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NEW SHAFER-FINK TYPE INEQUALITIES FOR INVERSE TRIGONOMETRIC, INVERSE HYPERBOLIC AND ARC LEMNISCATE FUNCTIONS

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In this paper, we present new Shafer-Fink type inequalities for the arc sine and arc tangent functions. We develop Shafer's inequality for the arc hyperbolic tangent function to produce sharp two-sided inequalities, and present a new lower bound for the arc hyperbolic tangent function. Also, we present new sharp inequalities for the arc lemniscate functions.

1. INTRODUCTION

For $0 \leq x \leq 1$, the following double inequality holds:

$$\frac{3x}{2+\sqrt{1-x^2}} \leq \arcsin x \leq \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{1}$$

The left-hand side inequality was presented by Shafer (see, *e.g.*, [**29**, p. 247]), while the right-hand side inequality was established by Fink [**14**]. Shafer-Fink's inequalities have attracted much interest of many mathematicians and have motivated a large number of papers involving various generalizations and improvements [**15, 18, 19, 20, 26, 39, 40, 51, 52**]. Zhu [**53**] provided a solution to an open

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problem posed by Oppenheim in [**38**], and deduced some Shafer-Fink inequalities from the solution of Oppenheim's problem. Chen and Cheung [**11**] gave a concise proof to Oppenheim's problem.

The double inequality

$$\frac{3x}{1+2\sqrt{1+x^2}} < \arctan x < \frac{2x}{1+\sqrt{1+x^2}} \tag{2}$$

holds for $x > 0$. The left-hand side inequality is due to Shafer [**42, 43**], while the right-hand side inequality can be found in, *e.g.*, [**16**, p. 288]. Shafer's inequality (2) was improved and generalized in [**5, 30, 36, 41, 44, 45, 46, 49, 51**].

The first aim of the present paper is to present new sharp inequalities for $\arcsin x$ and $\arctan x$ (Theorem 1 and Theorem 2).

Shafer [**44, 45, 46**] proved that for $0 < x < \sqrt{15}/4$,

$$\operatorname{arctanh} x < \frac{8x}{3+\sqrt{25-\frac{80}{3}x^2}}. \tag{3}$$

Zhu [**54**] provided an alternative proof of (3) in a concise way. The proof in [**54**] contains a small mistake. Bagul and Dhaigude [**4**] corrected this mistake and gave alternative proofs of (3).

The second aim of the present paper is to develop inequality (3) to produce sharp two-sided inequalities (Theorem 3 and Theorem 4), and present a new lower bound for $\operatorname{arctanh} x$ (Theorem 5).

The arc lemniscate sine function

$$\operatorname{arcsl} x = \int_0^x \frac{1}{\sqrt{1-t^4}}\mathrm{d}t, \qquad |x| \le 1 \tag{4}$$

and the hyperbolic arc lemniscate sine function

$$\operatorname{arcslh} x = \int_0^x \frac{1}{\sqrt{1+t^4}}\mathrm{d}t, \qquad x \in \mathbb{R} \tag{5}$$

have been studied by Gauss in 1797-1798. The limiting values of the above two functions are

$$\omega = \operatorname{arcsl}(1) = \frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right) = 1.311028777146\ldots$$

and

$$K = \operatorname{arcslh}(\infty) = \frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right) = 1.8540746773\ldots,$$

where $B(x,y)$ denotes the beta function. The arc lemniscate tangent

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), \qquad x \in \mathbb{R} \tag{6}$$

and the hyperbolic arc lemniscate tangent

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \qquad |x| < 1 \tag{7}$$

were introduced in [**31**]. These functions can be found (see [**47**, Ch. 1], [**7**, p. 259] and [**8, 9, 10, 13, 17, 32, 33, 34, 35, 49, 50**]). Direct computation gives

$$(\operatorname{arctl} x)' = \frac{1}{(1+x^4)^{3/4}}, \quad (\operatorname{arctlh} x)' = \frac{1}{(1-x^4)^{3/4}}.$$

Deng and Chen [**13**] established the following Shafer-Fink type inequalities for the arc lemniscate functions:

$$\frac{5}{4+\sqrt{1-x^4}} < \frac{\operatorname{arcsl} x}{x} < \frac{B\left(\frac{1}{4}, \frac{1}{2}\right)}{4+\sqrt{1-x^4}}, \quad 0 < |x| < 1, \tag{8}$$

$$\frac{\frac{10}{3}}{\frac{7}{3}+\sqrt{1-x^4}} < \frac{\operatorname{arctlh} x}{x} < \frac{\frac{7}{12}B\left(\frac{1}{4}, \frac{1}{4}\right)}{\frac{7}{3}+\sqrt{1-x^4}}, \qquad 0 < |x| < 1, \tag{9}$$

where $B(x, y)$ denotes the beta function. Sun and Chen [**49**] established the following inequalities:

$$\frac{10x}{5+\sqrt{25-10x^4}} < \operatorname{arcsl} x, \qquad 0 < x < 1, \tag{10}$$

$$\frac{10x}{5+\sqrt{25-15x^4}} < \operatorname{arctlh} x, \qquad 0 < x < 1 \tag{11}$$

and

$$\frac{95x}{80+\sqrt{225+285x^4}} < \operatorname{arcslh} x, \qquad x > 0. \tag{12}$$

The inequalities (10), (11) and (12) are analogues of (3). Recently, Wei et al [**50**] established Shafer-Fink type inequalities for the arc lemniscate functions. For example, the these authors proved that

$$\frac{\omega}{1+(\omega-1)\sqrt[4]{1-x^4}} < \frac{\operatorname{arcsl} x}{x} < \frac{5}{3+2\sqrt[4]{1-x^4}}, \quad 0 < |x| < 1, \tag{13}$$

$$\frac{5}{4+\sqrt{1-x^4}} < \frac{\operatorname{arcsl} x}{x} < \frac{\omega}{1+(\omega-1)\sqrt{1-x^4}}, \quad 0 < |x| < 1. \tag{14}$$

The last aim of the present paper is to present new sharp inequalities for $\operatorname{arcsl} x$, $\operatorname{arcslh} x$ and $\operatorname{arctl} x$ (Theorems 6, 7 and 8).

Remark 1. *The introduced inequalities could be improved and generalized using the recently introduced concept of stratified families of functions, see the references [**6, 23, 24, 25, 21, 27, 28, 22**].*

2. LEMMAS

The following lemmas are needed in our present investigation.

Lemma 1 ([**1, 2, 3**])**.** *Let $-\infty < a < b < \infty$, and $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Suppose $g' \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2. *(Sturm's theorem [**48**], Theorem 4.1 [**12**]) Let $p(x)$ be a real polynomial, and let $p_0, p_1, \ldots, p_r$ be the sequence of real polynomials given by*
(a) $p_0 = p$,
(b) $p_1 = p'$ (the derivative of p),
(c) for $0 < i < r$, there is a polynomial q_i such that $p_{i-1} = p_i q_i - p_{i+1}$ with $p_{i+1} \neq 0$ and $\deg(p_{i+1}) < \deg(p_i)$ (so that q_i and $-p_{i+1}$ are the quotient and remainder respectively when p_{i-1} is divided by p_i),
(d) $p_{r-1} = p_r q_r$.
For any real number c denote by $\delta(c)$ the number of sign changes in the sequence $p_0(c), \ldots, p_r(c)$ (ignoring zeros).
Suppose that a, b are real numbers that are not zeros of $p(x)$, and $a < b$. Then the number of zeros of $p(x)$ in the interval $[a, b]$ is $\delta(a) < \delta(b)$, (each zero being counted once only).

Lemma 3. *For $0 < x < \sqrt{15}/4$, we have*

$$\frac{1}{\sqrt{225-240x^2}} > \frac{1}{15} + \frac{8}{225}x^2 + \frac{32}{1125}x^4 + \frac{256}{10125}x^6 + \frac{3584}{151875}x^8, \tag{15}$$

$$17920x^8 - 97120x^6 + 159000x^4 - 103500x^2 + 23625 > 0 \tag{16}$$

and

$$\sqrt{225-240x^2} > \frac{15(15-16x^2)}{15-8x^2}. \tag{17}$$

Proof. The inequality (15) is equivalent to

$$P_{18}(x) = \frac{1}{\sqrt{225-240x^2}} > \left(\frac{1}{15}+\frac{8}{225}x^2+\frac{32}{1125}x^4+\frac{256}{10125}x^6+\frac{3584}{151875}x^8\right)^2(225-240x^2)-1<0$$

for $0<x<\sqrt{15}/4$. It holds that

$$P_{18}(x) = x^{10}\left(-\frac{57344}{84375}-\frac{229376}{759375}x^2-\frac{262144}{1265625}x^4-\frac{1835008}{11390625}x^6-\frac{205520896}{1537734375}x^8\right).$$

It is obvious that

$$-\frac{57344}{84375}-\frac{229376}{759375}x^2-\frac{262144}{1265625}x^4-\frac{1835008}{11390625}x^6-\frac{205520896}{1537734375}x^8<0$$

for $0<x<\sqrt{15}/4$. Therefore

$$P_{18}(x)<0$$

for $x\in(0,\sqrt{15}/4)$, i.e. the inequality (15) holds.

The proof of the inequality (16) follows from a direct application of the Sturm theorem. The proof of the inequality (17) is analogous to the proof of the inequality (15). The proof of Lemma 3 is complete. □

Lemma 4. *The function*

$$G(t) = 5t^{43}-218t^{30}+720t^{17}-455t^{13}-52$$

is negative on $[0,1)$ *and* $G(1)=0$.

Proof. By an application of the Sturm's theorem. □

3. SHARP INEQUALITIES FOR $\arcsin x$ AND $\arctan x$

In view of (1), we introduce the approximations family

$$\arcsin x \approx \frac{ax}{b+(1-x^2)^c}, \qquad x\to 0,$$

and $a,b,c\in\mathbb{R}$ are parameters. Our aim is to determine the values

$$a=\frac{13}{5},\quad b=\frac{8}{5},\quad c=\frac{13}{30}, \tag{18}$$

which provide the best approximation near the origin:

$$\arcsin x \approx \frac{\frac{13}{5}x}{\frac{8}{5}+(1-x^2)^{\frac{13}{30}}} = \frac{13x}{8+5(1-x^2)^{13/30}}. \tag{19}$$

We write $\arcsin x - \frac{ax}{b+(1-x^2)^c}$ as power series

$$\begin{aligned} \arcsin x - \frac{ax}{b+(1-x^2)^c} &= \frac{b-a+1}{b+1}x + \frac{b^2-6ac+2b+1}{6(b+1)^2}x^3 \\ &+ \frac{20abc^2-20ac^2-20abc-20ac+3b^3+9b^2+9b+3}{40(b+1)^3}x^5 + O(x^7). \end{aligned} \tag{20}$$

This produces the best approximation from (20):

$$\begin{cases} b-a+1=0 \\ b^2-6ac+2b+1=0 \\ 20abc^2-20ac^2-20abc-20ac+3b^3+9b^2+9b+3=0, \end{cases}$$

that is, by (18). We thus obtain the best approximation (19).

The formula (19) motivated us to establish Theorem 1.

Theorem 1. *For* $0<x<1$,

$$\frac{\alpha x}{8+5(1-x^2)^{13/30}} < \arcsin x < \frac{\beta x}{8+5(1-x^2)^{13/30}}, \tag{21}$$

with the best possible constants

$$\alpha = 4\pi = 12.56637\ldots \quad \textit{and} \quad \beta = 13.$$

Proof. For $0<x<1$, let

$$f(x) = \frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x) = \arcsin x \quad \text{and} \quad f_2(x) = \frac{x}{8+5(1-x^2)^{13/30}}.$$

Then,

$$\frac{1}{3}\frac{f_1'(x)}{f_2'(x)} = \frac{(1-x^2)^{1/15}(8+5(1-x^2)^{13/30})^2}{24(1-x^2)^{17/30}+15-2x^2} =: f_3(x).$$

Differentiating $f_3(x)$ with respect to x and applying Lemma 4 yield, for $0 < x < 1$,

$$\begin{aligned}
&-\frac{5(1-x^2)^{3/2}\big(24(1-x^2)^{17/30}+15-2x^2\big)^2}{4x\big(8+5(1-x^2)^{13/30}\big)}f_3'(x)\\
&=-720(1-x^2)(1-x^2)^{2/15}+(270-218x^2)(1-x^2)^{17/30}+450-445x^2-5x^4\\
&\overset{t=x^2}{=\!=}-720(1-t)(1-t)^{2/15}+(270-218t)(1-t)^{17/30}+450-445t-5t^2\\
&\overset{u=(1-t)^{1/30}}{=\!=\!=}-720u^{34}+(270-218(1-u^{30}))u^{17}+450-445(1-u^{30})-5(1-u^{30})^2\\
&=-u^{17}(5u^{43}-218u^{30}+720u^{17}-455u^{13}-52)>0, \qquad 0<u<1.
\end{aligned}$$

Hence, $f_3'(x) < 0$ for $x \in (0,1)$. Therefore, the functions $f_3(x)$ and $\frac{f_1'(x)}{f_2'(x)}$ are strictly decreasing on $(0,1)$. By Lemma 1, the function

$$f(x)=\frac{f_1(x)}{f_2(x)}=\frac{f_1(x)-f_1(0)}{f_2(x)-f_2(0)}$$

is strictly decreasing on $(0,1)$. And hence,

$$13=\lim_{x\to 0}f(x)>f(x)=\frac{\arcsin x}{\frac{x}{8+5(1-x^2)^{13/30}}}>\lim_{x\to 1}f(x)=4\pi$$

for $0 < x < 1$. The proof of Theorem 1 is complete. □

Remark 2. *In fact, we have the following approximation formulas near the origin:*

$$\begin{aligned}
\arcsin x-\frac{3x}{2+\sqrt{1-x^2}}&=O(x^5),\\
\arcsin x-\frac{\pi x}{2+\sqrt{1-x^2}}&=O(x),\\
\arcsin x-\frac{4\pi x}{8+5(1-x^2)^{13/30}}&=O(x),\\
\arcsin x-\frac{13x}{8+5(1-x^2)^{13/30}}&=O(x^7).
\end{aligned}$$

In inequalities (1) *and* (21)*, the upper bound*

$$\frac{13x}{8+5(1-x^2)^{13/30}}$$

is the best approximation near the origin, in the sense that it has the fastest approximation speed among the four bounds in (1) *and* (21)*.*

By expanding the corresponding functions into their Maclaurin series, we find that, as $x \to 0$,

$$\arcsin x=\frac{\frac{1459}{595}x}{\frac{864}{595}+\big(1-\frac{1741}{1785}x^2\big)^{\frac{1459}{3482}}}+O(x^9). \tag{22}$$

This fact led us to pose the following conjecture:

Conjecture 1. *Let* $0 < x < 1$. *Then*

$$\frac{\frac{1459}{595}x}{\frac{864}{595}+\left(1-\frac{1741}{1785}x^2\right)^{\frac{1459}{3482}}} < \arcsin x. \tag{23}$$

In view of (2), we introduce the approximations family

$$\arctan x \approx \frac{a_1 x}{b_1 + (1+x^2)^{c_1}}, \qquad x \to 0.$$

By expanding the corresponding functions into their Maclaurin series, we determine the values

$$a_1 = \frac{7}{5}, \quad b_1 = \frac{2}{5}, \quad c_1 = \frac{7}{15},$$

which provide the best approximation near the origin:

$$\arctan x \approx \frac{7x}{2+5(1+x^2)^{7/15}}. \tag{24}$$

The formula (24) motivated us to establish Theorem 2.

Theorem 2. *For* $x > 0$,

$$\arctan x < \frac{7x}{2+5(1+x^2)^{7/15}}. \tag{25}$$

Proof. For $x > 0$, let

$$g(x) = \frac{7x}{2+5(1+x^2)^{7/15}} - \arctan x.$$

Differentiation yields

$$\begin{aligned}
&3(2+5(1+x^2)^{7/15})^2(1+x^2)^{23/15}g'(x)\\
&= (30+42x^2)(1+x^2)^{8/15} + 45 + 52x^2 + 7x^4 - 75(1+x^2)(1+x^2)^{7/15}\\
&\overset{t=x^2}{=\!=} (30+42t)(1+t)^{8/15} + 45 + 52t + 7t^2 - 75(1+t)(1+t)^{7/15}\\
&\overset{v=(1+t)^{1/15}}{=\!=\!=} (30+42(v^{15}-1))v^8 + 45 + 52(v^{15}-1) + 7(v^{15}-1)^2 - 75v^{22}\\
&= v^8(-12+38v^7+42v^{15}-75v^{14}+7v^{22})\\
&= v^8(v-1)^3(12+36v+72v^2+120v^3+180v^4+252v^5+336v^6+394v^7\\
&\quad + 426v^8+432v^9+412v^{10}+366v^{11}+294v^{12}+196v^{13}+147v^{14}+105v^{15}\\
&\quad + 70v^{16}+42v^{17}+21v^{18}+7v^{19}) > 0, \qquad v > 1.
\end{aligned}$$

Therefore, the function $g(x)$ is strictly increasing on $(0,\infty)$. And hence,

$$g(x) = \frac{7x}{2+5(1+x^2)^{7/15}} - \arctan x > \lim_{x\to 0} g(x) = 0$$

for $x > 0$. The proof of Theorem 2 is complete. □

Remark 3. *In fact, we have the following approximation formulas near the origin:*

$$\arctan x - \frac{3x}{1+2\sqrt{1+x^2}} = O(x^5),$$
$$\arctan x - \frac{2x}{1+\sqrt{1+x^2}} = O(x^3),$$
$$\arctan x - \frac{7x}{2+5(1+x^2)^{7/15}} = O(x^7).$$

In inequalities (2) *and* (25), *the upper bound*

$$\frac{7x}{2+5(1+x^2)^{7/15}}$$

is the best approximation near the origin, in the sense that it has the fastest approximation speed among the three bounds in (2) *and* (25).

By expanding the corresponding functions into their Maclaurin series, we find that, as $x \to 0$,

$$\arctan x = \frac{\frac{53}{35}x}{\frac{18}{35}+\left(1+\frac{109}{105}x^2\right)^{\frac{53}{109}}} + O(x^9). \tag{26}$$

This fact led us to pose the following conjecture:

Conjecture 2. *Let* $x > 0$. *Then*

$$\arctan x < \frac{\frac{53}{35}x}{\frac{18}{35}+\left(1+\frac{109}{105}x^2\right)^{\frac{53}{109}}}. \tag{27}$$

4. SHARP INEQUALITIES FOR $\operatorname{arctanh} x$

Theorems 3 and 4 develop inequality (3) to produce sharp two-sided inequalities.

Theorem 3. *For* $0 < x < \sqrt{15}/4$, *we have*

$$\frac{\lambda x}{3+\sqrt{25-\frac{80}{3}x^2}} < \operatorname{arctanh} x < \frac{\mu x}{3+\sqrt{25-\frac{80}{3}x^2}}, \tag{28}$$

with the best possible constants

$$\lambda = \frac{4\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{\sqrt{15}}{4}\right) = 6.3933\ldots \quad \text{and} \quad \mu = 8.$$

Proof. Inequality (28) can be written as

$$\lambda < \frac{\operatorname{arctanh} x}{\frac{x}{3+\sqrt{25-\frac{80}{3}x^2}}} < \mu, \qquad 0 < x < \frac{\sqrt{15}}{4}.$$

Let

$$F(x) = \frac{\operatorname{arctanh} x}{\frac{x}{3+\sqrt{25-\frac{80}{3}x^2}}}, \quad F(0) = \lim_{x\to 0} F(x) = 8.$$

Direct computation gives

$$F\left(\frac{\sqrt{15}}{4}\right) = \frac{4\sqrt{15}}{5} \operatorname{arctanh}\left(\frac{\sqrt{15}}{4}\right) = 6.3933\ldots.$$

In order to prove Theorems 3, it suffices to show that $F(x)$ is strictly decreasing for $0 < x < \sqrt{15}/4$. Now let

$$F_1(x) = \operatorname{arctanh} x, \qquad F_2(x) = \frac{x}{3+\sqrt{25-\frac{80}{3}x^2}}.$$

Then, $F(x) = \frac{F_1(x)}{F_2(x)}$. Elementary calculations show that

$$27\frac{F_1'(x)}{F_2'(x)} = \frac{(9+\sqrt{225-240x^2})^2\sqrt{225-240x^2}}{(1-x^2)(25+\sqrt{225-240x^2})} =: F_3(x).$$

Differentiating $F_3(x)$ with respect to x and applying (17) yield, for $0 < x < \sqrt{15}/4$,

$$\begin{aligned}
&-\frac{(1-x^2)^2(25+\sqrt{225-240x^2})^2\sqrt{225-240x^2}}{60x(9+\sqrt{225-240x^2})}F_3'(x)\\
&= (45-28x^2)\sqrt{225-240x^2} + 780x^2 - 675\\
&> (45-28x^2)\frac{15(15-16x^2)}{15-8x^2} + 780x^2 - 675 = \frac{480x^4}{15-8x^2} > 0.
\end{aligned}$$

Hence, $F_3(x)$ and $\frac{F_1'(x)}{F_2'(x)}$ are both strictly decreasing for $0 < x < \sqrt{15}/4$. By Lemma 1, the function

$$F(x) = \frac{F_1(x)}{F_2(x)} = \frac{F_1(x)-F_1(0)}{F_2(x)-F_2(0)}$$

is strictly decreasing for $0 < x < \sqrt{15}/4$. The proof of Theorem 3 is complete. □

Theorem 4. *For* $0 < x < \sqrt{15}/4$, *we have*

$$\frac{8x - px^7}{3+\sqrt{25-\frac{80}{3}x^2}} < \operatorname{arctanh} x < \frac{8x - qx^7}{3+\sqrt{25-\frac{80}{3}x^2}}, \tag{29}$$

with the best possible constants

$$p = \frac{32768}{3375} - \frac{16384\sqrt{15}}{16875} \operatorname{arctanh}\left(\frac{\sqrt{15}}{4}\right) = 1.9499\ldots \quad \textit{and} \quad q = \frac{32}{4725} = 0.0067\ldots.$$

Proof. Inequality (29) can be written as

$$p > \frac{8x - \left(3 + \sqrt{25 - \frac{80}{3}x^2}\,\right) \operatorname{arctanh} x}{x^7} > q, \qquad 0 < x < \frac{\sqrt{15}}{4}.$$

Let

$$h(x) = \frac{8x - \left(3 + \sqrt{25 - \frac{80}{3}x^2}\,\right) \operatorname{arctanh} x}{x^7}.$$

Noting that

$$\begin{aligned} 3 + \sqrt{25 - \frac{80}{3}x^2} &= 3 + 5\left(1 - \frac{16}{15}x^2\right)^{1/2} \\ &= 3 + 5\left(1 - \frac{8}{15}x^2 - \frac{32}{225}x^4 - \frac{256}{3375}x^6 - \frac{512}{10125}x^8 - \ldots\right) \\ &= 8 - \frac{8}{3}x^2 - \frac{32}{45}x^4 - \frac{256}{675}x^6 - \frac{512}{2025}x^8 - \ldots, \end{aligned}$$

we obtain

$$\begin{aligned} h(x) &= \frac{8x - \left(8 - \frac{8}{3}x^2 - \frac{32}{45}x^4 - \frac{256}{675}x^6 - \frac{512}{2025}x^8 + O(x^{10})\right)\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + O(x^{11})\right)}{x^7} \\ &= \frac{32}{4725}x^7 + O(x^9) \end{aligned}$$

and

$$h(0) = \lim_{x \to 0} h(x) = \frac{32}{4725} = 0.0067\ldots.$$

Direct computation gives

$$h\left(\frac{\sqrt{15}}{4}\right) = \frac{32768}{3375} - \frac{16384\sqrt{15}}{16875} \operatorname{arctanh}\left(\frac{\sqrt{15}}{4}\right) = 1.9499\ldots.$$

In order to prove Theorems 4, it suffices to show that $h(x)$ is strictly increasing for $0 < x < \sqrt{15}/4$. Now let

$$h_1(x) = \frac{8x}{3 + \sqrt{25 - \frac{80}{3}x^2}} - \operatorname{arctanh} x, \qquad h_2(x) = \frac{x^7}{3 + \sqrt{25 - \frac{80}{3}x^2}}.$$

Then

$$h(x) = \frac{8x - \left(3 + \sqrt{25 - \frac{80}{3}x^2}\,\right) \operatorname{arctanh} x}{x^7} = \frac{\frac{8x}{3+\sqrt{25-\frac{80}{3}x^2}} - \operatorname{arctanh} x}{\frac{x^7}{3+\sqrt{25-\frac{80}{3}x^2}}} = \frac{h_1(x)}{h_2(x)}.$$

Elementary calculations show that

$$\frac{9}{2}\frac{h_1'(x)}{h_2'(x)} = \frac{225 - 180x^2 - (15 - 4x^2)\sqrt{225 - 240x^2}}{x^6(1 - x^2)(175 - 160x^2 + 7\sqrt{225 - 240x^2})} =: h_3(x).$$

Differentiating $h_3(x)$ with respect to x and noting that (15) and (16) hold, we obtain that for $0 < x < \sqrt{15}/4$,

$$\begin{aligned}
&\frac{x^7(1 - x^2)^2(175 - 160x^2 + 7\sqrt{225 - 240x^2})^2}{60} h_3'(x) \\
&= \frac{17920x^8 - 97120x^6 + 159000x^4 - 103500x^2 + 23625}{\sqrt{225 - 240x^2}} \\
&\quad + 3168x^6 - 7592x^4 + 6060x^2 - 1575 \\
&> (17920x^8 - 97120x^6 + 159000x^4 - 103500x^2 + 23625) \\
&\quad \times \left(\frac{1}{15} + \frac{8}{225}x^2 + \frac{32}{1125}x^4 + \frac{256}{10125}x^6 + \frac{3584}{151875}x^8\right) \\
&\quad + 3168x^6 - 7592x^4 + 6060x^2 - 1575 \\
&= x^8 P(x),
\end{aligned}$$

where

$$P(x) = \frac{1024}{5} - \frac{369664}{675}x^2 + \frac{3657728}{2025}x^4 - \frac{55853056}{30375}x^6 + \frac{12845056}{30375}x^8.$$

We now show that

$$P(x) > 0 \quad \text{for} \quad 0 < x < \frac{\sqrt{15}}{4},$$

it suffices to show that

$$\begin{aligned}
Q(t) = &\left(\frac{1024}{5} - \frac{369664}{675}t + \frac{1}{4}\frac{3657728}{2025}t^2\right) \\
&+ t^2\left(\frac{3}{4}\frac{3657728}{2025} - \frac{55853056}{30375}t + \frac{12845056}{30375}t^2\right) > 0
\end{aligned}$$

for $0 < t < 15/16$ which is true. We thus obtain that $P(x) > 0$ and $h_3'(x) > 0$ for $0 < x < \sqrt{15}/4$. Hence, $h_3(x)$ and $\frac{h_1'(x)}{h_2'(x)}$ are both strictly increasing for $0 < x < \sqrt{15}/4$. By Lemma 1, the function

$$h(x) = \frac{h_1(x)}{h_2(x)} = \frac{h_1(x) - h_1(0)}{h_2(x) - h_2(0)}$$

is strictly increasing for $0 < x < \sqrt{15}/4$. The proof of Theorem 4 is complete. □

By expanding the corresponding functions into their Maclaurin series, we find that, as $x \to 0$,

$$\operatorname{arctanh} x = \frac{3x}{1+2\sqrt{1-x^2}} + O(x^5), \tag{30}$$

$$\operatorname{arctanh} x = \frac{7x}{2+5(1-x^2)^{7/15}} + O(x^7), \tag{31}$$

$$\operatorname{arctanh} x = \frac{53x}{18+35\left(1-\frac{109}{105}x^2\right)^{\frac{53}{109}}} + O(x^9). \tag{32}$$

Remark 4. *Several developed bounds are dependent on the length of the interval. This is apparent, for example, in Theorems 1, 3, and 4. A smaller interval would result in a sharper bound. For* $0 < x < \sqrt{15}/4$, *we have*

$$\frac{8x - px^7}{3+\sqrt{25-\frac{80}{3}x^2}} - \frac{\lambda x}{3+\sqrt{25-\frac{80}{3}x^2}} = \frac{xq(x)}{3+\sqrt{25-\frac{80}{3}x^2}}$$

where

$$q(x) = 8 - \lambda - px^6$$

is strictly decreasing on $(0, \sqrt{15}/4)$ *and*

$$q(x) > q\left(\frac{\sqrt{15}}{4}\right) = 0, \quad 0 < x < \frac{\sqrt{15}}{4}.$$

Hence, the two bounds of inequality (29) *are sharper than those of inequality* (28).

The formula (31) motivated us to establish Theorem 5.

Theorem 5. *For* $0 < x < 1$,

$$\frac{7x}{2+5(1-x^2)^{7/15}} < \operatorname{arctanh} x. \tag{33}$$

Proof. For $0 < x < 1$, let

$$h(x) = \operatorname{arctanh} x - \frac{7x}{2+5(1-x^2)^{7/15}}.$$

Differentiation yields

$$\begin{aligned}
&3(2+5(1-x^2)^{7/15})^2(1-x^2)^{23/15}h'(x)\\
&\quad= -(30-42x^2)(1-x^2)^{8/15}+75(1-x^2)(1-x^2)^{7/15}-45+52x^2-7x^4\\
&\quad\overset{t=x^2}{=\!=} -(30-42t)(1-t)^{8/15}+75(1-t)(1-t)^{7/15}-45+52t-7t^2\\
&\quad\overset{w=(1-t)^{1/15}}{=\!=\!=} -(30-42(1-w^{15}))w^8+75w^{22}-45+52(1-w^{15})-7(1-w^{15})^2\\
&\quad= w^8(12-38w^7+75w^{14}-42w^{15}-7w^{22})\\
&\quad= w^8(1-w)^3(12+36w+72w^2+120w^3+180w^4+252w^5+336w^6+394w^7\\
&\qquad+426w^8+432w^9+412w^{10}+366w^{11}+294w^{12}+196w^{13}+147w^{14}\\
&\qquad+105w^{15}+70w^{16}+42w^{17}+21w^{18}+7w^{19})>0, \qquad 0<w<1.
\end{aligned}$$

Therefore, the function $h(x)$ is strictly increasing on $(0,\infty)$, and we have

$$h(x)=\operatorname{arctanh} x-\frac{7x}{2+5(1-x^2)^{7/15}}>\lim_{x\to 0}h(x)=0 \quad \text{for} \quad 0<x<1.$$

The proof of Theorem 5 is complete. □

Remark 5. *In view of* (30) *and* (31), *we find that for* $0<x<1$,

$$\begin{aligned}
&\frac{(2+5(1-x^2)^{7/15})(1+2\sqrt{1-x^2})}{x}\left(\frac{7x}{2+5(1-x^2)^{7/15}}-\frac{3x}{1+2\sqrt{1-x^2}}\right)\\
&= 1-15(1-x^2)^{7/15}+14\sqrt{1-x^2}\overset{p=(1-x^2)^{1/30}}{=\!=\!=}1-15p^{14}+14p^{15}\\
&= (14p^{13}+13p^{12}+12p^{11}+11p^{10}+10p^9+9p^8+8p^7+7p^6+6p^5\\
&\quad+5p^4+4p^3+3p^2+2p+1)(1-p)^2>0.
\end{aligned}$$

Hence, we have

$$\frac{3x}{1+2\sqrt{1-x^2}}<\frac{7x}{2+5(1-x^2)^{7/15}}<\operatorname{arctanh} x, \quad 0<x<1. \tag{34}$$

The formula (32) led us to pose the following conjecture:

Conjecture 3. *Let* $0<x<\sqrt{11445}/109$. *Then*

$$\operatorname{arctanh} x<\frac{53x}{18+35\left(1-\frac{109}{105}x^2\right)^{\frac{53}{109}}}. \tag{35}$$

Remark 6. *Some computer experiments indicate that among inequalities* (3), (29) *and* (35), *the upper bound*

$$\frac{53x}{18+35\left(1-\frac{109}{105}x^2\right)^{\frac{53}{109}}}$$

is the best, in the sense that it is the smallest one among the three upper bounds in (3), (29) *and* (35)*. Noting that* $\sqrt{15}/4 < \sqrt{11445}/109$*, computer experiment suggests that the inequality* (35) *holds for a larger range.*

5. SHARP INEQUALITIES FOR ARC LEMNISCATE FUNCTIONS

In view of (8), (13) and (14), we introduce the approximations family

$$\operatorname{arcsl} x \approx \frac{a_2 x}{b_2 + (1-x^4)^{c_2}}, \qquad x \to 0,$$

and $a_2, b_2, c_2 \in \mathbb{R}$ are parameters. By expanding the corresponding functions into their Maclaurin series, we determine the values

$$a_2 = \frac{11}{3}, \quad b_2 = \frac{8}{3}, \quad c_2 = \frac{11}{30},$$

which provide the best approximation near the origin:

$$\operatorname{arcsl} x = \frac{\frac{11}{3}x}{\frac{8}{3} + (1-x^4)^{\frac{11}{30}}} + O(x^{13}) = \frac{11x}{8 + 3(1-x^4)^{\frac{11}{30}}} + O(x^{13}). \tag{36}$$

The formula (36) motivated us to establish Theorem 6.

Theorem 6. *For* $0 < x < 1$*,*

$$\frac{\theta_1 x}{8 + 3\left(1-x^4\right)^{\frac{11}{30}}} < \operatorname{arcsl} x < \frac{\theta_2 x}{8 + 3\left(1-x^4\right)^{\frac{11}{30}}}, \tag{37}$$

with the best possible constants

$$\theta_1 = 2B\left(\frac{1}{4}, \frac{1}{2}\right) = 10.48823\ldots \quad \textit{and} \quad \theta_2 = 11.$$

Here $B(x,y)$ *denotes the beta function.*

Proof. Inequality (37) can be written as

$$\theta_1 < \frac{\operatorname{arcsl} x}{\frac{x}{8+3\left(1-x^4\right)^{\frac{11}{30}}}} < \theta_2, \qquad 0 < x < 1.$$

Let

$$I(x) = \frac{\operatorname{arcsl} x}{\frac{x}{8+3\left(1-x^4\right)^{\frac{11}{30}}}}, \quad I(0) = \lim_{x\to 0} I(x) = 11.$$

Direct computation gives

$$I(1) = 2B\left(\frac{1}{4}, \frac{1}{2}\right) = 10.48823\ldots.$$

In order to prove Theorem 6, it suffices to show that $I(x)$ is strictly decreasing for $0 < x < 1$. Now let

$$I_1(x) = \operatorname{arcsl} x, \qquad I_2(x) = \frac{x}{8 + 3\big(1 - x^4\big)^{\frac{11}{30}}}.$$

Then, $I(x) = \frac{I_1(x)}{I_2(x)}$. Elementary calculations show that

$$\frac{1}{5}\frac{I_1'(x)}{I_2'(x)} = \frac{\big(1 - x^4\big)^{2/15}\big(8 + 3(1 - x^4)^{11/30}\big)^2}{40\big(1 - x^4\big)^{19/30} + 15 + 7x^4} =: I_3(x).$$

Differentiating $I_3(x)$ with respect to x yields, for $0 < x < 1$,

$$\begin{aligned}
&-\frac{15\big(1 - x^4\big)^{3/2}\big(40\big(1 - x^4\big)^{19/30} + 15 + 7x^4\big)^2}{8x^3\big(8 + 3\big(1 - x^4\big)^{11/30}\big)} I_3'(x)\\
&= -1200(1 - x^4)(1 - x^4)^{4/15} + (750 - 574x^4)(1 - x^4)^{19/30}\\
&\quad + 450 - 471x^4 + 21x^8\\
&\overset{t=x^4}{=} -1200(1 - t)(1 - t)^{4/15} + (750 - 574t)(1 - t)^{19/30}\\
&\quad + 450 - 471t + 21t^2\\
&\overset{u=(1-t)^{1/30}}{=} -1200u^{38} + \big(750 - 574(1 - u^{30})\big)u^{19}\\
&\quad + 450 - 471(1 - u^{30}) + 21(1 - u^{30})^2\\
&= u^{19}(21u^{41} + 574u^{30} - 1200u^{19} + 429u^{11} + 176)\\
&= u^{19}(1 - u)^2\big(21u^{39} + 42u^{38} + 63u^{37} + 84u^{36} + 105u^{35} + 126u^{34}\\
&\quad + 147u^{33} + 168u^{32} + 189u^{31} + 210u^{30} + 231u^{29} + 826u^{28} + 1421u^{27}\\
&\quad + 2016u^{26} + 2611u^{25} + 3206u^{24} + 3801u^{23} + 4396u^{22} + 4991u^{21}\\
&\quad + 5586u^{20} + 6181u^{19} + 6776u^{18} + 6171u^{17} + 5566u^{16} + 4961u^{15}\\
&\quad + 4356u^{14} + 3751u^{13} + 3146u^{12} + 2541u^{11} + 1936u^{10} + 1760u^{9}\\
&\quad + 1584u^{8} + 1408u^{7} + 1232u^{6} + 1056u^{5} + 880u^{4} + 704u^{3}\\
&\quad + 528u^{2} + 352u + 176\big) > 0, \quad 0 < u < 1.
\end{aligned}$$

Hence, $I_3(x)$ and $\frac{I_1'(x)}{I_2'(x)}$ are both strictly decreasing for $0 < x < 1$. By Lemma 1, the function

$$I(x) = \frac{I_1(x)}{I_2(x)} = \frac{I_1(x) - I_1(0)}{I_2(x) - I_2(0)}$$

is strictly decreasing for $0 < x < 1$. The proof of Theorem 6 is complete. □

By expanding the corresponding functions into their Maclaurin series, we find that, as $x \to 0$,

$$\operatorname{arcslh} x = \frac{\frac{11}{3}x}{\frac{8}{3} + \left(1+x^4\right)^{\frac{11}{30}}} + O(x^{13}), \tag{38}$$

$$\operatorname{arctl} x = \frac{\frac{59}{27}x}{\frac{32}{27} + \left(1+x^4\right)^{\frac{59}{180}}} + O(x^{13}). \tag{39}$$

The formulas (38) and (39) motivated us to establish Theorems 7 and 8, respectively.

Theorem 7. *For $x > 0$,*

$$\frac{11x}{8 + 3\left(1+x^4\right)^{\frac{11}{30}}} < \operatorname{arcslh} x. \tag{40}$$

Proof. For $x > 0$, let

$$J_1(x) = \operatorname{arcslh} x - \frac{11x}{8 + 3\left(1+x^4\right)^{\frac{11}{30}}}.$$

Differentiation yields

$$\begin{aligned}
&5\left(8 + 3(1+x^4)^{11/30}\right)^2 (1+x^4)^{19/30} J_1'(x) \\
&= 320(1+x^4)^{2/15} + 240\sqrt{1+x^4} + 45(1+x^4)^{13/15} - 440(1+x^4)^{19/30} - 165 + 77x^4 \\
&\overset{t=x^4}{=\!=} 320(1+t)^{2/15} + 240\sqrt{1+t} + 45(1+t)^{13/15} - 440(1+t)^{19/30} - 165 + 77t \\
&\overset{v=(1+t)^{1/30}}{=\!=\!=} 320v^4 + 240v^{15} + 45v^{26} - 440v^{19} - 165 + 77(v^{30}-1) \\
&= 320v^4 + 240v^{15} + 45v^{26} - 440v^{19} - 242 + 77v^{30} \\
&= (v-1)^3(77v^{27} + 231v^{26} + 462v^{25} + 770v^{24} + 1200v^{23} + 1752v^{22} + 2426v^{21} \\
&\quad + 3222v^{20} + 4140v^{19} + 5180v^{18} + 6342v^{17} + 7186v^{16} + 7712v^{15} + 7920v^{14} \\
&\quad + 7810v^{13} + 7622v^{12} + 7356v^{11} + 7012v^{10} + 6590v^9 + 6090v^8 + 5512v^7 \\
&\quad + 4856v^6 + 4122v^5 + 3310v^4 + 2420v^3 + 1452v^2 + 726v + 242) > 0, \qquad v > 1.
\end{aligned}$$

Therefore, the function $J_1(x)$ is strictly increasing on $(0, \infty)$. And hence,

$$J_1(x) = \operatorname{arcslh} x - \frac{11x}{8 + 3\left(1+x^4\right)^{\frac{11}{30}}} > \lim_{x \to 0} J_1(x) = 0$$

for $x > 0$. The proof of Theorem 7 is complete. □

Theorem 8. *For $x > 0$,*

$$\frac{59x}{32 + 27\left(1+x^4\right)^{\frac{59}{180}}} < \operatorname{arctl} x. \tag{41}$$

Proof. For $x>0$, let

$$J_2(x)=\operatorname{arctl} x-\frac{59x}{32+27\big(1+x^4\big)^{\frac{59}{180}}}.$$

Differentiation yields

$$\begin{aligned}
&5(1+x^4)^{3/4}\big(32+27(1+x^4)^{59/180}\big)^2J_2'(x)\\
&=5120+8640(1+x^4)^{59/180}+3645(1+x^4)^{59/90}-9440(1+x^4)^{3/4}\\
&\qquad-7965(1+x^4)^{7/90}+2478x^4(1+x^4)^{7/90}\\
&\overset{t=x^4}{=\!=\!=}5120+8640(1+t)^{59/180}+3645(1+t)^{59/90}-9440(1+t)^{3/4}\\
&\qquad-7965(1+t)^{7/90}+2478t(1+t)^{7/90}\\
&\overset{v=(1+t)^{1/180}}{=\!=\!=\!=\!=}5120+8640v^{59}+3645v^{118}-9440v^{135}-7965v^{14}+2478(v^{180}-1)v^{14}\\
&\qquad\qquad=R(v),\qquad v>1,
\end{aligned}$$

where

$$R(v)=2478v^{194}-9440v^{135}+3645v^{118}+8640v^{59}-10443v^{14}+5120.$$

Elementary calculations show that

$$\begin{aligned}
R'(v)&=480732v^{193}-1274400v^{134}+430110v^{117}+509760v^{58}-146202v^{13},\\
R''(v)&=92781276v^{192}-170769600v^{133}+50322870v^{116}+29566080v^{57}-1900626v^{12},\\
R'''(v)&=17814004992v^{191}-22712356800v^{132}+5837452920v^{115}\\
&\qquad+1685266560v^{56}-22807512v^{11},\\
R^{(4)}(v)&=v^{131}(3402474953472v^{59}-2998031097600)+671307085800v^{114}\\
&\qquad+v^{10}(94374927360v^{45}-250882632)>0,\qquad v>1.
\end{aligned}$$

Hence, $R'''(v)$ is strictly increasing on $(1,\infty)$, and we have, for $v>1$,

$$\begin{aligned}
R'''(v)>R'''(1)=2601560160>0&\Longrightarrow R''(v)>R''(1)=0\\
&\Longrightarrow R'(v)>R'(1)=0\Longrightarrow R(v)>R(1)=0.
\end{aligned}$$

We thus obtain $J_2'(x)>0$ for $x>0$. Therefore, the function $J_2(x)$ is strictly increasing on $(0,\infty)$. And hence,

$$J_2(x)=\operatorname{arctl} x-\frac{59x}{32+27\big(1+x^4\big)^{\frac{59}{180}}}>\lim_{x\to0}J_2(x)=0$$

for $x>0$. The proof of Theorem 8 is complete. □

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