 15.** *In addition to (A1)-(A3), suppose that* $M = \|a\|_\infty$ *and*

$$EMe^\gamma \leq 1,$$

*where*

$$E = \sup_{t\in[0,1]} e^{-\gamma t} \frac{\alpha t^{\alpha-1} - (\alpha-\beta)t^\alpha}{\alpha(\alpha-\beta)\Gamma(\alpha)},$$

*then the FBVP given in (12) has a unique solution.*

*Proof.* First of all, we know by Lemma 3.1 of [**7**] that, $G(t,s) \geq 0$ for all $t, s \in [0,1]$ and

$$\sup_{t\in[0,1]} e^{-\gamma t} \int_0^1 G(t,s)ds = \sup_{t\in[0,1]} e^{-\gamma t} \frac{\alpha t^{\alpha-1} - (\alpha-\beta)t^\alpha}{\alpha(\alpha-\beta)\Gamma(\alpha)} = E.$$

Consider the operator $T : C[0,1] \to C[0,1]$ defined by

$$T\xi(t) = \int_0^1 G(t,s)a(s)f(\xi(s))ds$$

Then for any $\xi, \zeta \in C[0,1]$ and $t \in [0,1]$ we have

$$\begin{aligned}
|T\xi(t) - T\zeta(t)| &= \left| \int_0^1 G(t,s)a(s)f(\xi(s))ds - \int_0^1 G(t,s)a(s)f(\zeta(s))ds \right| \\
&\leq \int_0^1 G(t,s)\,|a(s)|\,|f(\xi(s)) - f(\zeta(s))|\,ds \\
&\leq \int_0^1 G(t,s)\,|a(s)|\,h(\xi(s),\zeta(s))\,|\xi(s)-\zeta(s)|\,ds \\
&= \int_0^1 G(t,s)\,|a(s)|\,h(\xi(s),\zeta(s))e^{\gamma s}e^{-\gamma s}\,|\xi(s)-\zeta(s)|\,ds \\
&\leq Mp_\gamma(\xi,\zeta) \int_0^1 G(t,s)h(\xi(s),\zeta(s))e^{\gamma s}ds \\
&\leq Me^\gamma \beta(p_\gamma(\xi,\zeta))p_\gamma(\xi,\zeta) \int_0^1 G(t,s)ds
\end{aligned}$$

and then we get

$$e^{-\gamma t}\left|T\xi(t)-T\zeta(t)\right| \leq Me^{\gamma}\Omega(p_{\gamma}(\xi,\zeta))p_{\gamma}(\xi,\zeta)e^{-\gamma t}\int_{0}^{1}G(t,s)ds.$$

By taking supremum over $t \in [0,1]$ we have

$$p_{\gamma}(T\xi,T\zeta) \leq EMe^{\gamma}\Omega(p_{\gamma}(\xi,\zeta))p_{\gamma}(\xi,\zeta) \leq \Omega(p_{\gamma}(\xi,\zeta))p_{\gamma}(\xi,\zeta). \tag{14}$$

Therefore the condition (i) of Theorem 13 is satisfied. Now let $\xi \in C[0,1]$ be an arbitrary function. Define a sequence of functions $\{\xi_n\}$ as $\xi_n = T^n\xi$. By the proof of Theorem 13 and by (14), we have

$$p_{\gamma}(\xi_n,\xi_m) \to 0$$

as $m,n \to +\infty$. On the other hand, since

$$e^{-\gamma}\rho_{\infty}(\xi,\zeta) \leq p_{\gamma}(\xi,\zeta) \leq \rho_{\infty}(\xi,\zeta)$$

for all $\xi,\zeta \in C\,[0,1]$, we have

$$\rho_{\infty}(\xi_n,\xi_m) \to 0$$

as $m,n \to +\infty$. That is, $\{\xi_n\}$ is a Cauchy sequence and $(X,\rho_{\infty})$ is complete, so it has a convergent subsequence w.r.t. $\rho_{\infty}$. Hence, the assumption (ii) of Theorem 13 is satisfied. As a result, there exists unique fixed point $\eta \in C\,[0,1]$ of $T$ and $p_{\gamma}(\eta,\eta) = 0$. Hence, the FBVP (13) has a unique solution. □

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