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SEVERAL NEW HUYGENS TYPE INEQUALITIES

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In the paper, the authors establish several inequalities for bounding the sums of $\operatorname{sinc}(2x)$, $\operatorname{tanc} x$, and their reciprocals. This is achieved by using the Maclaurin power series expansions of the functions $\sin x$ and $\cos x$, as well as the stratification method developed by Malešević and his coauthors. These results provide a significant and nontrivial extension of the classical Huygens-type inequalities.

1. INTRODUCTION

For $z \in \mathbb{R}$, the functions

$$\operatorname{sinc} z = \begin{cases} \dfrac{\sin z}{z}, & z \neq 0; \\ 1, & z = 0, \end{cases} \qquad \operatorname{sinhc} z = \begin{cases} \dfrac{\sinh z}{z}, & z \neq 0; \\ 1, & z = 0, \end{cases}$$

$$\operatorname{tanc} z = \begin{cases} \dfrac{\tan z}{z}, & z \neq 0; \\ 1, & z = 0, \end{cases} \qquad \operatorname{tanhc} z = \begin{cases} \dfrac{\tanh z}{z}, & z \neq 0; \\ 1, & z = 0 \end{cases}$$

are called the sinc function, the hyperbolic sinc function, the tanc function, and the hyperbolic tanc function, respectively. The function $\operatorname{sinc} z$ is also called the sine cardinal or sampling function, as well as the function $\operatorname{sinhc} z$ is also called the hyperbolic sine cardinal; see the papers [**4, 7, 11, 23, 24**].

In [**26**], Wilker proposed the following two open problems:

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1. Let $0 < x < \frac{\pi}{2}$. Then

$$\operatorname{sinc}^2 x + \operatorname{tanc} x > 2; \tag{1}$$

2. There exists a largest constant c such that

$$\operatorname{sinc}^2 x + \operatorname{tanc} x > 2 + cx^3 \tan x \tag{2}$$

for $0 < x < \frac{\pi}{2}$.

In [**25**], Sumner et al. affirmed these two problems and set up a double inequality

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \operatorname{sinc}^2 x + \operatorname{tanc} x < 2 + \frac{8}{45} x^3 \tan x \tag{3}$$

for $0 < x < \frac{\pi}{2}$, where $\left(\frac{2}{\pi}\right)^4$ and $\frac{8}{45}$ in (3) are the best constants.

In [**5**], Guo and her coauthors gave new proofs of the inequalities (1) and (2). In [**22**], using L'Hospital rules for monotonicity, Pinelis showed another proof of the inequalities (3). In [**29**], Zhang and Zhu presented a new elementary proof of the double inequality (3).

In [**6**], the Huygens inequality reads that

$$2 \operatorname{sinc} x + \operatorname{tanc} x > 3, \quad 0 < |x| < \frac{\pi}{2}. \tag{4}$$

In [**27**, Lemma 3], Wu and Srivastava established the inequality

$$\left(\frac{1}{\operatorname{sinc} x}\right)^2 + \frac{1}{\operatorname{tanc} x} > 2, \quad 0 < |x| < \frac{\pi}{2}.$$

In [**2**, Theorem 3], Chen and Cheung established two sharp Huygens type inequalities

$$3 + \frac{3}{20} x^3 \tan x < 2 \operatorname{sinc} x + \operatorname{tanc} x < 3 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x \tag{5}$$

and

$$3 + \frac{3}{20} x^4 + \frac{3}{56} x^5 \tan x < 2 \operatorname{sinc} x + \operatorname{tanc} x < 3 + \frac{3}{20} x^4 + \left(\frac{2}{\pi}\right)^4 x^5 \tan x$$

for $0 < x < \frac{\pi}{2}$. In [**3**, Remark 3.4], Chen and Paris showed that

$$3 < \frac{2}{\operatorname{sinc} x} + \frac{1}{\operatorname{tanc} x} < 3 + \frac{1}{60} x^3 \tan x, \quad 0 < x < \frac{\pi}{2}. \tag{6}$$

In [**7**, Theorem 1], among other things, Li and Guo proved that

$$\frac{4}{15}\left(\cos x + \frac{11}{4}\right)^2 - \frac{3}{4} \le \operatorname{sinc}(2x) + 2 \operatorname{sinc} x \le \frac{4}{15}\left(\cos x + \frac{11}{4}\right)^2 - \frac{3}{4} + \frac{1}{1260} x^6 \tag{7}$$

for $|x| < \frac{\pi}{2}$.

In [**28**, Theorem 4], the authors derived a series of inequalities

$$\operatorname{tanc} x + (2n+2)\frac{\ln\cos x}{x^2} > \frac{2^{2n+3}\left(2^{2n+4}-1\right)|B_{2n+4}|}{(n+2)^2(2n+3)!}x^{2n+2} - \sum_{k=0}^{n-1}\frac{(n-k)2^{2k+2}\left(2^{2k+2}-1\right)|B_{2k+2}|}{(k+1)(2k+2)!}x^{2k} \tag{8}$$

for $n \in \mathbb{N}_0$ and $x \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$, where an empty sum is conventionally regarded as 0 and the Bernoulli numbers B_{2j} are generated by the Maclaurin power series expansion

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!} = 1 - \frac{x}{2} + \sum_{j=1}^{\infty} B_{2j}\frac{x^{2j}}{(2j)!}, \quad 0 < |x| < 2\pi.$$

In particular, taking $n = 0, 1$ in (8) gives

$$\operatorname{tanc} x + \frac{2\ln\cos x}{x^2} > \frac{x^2}{6} \quad \text{and} \quad \operatorname{tanc} x + \frac{4\ln\cos x}{x^2} > \frac{2x^4}{45} - 1$$

for $x \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$. For more information on this topic, please refer to the newly published articles [**8, 9, 10, 11, 23**] and closely related references therein.

In this paper, inspired by the inequalities in (4), (5), (6), and (7), basing on the Maclaurin power series expansions of the functions $\sin x$ and $\cos x$, and employing the stratification method developed by Malešević and his coauthors, we establish several new extensions for Huygens type inequalities involving the functions $\operatorname{sinc}(2x)$ and $\operatorname{tanc} x$.

2. INEQUALITIES

Now we are in a position to state and prove our main results.

Theorem 1. *The function*

$$H(x) = \operatorname{sinc}(2x) + 2\operatorname{tanc} x$$

is increasing and $H(x) > 3$ in $x \in (0, \frac{\pi}{2})$.

Proof. Let $h_1(x) = \sin x\cos^2 x + 2\sin x$ and $h_2(x) = x\cos x$. Then

$$H(x) = \frac{\sin x\cos x}{x} + \frac{2\sin x}{x\cos x} = \frac{h_1(x)}{h_2(x)}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

A direct computation yields

$$h_1'(x) = \cos^3 x + 2\cos x - 2\sin^2 x\cos x, \quad h_2'(x) = \cos x - x\sin x,$$

and

$$\frac{H'(x)}{H(x)}=\frac{h_1'(x)h_2(x)-h_1(x)h_2'(x)}{h_1(x)h_2(x)}=\frac{h_3(x)}{8h_1(x)h_2(x)}, \tag{9}$$

where

$$h_3(x)=18x-10\sin(2x)-\sin(4x)+4x\cos(2x)+2x\cos(4x).$$

Using the Maclaurin power series expansions of $\sin x$ and $\cos x$, we have

$$\begin{aligned}
h_3(x)&=18x-10\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}}{(2n+1)!}x^{2n+1}-\sum_{n=0}^{\infty}(-1)^n\frac{4^{2n+1}}{(2n+1)!}x^{2n+1}\\
&\quad+\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+2}}{(2n)!}x^{2n+1}+\sum_{n=0}^{\infty}(-1)^n\frac{2^{4n+1}}{(2n)!}x^{2n+1}\\
&=\sum_{n=2}^{\infty}(-1)^n\frac{2^{2n+2}}{(2n+1)!}\big[(2n-1)2^{2n-1}+2n-4\big]x^{2n+1}.
\end{aligned}$$

Let

$$w_n=\frac{2^{2n+2}}{(2n+1)!}\big[(2n-1)2^{2n-1}+2n-4\big],\quad n\ge 2.$$

It is easy to see that the sequence w_n is positive in $n\ge 2$ and

$$\begin{aligned}
h_3(x)&=\frac{64x^5}{5}-\frac{288x^7}{35}+\frac{160x^9}{63}-\frac{24608x^{11}}{51975}+\sum_{n=6}^{\infty}(-1)^n w_n x^{2n+1}\\
&=\frac{32x^5\big(20790-13365x^2+4125x^4-769x^6\big)}{51975}\\
&\quad+\sum_{k=3}^{\infty}w_{2k+1}\left(\frac{w_{2k}}{w_{2k+1}}-x^2\right)x^{4k+1}.
\end{aligned}$$

In what follows, we show the inequality

$$\frac{w_{2k}}{w_{2k+1}}>\frac{5}{2}>\left(\frac{\pi}{2}\right)^2=2.467\dots,\quad k\ge 3. \tag{10}$$

The inequality (10) is equivalent to

$$2w_{2k}-5w_{2k+1}>0,\quad k\ge 3$$

which can be reformulated as the inequality

$$2^{4k}(32k^3+32k^2-78k-23)+64k^3+16k^2-96k-4>0.$$

Since

$$32k^3+32k^2-78k-23=32(k-3)^3+320(k-3)^2+978(k-3)+895>0$$

and

$$64k^3+16k^2-96k-4=64(k-3)^3+592(k-3)^2+1728(k-3)+1580>0$$

for $k\ge 3$, the inequality (10) is thus valid.

On the other hand, it is easy to see that

$$\sum_{k=1}^{2} w_{2k+1}\left(\frac{w_{2k}}{w_{2k+1}}-x^2\right)x^{4k+1}=\frac{32x^5}{51975}W\big(x^2\big), \tag{11}$$

where, by the Cardano formula for the general cubic equation, the function

$$W(t)=-769t^3+4125t^2-13365t+20790$$

has only one real zero

$$\frac{1375+\sqrt[3]{\frac{55(61123583+29991\sqrt{9473649}\,)}{2}}-27914\sqrt[3]{\frac{6050}{61123583+29991\sqrt{9473649}}}}{769}$$
$$=2.653\cdots>2.467\cdots=\left(\frac{\pi}{2}\right)^2.$$

This means that the polynomial in (11) of degree 11 is positive for $x\in\big(0,\frac{\pi}{2}\big)$. As a result, by the inequality (10) and (11), the function

$$h_3(x)=\sum_{k=1}^{2} w_{2k+1}\left(\frac{w_{2k}}{w_{2k+1}}-x^2\right)x^{4k+1}+\sum_{k=3}^{\infty} w_{2k+1}\left(\frac{w_{2k}}{w_{2k+1}}-x^2\right)x^{4k+1}$$

is positive for $x\in\big(0,\frac{\pi}{2}\big)$. Consequently, from the relation (9), it follows that the first derivative $H'(x)$ is positive, and then the function $H(x)$ is increasing, on $\big(0,\frac{\pi}{2}\big)$. Moreover, it is straightforward that

$$H(x)>\lim_{x\to0^+}H(x)=3.$$

The proof of Theorem 1 is thus complete. □

Theorem 2. *The function*

$$G(x)=\frac{1}{\operatorname{sinc}(2x)}+\frac{2}{\operatorname{tanc}x}$$

is increasing and $G(x)>3$ *in* $x\in(0,\frac{\pi}{2})$.

Proof. Let $g_1(x)=x+2x\cos^2x$ and $g_2(x)=\sin x\cos x$. Then

$$G(x)=\frac{x}{\sin x\cos x}+\frac{2x\cos x}{\sin x}=\frac{g_1(x)}{g_2(x)},\quad x\in\left(0,\frac{\pi}{2}\right).$$

Direct computations yield

$$g_1'(x) = 2\cos^2 x - 4x\sin x\cos x + 1, \quad g_2'(x) = \cos^2 x - \sin^2 x,$$

and

$$\frac{G'(x)}{G(x)} = \frac{g_1'(x)g_2(x) - g_1(x)g_2'(x)}{g_1(x)g_2(x)} = \frac{g_3(x)}{4g_1(x)g_2(x)}, \tag{12}$$

where

$$g_3(x) = -4x + 4\sin(2x) + \sin(4x) - 8x\cos(2x).$$

Using the Maclaurin power series expansions of $\sin x$ and $\cos x$, we have

$$\begin{aligned} g_3(x) &= -4x + \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+3}}{(2n+1)!}x^{2n+1} + \sum_{n=0}^{\infty}(-1)^n\frac{4^{2n+1}}{(2n+1)!}x^{2n+1} \\ &\quad - \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+3}}{(2n)!}x^{2n+1} \\ &= \sum_{n=2}^{\infty}(-1)^n\frac{2^{2n+3}}{(2n+1)!}\bigl(2^{2n-1} - 2n\bigr)x^{2n+1}. \end{aligned}$$

Let

$$v_n = \frac{2^{2n+3}}{(2n+1)!}\bigl(2^{2n-1} - 2n\bigr), \quad n \ge 2.$$

It is obvious that the sequence v_n is positive in $n \ge 2$ and

$$\begin{aligned} g_3(x) &= \frac{64x^5}{15} - \frac{832x^7}{315} + \frac{128x^9}{189} - \frac{16064x^{11}}{155925} + \sum_{n=6}^{\infty}(-1)^{n+1}v_n x^{2n+1} \\ &= \frac{64x^5\bigl(10395 - 6435x^2 + 1650x^4 - 251x^6\bigr)}{155925} \\ &\quad + \sum_{k=3}^{\infty}v_{2k+1}\left(\frac{v_{2k}}{v_{2k+1}} - x^2\right)x^{4k+1}. \end{aligned}$$

In what follows, we show the inequality

$$\frac{v_{2k}}{v_{2k+1}} > \frac{5}{2} > \left(\frac{\pi}{2}\right)^2 = 2.467\dots, \quad k \ge 3. \tag{13}$$

This inequality is equivalent to

$$2v_{2k} - 5v_{2k+1} > 0, \quad k \ge 3,$$

which can be rewritten as the inequality

$$2^{4k-2}\bigl(8k^2 + 10k - 17\bigr) - (16k^3 + 20k^2 - 4k - 5) > 0, \quad k \ge 3.$$

Since

$$8k^2 + 10k - 17 = 8(k-3)^2 + 58(k-3) + 85 > 0$$

and

$$16k^3 + 20k^2 - 4k - 5 = 16(k-3)^3 + 164(k-3)^2 + 548(k-3) + 595 > 0$$

for $k \ge 3$, it follows that

$$\begin{aligned} &2^{4k-2}\big(8k^2 + 10k - 17\big) - \big(16k^3 + 20k^2 - 4k - 5\big) \\ &= \big(2^{4k-1} - 4k - 29\big)(k-3)^2 + \big(58 \times 2^{4k-2} - 548\big)(k-3) + 85 \times 2^{4k-2} - 595 \\ &\ge 86445 \end{aligned}$$

for $k \ge 3$. Therefore, the inequality (13) is valid.

On the other hand, it is easy to see that

$$\sum_{k=1}^{2} v_{2k+1}\left(\frac{v_{2k}}{v_{2k+1}} - x^2\right)x^{4k+1} = \frac{32x^5}{51975}V\big(x^2\big), \tag{14}$$

where, by the Cardano formula for the general cubic equation, the function

$$V(t) = -251t^3 + 1650t^2 - 6435t + 10395$$

has only one real zero

$$\begin{aligned} &\frac{550 + \sqrt[3]{\frac{55(1805339+753\sqrt{36360789}\,)}{2}} - 4289\sqrt[3]{\frac{6050}{1805339+753\sqrt{36360789}}}}{251} \\ &\qquad = 2.735\cdots > 2.467\cdots = \left(\frac{\pi}{2}\right)^2. \end{aligned}$$

This means that the polynomial in (14) of degree 11 is positive for $x \in \big(0, \frac{\pi}{2}\big)$. Accordingly, by the inequalities (13) and (14), the function

$$h_3(x) = \sum_{k=1}^{2} v_{2k+1}\left(\frac{v_{2k}}{v_{2k+1}} - x^2\right)x^{4k+1} + \sum_{k=3}^{\infty} v_{2k+1}\left(\frac{v_{2k}}{v_{2k+1}} - x^2\right)x^{4k+1}$$

is positive for $x \in \big(0, \frac{\pi}{2}\big)$. As a result, from the relation (12), it follows that the first derivative $G'(x)$ is positive, and then the function $G(x)$ is increasing, on $\big(0, \frac{\pi}{2}\big)$. Moreover, it is straightforward that

$$G(x) > \lim_{x \to 0^+} G(x) = 3.$$

The proof of Theorem 2 is thus complete. □

Theorem 3. *For $0<x<\frac{\pi}{2}$, we have*

$$3+\frac{32}{\pi^4}x^3\tan x<\operatorname{sinc}(2x)+2\operatorname{tanc}x<3+\frac{2}{5}x^3\tan x,$$

where $\frac{32}{\pi^4}$ and $\frac{2}{5}$ are the best possible constants.

Proof. For $x\in\left(0,\frac{\pi}{2}\right)$, let $f_1(x)=\sin x\cos^2x+2\sin x-3x\cos x$ and $f_2(x)=x^4\sin x$. Then

$$F(x)=\frac{\frac{\sin(2x)}{2x}+2\frac{\tan x}{x}-3}{x^3\tan x}=\frac{\sin(2x)+4\tan x-6x}{2x^4\tan x}=\frac{f_1(x)}{f_2(x)}. \tag{15}$$

Direct computations yield

$$\begin{aligned}f_1'(x)&=\cos^3x-2\sin^2x\cos x-\cos x+3x\sin x,\\ f_2'(x)&=x^4\cos x+4x^3\sin x,\end{aligned}$$

and

$$\frac{F'(x)}{F(x)}=\frac{f_1'(x)x\sin x-f_1(x)(x\cos x+4\sin x)}{f_1(x)x\sin x}=\frac{f_3(x)}{4f_1(x)x\sin x}, \tag{16}$$

where

$$f_3(x)=12x^2+16x\sin(2x)+x\sin(4x)+16\cos(2x)+2\cos(4x)-18.$$

Making use of the Maclaurin power series expansions of $\sin x$ and $\cos x$, we obtain

$$\begin{aligned}f_3(x)&=12x^2+16\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+2}}{(2n+1)!}+\sum_{n=0}^{\infty}(-1)^n\frac{4^{2n+1}x^{2n+2}}{(2n+1)!}\\ &\quad+16\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n}}{(2n)!}+2\sum_{n=0}^{\infty}(-1)^n\frac{4^{2n}x^{2n}}{(2n)!}-18\\ &=12x^2+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n+3}x^{2n}}{(2n-1)!}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{4n-2}x^{2n}}{(2n-1)!}\\ &\quad+\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n+4}x^{2n}}{(2n)!}+\sum_{n=1}^{\infty}(-1)^n\frac{2^{4n+1}x^{2n}}{(2n)!}\\ &=\sum_{n=4}^{\infty}(-1)^{n+1}\frac{2^{2n+4}}{(2n)!}\left[2^{2n-5}(n-4)+n-1\right]x^{2n}.\end{aligned}$$

Let

$$u_n=\frac{2^{2n+4}\left[2^{2n-5}(n-4)+n-1\right]}{(2n)!},\quad n\ge 4.$$

It is apparent that the sequence u_n is positive in $n \ge 4$ and

$$\begin{aligned}
f_3(x) &= -\frac{32x^8}{105}+\frac{256x^{10}}{1575}-\frac{1856x^{12}}{51975}+\frac{65792x^{14}}{14189175}+\sum_{n=8}^{\infty}(-1)^{n+1}u_n x^{2n}\\
&= \frac{32x^8\bigl(2056x^6-15834x^4+72072x^2-135135\bigr)}{14189175}-\sum_{k=4}^{\infty}\bigl(u_{2k}-u_{2k+1}x^2\bigr)x^{4k}\\
&= \frac{32x^8\bigl(2056x^6-15834x^4+72072x^2-135135\bigr)}{14189175}\\
&\quad -\sum_{k=4}^{\infty}u_{2k+1}\left(\frac{u_{2k}}{u_{2k+1}}-x^2\right)x^{4k}.
\end{aligned}$$

The inequality

$$\frac{u_{2k}}{u_{2k+1}} > \frac{5}{2} > \left(\frac{\pi}{2}\right)^2, \quad k \ge 4 \tag{17}$$

is equivalent to

$$2u_{2k}-5u_{2k+1}>0, \quad k\ge 4$$

which can be reformulated as the inequality

$$2^{4k-4}(8k^3-10k^2-31k+28)+16k^3+4k^2-14k-1>0.$$

Since

$$8k^3-10k^2-31k+28=8(k-4)^3+86(k-4)^2+273(k-4)+256>0$$

and

$$16k^3+4k^2-14k-1=16(k-4)^3+196(k-4)^2+786(k-4)+1031>0$$

for $k \ge 4$, the inequality (17) is valid.

On the other hand,

$$\sum_{k=2}^{3}u_{2k+1}\left(\frac{u_{2k}}{u_{2k+1}}-x^2\right)x^{4k}=\frac{32x^8}{14189175}U\bigl(x^2\bigr), \tag{18}$$

where, by the Cardano formula for the general cubic equation, the function

$$U(t)=2056t^3-15834t^2+72072t-135135$$

has only one real zero

$$\begin{aligned}
&\frac{2639+\sqrt[3]{91\bigl(1542\sqrt{9299551206}+57143753\bigr)}-\frac{59165\sqrt[3]{8281}}{\sqrt[3]{1542\sqrt{9299551206}+57143753}}}{1028}\\
&\qquad = 3.178\cdots > 2.467\cdots = \left(\frac{\pi}{2}\right)^2.
\end{aligned}$$

This means that the polynomial in (18) of degree 14 is negative for $x \in \left(0, \frac{\pi}{2}\right)$. Accordingly, by the inequality (17) and (18), the function

$$f_3(x) = \sum_{k=2}^{3} u_{2k+1}\left(\frac{u_{2k}}{u_{2k+1}} - x^2\right)x^{4k} - \sum_{k=4}^{\infty} u_{2k+1}\left(\frac{u_{2k}}{u_{2k+1}} - x^2\right)x^{4k}$$

is negative for $x \in \left(0, \frac{\pi}{2}\right)$.

The inequality $\operatorname{sinc}(2x) + 2\operatorname{tanc} x > 3$ in Theorem 1 is equivalent to

$$f_1(x) = [\operatorname{sinc}(2x) + 2\operatorname{tanc} x - 3]x\cos x > 0, \quad 0 < x < \frac{\pi}{2}.$$

As a result, from the relation (16), it follows that the first derivative $F'(x)$ is negative, and then the function $F(x)$ is decreasing, on $\left(0, \frac{\pi}{2}\right)$. Moreover, it is straightforward that

$$\frac{32}{\pi^4} = \lim_{x\to(\pi/2)^-} F(x) < F(x) < \lim_{x\to 0^+} F(x) = \frac{2}{5}, \quad 0 < x < \frac{\pi}{2}. \tag{19}$$

The proof of Theorem 3 is thus complete. □

Theorem 4. *For $0 < x < \frac{\pi}{2}$, we have*

$$3 + \frac{4}{15}x^3 \tan x < \frac{1}{\operatorname{sinc}(2x)} + \frac{2}{\operatorname{tanc} x} < 3 + \frac{4}{\pi^2}x^3 \tan x,$$

where $\frac{4}{15}$ and $\frac{4}{\pi^2}$ are the best possible constants.

Proof. Let $\ell_1(x) = x + 2x\cos^2 x - 3\sin x\cos x$ and $\ell_2(x) = x^3\sin^2 x$. Then

$$L(x) = \frac{\frac{x}{\sin x\cos x} + \frac{2x\cos x}{\sin x} - 3}{x^3\tan x} = \frac{\ell_1(x)}{\ell_2(x)}. \tag{20}$$

Direct computations yield

$$\ell_1'(x) = 3\sin^2 x - \cos^2 x - 4x\sin x\cos x + 1,$$
$$\ell_2'(x) = 2x^3\cos x\sin x + 3x^2\sin^2 x,$$

and

$$\frac{L'(x)}{L(x)} = \frac{\ell_1'(x)\ell_2(x) - \ell_1(x)\ell_2'(x)}{\ell_1(x)\ell_2(x)} = \frac{\ell_3(x)}{4\ell_1(x)x\sin x}, \tag{21}$$

where

$$\ell_3(x) = 9\cos x - 9\cos(3x) - 24x^2\cos x - 4x\sin(3x).$$

Using the Maclaurin power series expansions of $\sin x$ and $\cos x$, we acquire

$$\begin{aligned}\ell_3(x) &= 9\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n} - 9\sum_{n=0}^{\infty}(-1)^n\frac{3^{2n}}{(2n)!}x^{2n} - 24\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n+2}\\ &\quad - 4\sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}}{(2n+1)!}x^{2n+2}\\ &= 9\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n} - 9\sum_{n=1}^{\infty}(-1)^n\frac{3^{2n}}{(2n)!}x^{2n}\\ &\quad - 24\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-2)!}x^{2n} - 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{2n-1}}{(2n-1)!}x^{2n}\\ &= \sum_{n=4}^{\infty}(-1)^n\frac{1}{(2n)!}\left[3^{2n-1}(8n-27)+96n^2-48n+9\right]x^{2n}.\end{aligned}$$

Let

$$m_n = \frac{3^{2n-1}(8n-27)+96n^2-48n+9}{(2n)!}, \quad n \geq 4.$$

It is clear that the sequence m_n is positive in $n \geq 4$ and

$$\ell_3(x) = \sum_{k=2}^{\infty} m_{2k+1}\left(\frac{m_{2k}}{m_{2k+1}} - x^2\right)x^{4k}.$$

In what follows, we show the inequality

$$\frac{m_{2k}}{m_{2k+1}} > \frac{5}{2} > \left(\frac{\pi}{2}\right)^2, \quad k \geq 2. \tag{22}$$

This inequality is equivalent to

$$2m_{2k} - 5m_{2k+1} > 0, \quad k \geq 2$$

which can be rearranged as the inequality

$$3^{4k-2}(512k^3 - 480k^2 - 1304k + 747) + 4096k^4 + 2048k^3 - 800k^2 - 536k - 83 > 0.$$

Since

$$\begin{aligned}512k^3 - 480k^2 - 1304k + 747 &= 512(k-2)^3 + 2592(k-2)^2 + 2920(k-2) + 315\\ &> 0\end{aligned}$$

and

$$\begin{aligned}&4096k^4 + 2048k^3 - 800k^2 - 536k - 83\\ &= 4096(k-2)^4 + 34816(k-2)^3 + 109792(k-2)^2 + 151912(k-2) + 77565\\ &> 0\end{aligned}$$

for $k \geq 2$, the inequality in (22) is valid. This implies that $\ell_3(x)$ is positive for $x \in \left(0, \frac{\pi}{2}\right)$. As a result, from the relation (21) and Theorem 2, it follows that the first derivative $L'(x)$ is positive, and then the function $L(x)$ is increasing, on $\left(0, \frac{\pi}{2}\right)$. Moreover, it is straightforward that

$$\frac{4}{15} = \lim_{x \to 0^+} L(x) < L(x) < \lim_{x \to (\pi/2)^-} L(x) = \frac{4}{\pi^2}. \tag{23}$$

The proof of Theorem 4 is thus complete. □

3. GENERALIZATIONS OF THEOREMS 3 AND 4

To generalize Theorems 3 and 4, we now introduce the concept of stratification and two key preliminary theorems. The concept of stratification has been invented and applied by Malešević and his coauthors in numerous publications such as [**1, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21**].

Definition 1 (Stratification of a family of functions)**.** Let $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ be a family of functions, where the independent variable $x \in \mathbb{S} \subseteq \mathbb{R}$, the parameter $p \in \mathbb{P} \subseteq \mathbb{R}$, with $\mathbb{S} \neq \emptyset$ and $\mathbb{P} \neq \emptyset$.

1. For any $x \in \mathbb{S}$ and any $p_1, p_2 \in \mathbb{P}$, if

$$p_1 < p_2 \quad \Longleftrightarrow \quad \Phi_{p_1}(x) < \Phi_{p_2}(x),$$

   then the family $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ is said to be increasingly stratified on the set $\mathbb{S}$ with respect to the parameter $p \in \mathbb{P}$.

2. For any $x \in \mathbb{S}$ and any $p_1, p_2 \in \mathbb{P}$, if

$$p_1 < p_2 \quad \Longleftrightarrow \quad \Phi_{p_1}(x) > \Phi_{p_2}(x),$$

   then the family $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ is said to be decreasingly stratified on the set $\mathbb{S}$ with respect to the parameter $p \in \mathbb{P}$.

Stratification describes the consistent monotonicity of a function family as the parameter changes. This property provides a systematic method of analyzing and generalizing numerous analytic inequalities.

The following two lemmas are the cores of the stratification method. They utilize the limiting behaviors of stratified families to determine the optimal range of constants for which an inequality holds. Their proofs can be found in [**15**, Theorems 5 and 6].

Lemma 1 ([**15**, Theorem 6])**.** *Let* $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ *be a family of functions with* $x \in (a,b) \subseteq \mathbb{R}$ *and* $\mathbb{P} \subseteq \mathbb{R}$ *such that* $\mathbb{P} \neq \emptyset$. *If*

1) the family of functions $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ *is increasingly (or decreasingly, respectively) stratified on* (a,b)*,*

2) there exists a continuous and decreasing function $g:(a,b)\to\mathbb{P}$ *satisfying*

$$g(x)=p \quad\Longleftrightarrow\quad \Phi_p(x)=0; \tag{24}$$

3) there exists the limits $\lim_{x\to b} g(x)=A\in(-\infty,\infty)\cup\{\pm\infty\}$ *and* $\lim_{x\to a} g(x)=B\in(-\infty,\infty)\cup\{\pm\infty\}$ *such that* $(A,B)\subseteq\mathbb{P}$*,*

then the following conclusions are valid:

i) For $p\le A$ *and* $x\in(a,b)$*, we have*

$$\Phi_p(x)\le\Phi_A(x)<0 \quad (\text{or } \Phi_p(x)\ge\Phi_A(x)>0, \text{ respectively});$$

ii) For $p\in(A,B)$ *and* $x\in(a,b)$*, the equation* $\Phi_p(x)=0$ *has a unique solution* $x_0^{(p)}\in(a,b)$ *such that* $\Phi_p(x)<0$ *on the interval* $\big(a,x_0^{(p)}\big)$ *and* $\Phi_p(x)>0$ *on the interval* $\big(x_0^{(p)},b\big)$ *(or such that* $\Phi_p(x)>0$ *on* $\big(a,x_0^{(p)}\big)$ *and* $\Phi_p(x)<0$ *on* $\big(x_0^{(p)},b\big)$*, respectively);*

iii) For $p\ge B$ *and* $x\in(a,b)$*, we have*

$$\Phi_p(x)\ge\Phi_B(x)>0 \quad (\text{or } \Phi_p(x)\le\Phi_B(x)<0, \text{ respectively}).$$

Lemma 2 ([**15**, Theorem 5])**.** *Let* $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ *be a family of functions with* $x\in(a,b)\subseteq\mathbb{R}$ *and* $\mathbb{P}\subseteq\mathbb{R}$ *such that* $\mathbb{P}\ne\emptyset$*. If*

1. *the family of functions* $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ *is increasingly (or decreasingly, respectively) stratified on* (a,b)*;*
2. *there exists a continuous and increasing function* $g:(a,b)\to\mathbb{P}$ *satisfying the equivalence* (24)*;*
3. *there exists the limits* $\lim_{x\to a} g(x)=A\in(-\infty,\infty)\cup\{\pm\infty\}$ *and* $\lim_{x\to b} g(x)=B\in(-\infty,\infty)\cup\{\pm\infty\}$ *such that* $(A,B)\subseteq\mathbb{P}$*;*

then the following conclusions are valid:

(1) For $p\le A$ *and* $x\in(a,b)$*, we have*

$$\Phi_p(x)\le\Phi_A(x)<0 \quad (\text{or } \Phi_p(x)\ge\Phi_A(x)>0, \text{ respectively}).$$

(2) For $p\in(A,B)$ *and* $x\in(a,b)$*, the equation* $\Phi_p(x)=0$ *has a unique solution* $x_0^{(p)}\in(a,b)$ *such that* $\Phi_p(x)>0$ *on the interval* $\big(a,x_0^{(p)}\big)$ *and* $\Phi_p(x)<0$ *on the interval* $\big(x_0^{(p)},b\big)$ *(or such that* $\Phi_p(x)<0$ *on* $\big(a,x_0^{(p)}\big)$ *and* $\Phi_p(x)>0$ *on* $\big(x_0^{(p)},b\big)$*, respectively).*

(3) For $p \ge B$ and $x \in (a,b)$, we have

$$\Phi_p(x) \ge \Phi_B(x) > 0 \quad (\textit{or } \Phi_p(x) \le \Phi_B(x) < 0, \textit{ respectively}).$$

In light of Lemma 1, we can generalize Theorems 3 and 4 as follows.

Theorem 5. *Let $A = \frac{32}{\pi^4}$ and $B = \frac{2}{5}$.*

(i) If $p \in (-\infty, A)$, then

$$\operatorname{sinc}(2x) + 2\operatorname{tanc} x > 3 + A\,x^3 \tan x > 3 + p\,x^3 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right)$$

and the constant A is the best possible.

(ii) If $p \in (A,B)$, the equation $\Phi_p(x) = 0$ has a unique solution $x_0^{(p)}$ such that

$$\operatorname{sinc}(2x) + 2\operatorname{tanc} x > 3 + p\,x^3 \tan x, \quad x \in \left(0, x_0^{(p)}\right)$$

and

$$\operatorname{sinc}(2x) + 2\operatorname{tanc} x < 3 + p\,x^3 \tan x \quad x \in \left(x_0^{(p)}, \frac{\pi}{2}\right).$$

(iii) If $p \in (B, \infty)$, then

$$\operatorname{sinc}(2x) + 2\operatorname{tanc} x < 3 + B\,x^3 \tan x < 3 + p\,x^3 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right)$$

and the constant B is the best possible.

Proof. Define the family of functions $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ by

$$\Phi_p(x) = \operatorname{sinc}(2x) + 2\operatorname{tanc} x - 3 - p\,x^3 \tan x \tag{25}$$

for $x = \left(0, \frac{\pi}{2}\right)$ and $p \in \mathbb{P} = \mathbb{R}$. A direct differentiation yields

$$\frac{\partial \Phi_p(x)}{\partial p} = -x^3 \tan x < 0$$

for $x > 0$. Therefore, the family $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ is decreasingly stratified on the interval $(0,\infty)$ with respect to the parameter $p \in \mathbb{P} = \mathbb{R}$.

Let $g(x) = F(x)$ for $x \in (0,\infty)$, where $F(x)$ is defined by (15). In the proof of Theorem 3, we proved that the function $F(x)$ is decreasing on $(0,\infty)$. Therefore, from the limits in (19), it follows that the relation in (24) holds for $p = \frac{32}{\pi^4}$ and $p = \frac{2}{5}$.

From the limits in (19), we conclude that $A = \frac{32}{\pi^4}$ and $B = \frac{2}{5}$ in Lemma 1.

Combining the above discussion and making use of Lemma 1, we derive that, if and only if $p \le \frac{32}{\pi^4}$,

$$\Phi_p(x) \ge \Phi_{32/\pi^4}(x) > 0, \quad x \in \left(0, \frac{\pi}{2}\right),$$

and if and only if $p \ge 2/5$,

$$\Phi_p(x) \le \Phi_{2/5}(x) < 0, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Accordingly, the inequalities formulated in Theorem 5 and the best possibility of the constant $A = \frac{32}{\pi^4}$ and $B = \frac{2}{5}$ are verified. The proof of Theorem 5 is complete. □

Remark 1. According to [**21**], for the stratified family of functions $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ defined by (25), it is also significant to determine, if it exists, the minimax approximant $\Phi_{p_0}(x)$ for some $p_0 \in \mathbb{P}$ as a function such that

$$\inf_{p\in\mathbb{P}} \sup_{x\in\mathbb{S}} |\Phi_p(x)| = \sup_{x\in\mathbb{S}} |\Phi_{p_0}(x)|.$$

The function $\Phi_p(x)$ for $p \in \mathbb{R}$ are continuous with respect to $x \in \left(0, \frac{\pi}{2}\right)$ and $\lim_{x\to0^+} \Phi_p(x) = 0$ for $p \in \mathbb{R}$. Hence, since

$$\lim_{x\to(\frac{\pi}{2})^-} \Phi_p(x) = \infty, \quad p < A = \frac{32}{\pi^4};$$
$$\lim_{x\to(\frac{\pi}{2})^-} \Phi_p(x) = \frac{32-3\pi^2}{\pi^2}, \quad p = A = \frac{32}{\pi^4};$$
$$\lim_{x\to(\frac{\pi}{2})^-} \Phi_p(x) = -\infty, \quad p > A = \frac{32}{\pi^4},$$

the function

$$\Phi_A(x) = \operatorname{sinc}(2x) + 2\operatorname{tanc} x - 3 - \frac{32}{\pi^4}x^3 \tan x$$

is the minimax approximant of the family $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ defined by (25), which determines the corresponding minimax approximation

$$\operatorname{sinc}(2x) + 2\operatorname{tanc} x \approx 3 + \frac{32}{\pi^4}x^3 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right).$$

For the value

$$d_0 = \lim_{x\to(\frac{\pi}{2})^-} \Phi_A(x) = \frac{32-3\pi^2}{\pi^2},$$

we have

$$d_0 = \inf_{p\in\mathbb{R}} \sup_{x\in(0,\frac{\pi}{2})} |\Phi_p(x)|.$$

Theorem 6. *Let* $A = \frac{4}{15}$ *and* $B = \frac{4}{\pi^2}$.

(i) If $p \in (-\infty, A)$, then

$$\frac{1}{\operatorname{sinc}(2x)} + \frac{2}{\operatorname{tanc} x} > 3 + A\,x^3 \tan x > 3 + p\,x^3 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right)$$

and the constant A is the best possible.

(ii) If $p \in (A, B)$, the equation $\Phi_p(x) = 0$ has a unique solution $x_0^{(p)}$ such that

$$\frac{1}{\operatorname{sinc}(2x)} + \frac{2}{\operatorname{tanc} x} < 3 + p\,x^3 \tan x, \quad x \in \left(0, x_0^{(p)}\right)$$

and

$$\frac{1}{\operatorname{sinc}(2x)} + \frac{2}{\operatorname{tanc} x} > 3 + p\,x^3 \tan x \quad x \in \left(x_0^{(p)}, \frac{\pi}{2}\right).$$

(iii) If $p \in (B, \infty)$, then

$$\frac{1}{\operatorname{sinc}(2x)} + \frac{2}{\operatorname{tanc} x} < 3 + B\,x^3 \tan x < 3 + p\,x^3 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right)$$

and the constant B is the best possible.

Proof. Define the family of functions $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ by

$$\Phi_p(x) = \frac{1}{\operatorname{sinc}(2x)} + \frac{2}{\operatorname{tanc} x} - 3 - p\,x^3 \tan x \tag{26}$$

for $x \in \left(0, \frac{\pi}{2}\right)$ and $p \in \mathbb{P} = \mathbb{R}$. A straightforward differentiation gives

$$\frac{\partial \Phi_p(x)}{\partial p} = -x^3 \tan x < 0$$

for $x \in \left(0, \frac{\pi}{2}\right)$. Therefore, the family $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ is decreasingly stratified on the interval $\left(0, \frac{\pi}{2}\right)$ with respect to the parameter $p \in \mathbb{P} = \mathbb{R}$.

Let $g(x) = L(x)$ for $x \in \left(0, \frac{\pi}{2}\right)$, where $L(x)$ is defined by (20). In the proof of Theorem 4, we proved that $L(x)$ is increasing on $\left(0, \frac{\pi}{2}\right)$. Hence, from the limits in (23), it follows that the relation in (24) holds for $p = \frac{4}{15}$ and $p = \frac{4}{\pi^2}$.

From the limits in (23), we conclude that $A = \frac{4}{15}$ and $B = \frac{4}{\pi^2}$ in Lemma 2.

Combining the above discussion and making use of Lemma 2, we derive that, if and only if $p \le \frac{4}{15}$,

$$\Phi_p(x) \ge \Phi_{4/15}(x) > 0, \quad x \in \left(0, \frac{\pi}{2}\right),$$

and if and only if $p \ge \frac{4}{\pi^2}$,

$$\Phi_p(x) \le \Phi_{4/\pi^2}(x) < 0, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Accordingly, the inequalities stated in Theorem 6 and the best possibility of the constant $A = \frac{4}{15}$ and $B = \frac{4}{\pi^2}$ are verified. The proof of Theorem 6 is complete. □

Remark 2. The function $\Phi_p(x)$ defined by (26) for $p \in \mathbb{R}$ are continuous with respect to $x \in \left(0, \frac{\pi}{2}\right)$ and $\lim_{x\to0^+} \Phi_p(x) = 0$ for $p \in \mathbb{R}$. Hence, since

$$\lim_{x\to(\frac{\pi}{2})^-} \Phi_p(x) = \infty, \quad p < B = \frac{4}{\pi^2};$$
$$\lim_{x\to(\frac{\pi}{2})^-} \Phi_p(x) = -1, \quad p = B = \frac{4}{\pi^2};$$
$$\lim_{x\to(\frac{\pi}{2})^-} \Phi_p(x) = -\infty, \quad p > B = \frac{4}{\pi^2},$$

the function

$$\Phi_B(x) = \frac{1}{\operatorname{sinc}(2x)} + \frac{2}{\operatorname{tanc} x} - 3 - \frac{4}{\pi^2} x^3 \tan x$$

is the minimax approximant of the family $\{\Phi_p(x)\}_{p\in\mathbb{P}}$ defined by (26), which determines the corresponding minimax approximation

$$\frac{1}{\operatorname{sinc}(2x)} + 2\frac{1}{\operatorname{tanc} x} \approx 3 + \frac{4}{\pi^2} x^3 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right).$$

For the value

$$d_0 = \left| \lim_{x\to(\frac{\pi}{2})^-} \Phi_p(x) \right| = 1,$$

we have

$$d_0 = \inf_{p\in\mathbb{R}} \sup_{x\in(0,\frac{\pi}{2})} |\Phi_p(x)|.$$

Remark 3. The families of functions in (25) and (26) can be showed by Figure 1 and 2 respectively, which are plotted for

$$p = -0.2, \quad 0.3, \quad \frac{32}{\pi^4}, \quad 0.34, \quad 0.36, \quad 0.37, \quad \frac{2}{5}, \quad 0.42, \quad 0.43$$

and

$$p = 0, \quad 0.1, \quad 0.2, \quad \frac{4}{15}, \quad 0.3, \quad 0.35, \quad 0.4, \quad \frac{4}{\pi^2}, \quad 0.5$$

respectively, using the software Mathematica 14.0.0.0.

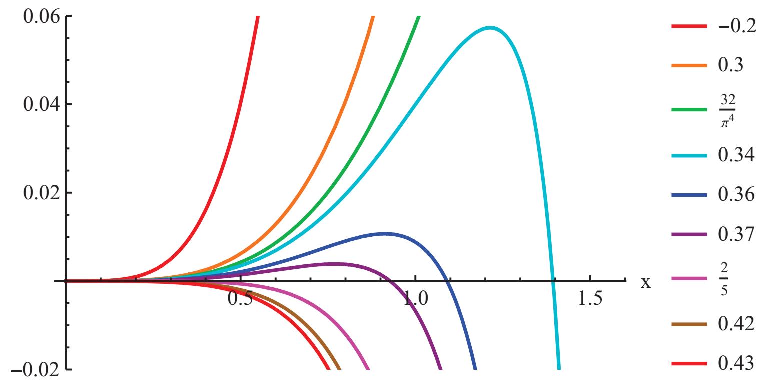

Figure 1: Stratified family of functions in (25)

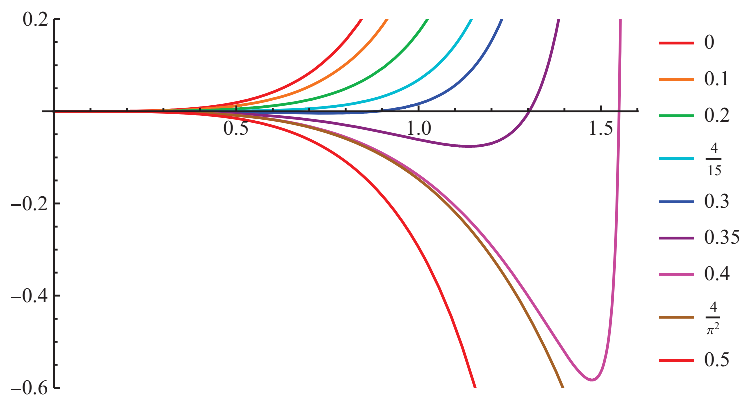

Figure 2: Stratified family of functions in (26)

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