

HUYGENS TYPE INEQUALITIES FOR SINHC AND TANHC FUNCTIONS

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In the paper, with aid of the Maclaurin power series expansions of the hyperbolic sine and cosine functions, and in light of stratification method invented and applied by Malešević and his coauthors, the authors establish several inequalities for bounding the sums of the hyperbolic functions $\sinhc(2x)$, $\tanhc x$, and their reciprocals. These inequalities are extensions of the Huygens-type inequalities.

1. INTRODUCTION

For $x \in \mathbb{R}$, the functions

$$\begin{aligned} \operatorname{sinc} x &= \begin{cases} \frac{\sin x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases} & \sinhc x &= \begin{cases} \frac{\sinh x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases} \\ \operatorname{tanc} x &= \begin{cases} \frac{\tan x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases} & \tanhc x &= \begin{cases} \frac{\tanh x}{x}, & x \neq 0; \\ 1, & x = 0 \end{cases} \end{aligned}$$

are called the sinc function, the hyperbolic sinc function, the tanc function, and the hyperbolic tanc function, respectively. The function $\operatorname{sinc} x$ is also called the

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sine cardinal or sampling function, as well as the function $\operatorname{sinhc} x$ is also called the hyperbolic sine cardinal; see the papers [4, 10, 14, 28] and closely related references therein.

In [30], Wilker proposed two open problems below:

1. Let $0 < x < \frac{\pi}{2}$. Then

$$(1) \quad \operatorname{sinc}^2 x + \operatorname{tanc} x > 2;$$

2. There exists a largest constant c such that

$$(2) \quad \operatorname{sinc}^2 x + \operatorname{tanc} x > 2 + cx^3 \tan x$$

for $0 < x < \frac{\pi}{2}$.

In [29], Sumner et al. affirmed the truth of the above problems and obtained

$$(3) \quad 2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \operatorname{sinc}^2 x + \operatorname{tanc} x < 2 + \frac{8}{45} x^3 \tan x, \quad 0 < x < \frac{\pi}{2},$$

where the constants $\left(\frac{2}{\pi}\right)^4$ and $\frac{8}{45}$ in (3) are the best possible.

In [5, 6], Guo and her coauthors gave new proofs of the inequalities (1) and (2).

In [26], using the L'Hospital rules for monotonicity, Pinelis showed alternative proofs of the double inequality (3). In [34], Zhang and Zhu gave a new elementary proof of the double inequality (3).

In [32], Wu and Srivastava established

$$(4) \quad \left(\frac{1}{\operatorname{sinc} x}\right)^2 + \frac{1}{\operatorname{tanc} x} > 2, \quad 0 < |x| < \frac{\pi}{2}.$$

In [35], Zhu presented the hyperbolic counterpart of (1) by

$$(5) \quad (\operatorname{sinhc} x)^2 + \operatorname{tanhc} x > 2, \quad x \in \mathbb{R} \setminus \{0\}.$$

In [31], Wu and Debnath proved the inequality

$$(6) \quad \left(\frac{1}{\operatorname{sinhc} x}\right)^2 + \frac{1}{\operatorname{tanhc} x} > 2, \quad x \in \mathbb{R} \setminus \{0\},$$

which is the hyperbolic counterpart of (4).

In [8], another inequality of interest is the Huygens inequality, which asserts that

$$(7) \quad 2 \operatorname{sinc} x + \operatorname{tanc} x > 3, \quad 0 < |x| < \frac{\pi}{2}.$$

Its hyperbolic counterpart gave in [25] by

$$(8) \quad 2 \sinh x + \tanh x > 3, \quad x \in \mathbb{R} \setminus \{0\}.$$

In [7, 9], Hua and Jiang obtained the sharp inequalities

$$3 + \frac{1}{40}x^3 \sin x < \operatorname{sinc} x + 2 \operatorname{tanc} \frac{x}{2} < 3 + \frac{80 - 24\pi}{\pi^4}x^3 \sin x$$

and

$$3 + \frac{3}{20}x^3 \tanh x < 2 \sinh x + \tanh x < 3 + \frac{3}{20}x^3 \sinh x$$

for $0 < |x| < \frac{\pi}{2}$.

In [2, Theorem 3], Chen and Cheung established the sharp Huygens-type inequalities

$$(9) \quad 3 + \frac{3}{20}x^3 \tan x < 2 \operatorname{sinc} x + \operatorname{tanc} x < 3 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x$$

and

$$3 + \frac{3}{20}x^4 + \frac{3}{56}x^5 \tan x < 2 \operatorname{sinc} x + \operatorname{tanc} x < 3 + \frac{3}{20}x^4 + \left(\frac{2}{\pi}\right)^4 x^5 \tan x$$

for $0 < x < \frac{\pi}{2}$.

In [3, Remark 3.4], Chen and Paris showed the inequality

$$(10) \quad 3 < \frac{2}{\operatorname{sinc} x} + \frac{1}{\operatorname{tanc} x} < 3 + \frac{1}{60}x^3 \tan x, \quad 0 < x < \frac{\pi}{2}.$$

In [10], Li and Guo proved

$$\frac{4}{15} \left(\cos x + \frac{11}{4} \right)^2 - \frac{3}{4} \leq \operatorname{sinc}(2x) + 2 \operatorname{sinc} x \leq \frac{4}{15} \left(\cos x + \frac{11}{4} \right)^2 - \frac{3}{4} + \frac{1}{1260}x^6$$

for $|x| < \frac{\pi}{2}$ and

$$(11) \quad 1 + 2 \cosh x + \sum_{k=2}^n b_k x^{2k} \leq \sinh x + 2 \sinh x \leq \frac{4}{15} \left(\cosh x + \frac{11}{4} \right)^2 - \frac{3}{4}$$

for $x \in \mathbb{R}$, where $b_k = \frac{2^{2k}-4k}{(2k+1)!}$ for $k \geq 2$.

In [33, Theorem 4], the authors derived the series of inequalities

$$(12) \quad \operatorname{tanc} x + (2n+2) \frac{\ln \cos x}{x^2} > \frac{2^{2n+3}(2^{2n+4}-1)|B_{2n+4}|}{(n+2)^2(2n+3)!} x^{2n+2} - \sum_{k=0}^{n-1} \frac{(n-k)2^{2k+2}(2^{2k+2}-1)|B_{2k+2}|}{(k+1)(2k+2)!} x^{2k}$$

for $n \in \mathbb{N}_0$ and $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, where an empty sum is conventionally regarded as 0 and the Bernoulli numbers B_j are generated by the Maclaurin power series expansion

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!} = 1 - \frac{x}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{x^{2j}}{(2j)!}, \quad 0 < |x| < 2\pi.$$

In particular, taking $n = 0, 1$ in (12) gives

$$\operatorname{tanc} x + \frac{2 \ln \cos x}{x^2} > \frac{x^2}{6} \quad \text{and} \quad \operatorname{tanc} x + \frac{4 \ln \cos x}{x^2} > \frac{2x^4}{45} - 1$$

for $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$. For more information on this topic, please refer to the newly published articles [11, 12, 13, 14, 27] and closely related references therein.

In this work, inspired by the results in (1), (4), (5), (6), (7), (8), (9), (10) and (11), basing on the Maclaurin series expansions of the functions $\sinh x$ and $\cosh x$, and using stratification method invented and applied by Malešević and his coauthors, we will establish analogous inequalities involving the hyperbolic functions $\sinhc(2x)$ and $\tanhc x$. These newly established inequalities are extensions of the Huygens-type inequalities mentioned above.

2. INEQUALITIES

Now we are in a position to state and prove our main results.

Theorem 1. *The function*

$$H(x) = \sinhc(2x) + 2 \tanhc x$$

is increasing and $H(x) > 3$ on $x \in (0, \infty)$.

Proof. Let $h_1(x) = \sinh x \cosh^2 x + 2 \sinh x$ and $h_2(x) = x \cosh x$. Then

$$H(x) = \frac{\sinh x \cosh x}{x} + \frac{2 \sinh x}{x \cosh x} = \frac{h_1(x)}{h_2(x)}, \quad x \in (0, \infty).$$

Direct computation yields

$$h'_1(x) = 3 \cosh^3 x, \quad h'_2(x) = \cosh x + x \sinh x,$$

and

$$(13) \quad \frac{H'(x)}{H(x)} = \frac{h'_1(x)h_2(x) - h_1(x)h'_2(x)}{h_1(x)h_2(x)} = \frac{h_3(x)}{8h_1(x)h_2(x)},$$

where

$$h_3(x) = 18x - 10 \sinh(2x) - \sinh(4x) + 4x \cosh(2x) + 2x \cosh(4x).$$

Using the Maclaurin power series of $\sinh x$ and $\cosh x$, we acquire

$$\begin{aligned} h_3(x) &= 18x - 10 \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1} - \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(2n+1)!} x^{2n+1} \\ &\quad + \sum_{n=0}^{\infty} \frac{2^{2n+2}}{(2n)!} x^{2n+1} + \sum_{n=0}^{\infty} \frac{2^{4n+1}}{(2n)!} x^{2n+1} \\ &= \sum_{n=2}^{\infty} \frac{2^{2n+2}}{(2n+1)!} [(2n-1)2^{2n-1} + 2n-4] x^{2n+1} \\ &> 0. \end{aligned}$$

As a result, from the relation (13), it follows that the first derivative $H'(x)$ is positive on $(0, \infty)$. Accordingly, the function $H(x)$ is increasing on $(0, \infty)$. Moreover, it is straightforward that

$$H(x) > \lim_{x \rightarrow 0^+} H(x) = 3.$$

The proof of Theorem 1 is thus complete. $\square$

Theorem 2. *The function*

$$G(x) = \frac{1}{\sinh(2x)} + \frac{2}{\tanh x}$$

is increasing and $G(x) > 3$ on $x \in (0, \infty)$.

Proof. Let $g_1(x) = x + 2x \cosh^2 x$ and $g_2(x) = \sinh x \cosh x$. Then

$$G(x) = \frac{x}{\sinh x \cosh x} + \frac{2x \cosh x}{\sinh x} = \frac{g_1(x)}{g_2(x)}.$$

Direct computation yields

$$g_1'(x) = 2 \cosh^2 x + 4x \sinh x \cosh x + 1, \quad g_2'(x) = \cosh^2 x + \sinh^2 x,$$

and

$$(14) \quad \frac{G'(x)}{G(x)} = \frac{g_1'(x)g_2(x) - g_1(x)g_2'(x)}{g_1(x)g_2(x)} = \frac{g_3(x)}{4g_1(x)g_2(x)},$$

where

$$g_3(x) = -4x + 4 \sinh 2x + \sinh(4x) - 8x \cosh(2x).$$

Employing the Maclaurin power series of $\sinh x$ and $\cosh x$ leads to

$$\begin{aligned} g_3(x) &= -4x + \sum_{n=0}^{\infty} \frac{2^{2n+3}}{(2n+1)!} x^{2n+1} + \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(2n+1)!} x^{2n+1} - \sum_{n=0}^{\infty} \frac{2^{2n+3}}{(2n)!} x^{2n+1} \\ &= \sum_{n=2}^{\infty} \frac{2^{2n+3}}{(2n+1)!} (2^{2n-1} - 2n) x^{2n+1} \\ &> 0. \end{aligned}$$

As a result, from the relation (14), it follows that the first derivative $G'(x)$ is positive, and then the function $G(x)$ is increasing, on $(0, \infty)$. Moreover, it is straightforward that

$$G(x) > \lim_{x \rightarrow 0^+} G(x) = 3.$$

The proof of Theorem 2 is thus complete. $\square$

Theorem 3. For $0 < x < \infty$, we have

$$\sinh(2x) + 2 \tanh x > 3 + \frac{2}{5} x^3 \tanh x,$$

where $\frac{2}{5}$ is the best possible constant.

Proof. Let $f_1(x) = \sinh x \cosh^2 x + 2 \sinh x - 3x \cosh x$ and $f_2(x) = x^4 \sinh x$. Then

$$(15) \quad F(x) = \frac{\frac{\sinh(2x)}{2x} + 2 \frac{\tanh x}{x} - 3}{x^3 \tanh x} = \frac{\sinh(2x) + 4 \tanh x - 6x}{2x^4 \tanh x} = \frac{f_1(x)}{f_2(x)}.$$

Direct computation gives

$$\begin{aligned} f_1'(x) &= \cosh^3 x + 2 \sinh^2 x \cosh x - \cosh x - 3x \sinh x, \\ f_2'(x) &= x^4 \cosh x + 4x^3 \sinh x, \end{aligned}$$

and

$$(16) \quad \frac{F'(x)}{F(x)} = \frac{f_1'(x)x \sinh x - f_1(x)(x \cosh x + 4 \sinh x)}{f_1(x)x \sinh x} = \frac{f_3(x)}{4f_1(x)x \sinh x},$$

where

$$f_3(x) = 12x^2 + 16x \sinh(2x) + x \sinh(4x) - 16 \cosh(2x) - 2 \cosh(4x) + 18.$$

Making use of the Maclaurin power series of $\sinh x$ and $\cosh x$, we arrive at

$$\begin{aligned} f_3(x) &= 12x^2 + \sum_{n=0}^{\infty} \frac{2^{2n+5} x^{2n+2}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{2^{4n+2} x^{2n+2}}{(2n+1)!} \\ &\quad - \sum_{n=0}^{\infty} \frac{2^{2n+4} x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{2^{4n+1} x^{2n}}{(2n)!} - 18 \\ &= 12x^2 + \sum_{n=1}^{\infty} \frac{2^{2n+3} x^{2n}}{(2n-1)!} + \sum_{n=1}^{\infty} \frac{2^{4n-2} x^{2n}}{(2n-1)!} - \sum_{n=1}^{\infty} \frac{2^{2n+4} x^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{2^{4n+1} x^{2n}}{(2n)!} \\ &= \sum_{n=4}^{\infty} \frac{2^{2n+4}}{(2n)!} [2^{2n-5} (n-4) + n-1] x^{2n} \\ &> 0. \end{aligned}$$

As a result, from the relation (16), it follows that the first derivative $F'(x)$ is positive, and then the function $F(x)$ is increasing, on $(0, \infty)$. Moreover, it is straightforward that

$$(17) \quad \lim_{x \rightarrow 0^+} F(x) = \frac{2}{5} < F(x) < \lim_{x \rightarrow \infty} F(x) = \infty.$$

The proof of Theorem 3 is thus complete. $\square$

Theorem 4. For $0 < x < \infty$, the double inequality

$$3 + \alpha x^3 \tanh x < \frac{1}{\sinh(2x)} + \frac{2}{\tanh x} < 3 + \beta x^3 \tanh x,$$

is valid if and only if $\alpha \leq 0$ and $\beta \geq \frac{4}{15}$.

Proof. Let $\ell_1(x) = x + 2x \cosh^2 x - 3 \sinh x \cosh x$ and $\ell_2(x) = x^3 \sinh^2 x$. Then

$$(18) \quad L(x) = \frac{\frac{x}{\sinh x \cosh x} + \frac{2x \cosh x}{\sinh x} - 3}{x^3 \tanh x} = \frac{\ell_1(x)}{\ell_2(x)}.$$

Direct computation results in

$$\begin{aligned} \ell_1'(x) &= -3 \sinh^2 x - \cosh^2 x + 4x \sinh x \cosh x + 1, \\ \ell_2'(x) &= 2x^3 \sinh x \cosh x + 3x^2 \sinh^2 x, \end{aligned}$$

and

$$(19) \quad \frac{L'(x)}{L(x)} = \frac{\ell_1'(x)\ell_2(x) - \ell_1(x)\ell_2'(x)}{\ell_1(x)\ell_2(x)} = -\frac{\ell_3(x)}{4\ell_1(x)x \sinh x},$$

where

$$\ell_3(x) = 9 \cosh x - 9 \cosh(3x) + 24x^2 \cosh x + 4x \sinh(3x).$$

Using the Maclaurin power series of $\sinh x$ and $\cosh x$ leads to

$$\begin{aligned} \ell_3(x) &= \sum_{n=0}^{\infty} \frac{9}{(2n)!} x^{2n} - \sum_{n=0}^{\infty} \frac{3^{2n+2}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{24}{(2n)!} x^{2n+2} + 4 \sum_{n=0}^{\infty} \frac{3^{2n+1}}{(2n+1)!} x^{2n+2} \\ &= \sum_{n=1}^{\infty} \frac{9}{(2n)!} x^{2n} - \sum_{n=1}^{\infty} \frac{3^{2n+2}}{(2n)!} x^{2n} + \sum_{n=1}^{\infty} \frac{24}{(2n-2)!} x^{2n} + 4 \sum_{n=1}^{\infty} \frac{3^{2n-1}}{(2n-1)!} x^{2n} \\ &= \sum_{n=4}^{\infty} \frac{1}{(2n)!} [3^{2n-1}(8n-27) + 96n^2 - 48n + 9] x^{2n} \\ &> 0. \end{aligned}$$

Consequently, from the relation (19) and Theorem 2, it follows that the first derivative $L'(x)$ is negative, and then the function $L(x)$ is decreasing, on $(0, \infty)$. Moreover, it is straightforward that

$$(20) \quad \lim_{x \rightarrow \infty} L(x) = 0 < L(x) < \frac{4}{15} = \lim_{x \rightarrow 0^+} L(x).$$

The proof of Theorem 4 is thus complete. $\square$

3. GENERALIZATIONS OF THEOREMS 3 AND 4

To generalize Theorems 3 and 4, we now introduce the concept of stratification and two key preliminary theorems. The concept of stratification has been invented and applied by Malešević and his coauthors in numerous publications such as [1, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

Definition 1 (Stratification of a family of functions). Let $\{\phi_p(x)\}_{p \in \mathbb{P}}$ be a family of functions, where the independent variable $x \in \mathbb{S} \subseteq \mathbb{R}$, the parameter $p \in \mathbb{P} \subseteq \mathbb{R}$, with $\mathbb{S} \neq \emptyset$ and $\mathbb{P} \neq \emptyset$.

1. For any $x \in \mathbb{S}$ and any $p_1, p_2 \in \mathbb{P}$, if

$$p_1 < p_2 \iff \phi_{p_1}(x) < \phi_{p_2}(x),$$

then the family $\{\phi_p(x)\}_{p \in \mathbb{P}}$ is said to be increasingly stratified on the set $\mathbb{S}$ with respect to the parameter $p \in \mathbb{P}$.

2. For any $x \in \mathbb{S}$ and any $p_1, p_2 \in \mathbb{P}$, if

$$p_1 < p_2 \iff \phi_{p_1}(x) > \phi_{p_2}(x),$$

then the family $\{\phi_p(x)\}_{p \in \mathbb{P}}$ is said to be decreasingly stratified on the set $\mathbb{S}$ with respect to the parameter $p \in \mathbb{P}$.

Stratification describes the consistent monotonicity of a function family as the parameter changes. This property provides a systematic method of analyzing and generalizing numerous analytic inequalities.

The following two lemmas are the cores of the stratification method. They utilize the limiting behaviors of stratified families to determine the optimal range of constants for which an inequality holds. Their proofs can be found in [18, Theorems 5 and 6].

Lemma 1 ([18, Theorem 5]). Let $\{\phi_p(x)\}_{p \in \mathbb{P}}$ be a family of functions with $x \in (a, b) \subseteq \mathbb{R}$ and $\mathbb{P} \subseteq \mathbb{R}$ such that $\mathbb{P} \neq \emptyset$. If

1. the family of functions $\{\phi_p(x)\}_{p \in \mathbb{P}}$ is increasingly (or decreasingly, respectively) stratified on (a, b) ;
2. there exists a continuous and increasing function $g : (a, b) \rightarrow \mathbb{P}$ satisfying

$$(21) \quad g(x) = p \iff \phi_p(x) = 0;$$

3. there exists the limits $\lim_{x \rightarrow a} g(x) = A \in (-\infty, \infty) \cup \{\pm\infty\}$ and $\lim_{x \rightarrow b} g(x) = B \in (-\infty, \infty) \cup \{\pm\infty\}$ such that $(A, B) \subseteq \mathbb{P}$;

then the following conclusions are valid:

(1) For $p \leq A$ and $x \in (a, b)$, we have

$$\phi_p(x) \leq \phi_A(x) < 0 \quad (\text{or } \phi_p(x) \geq \phi_A(x) > 0, \text{ respectively}).$$

(2) For $p \in (A, B)$ and $x \in (a, b)$, the equation $\phi_p(x) = 0$ has a unique solution $x_0^{(p)} \in (a, b)$ such that $\phi_p(x) > 0$ in $x \in (a, x_0^{(p)})$ and $\phi_p(x) < 0$ in $x \in (x_0^{(p)}, b)$ (or such that $\phi_p(x) < 0$ in $x \in (a, x_0^{(p)})$ and $\phi_p(x) > 0$ in $x \in (x_0^{(p)}, b)$, respectively).

(3) For $p \geq B$ and $x \in (a, b)$, we have

$$\phi_p(x) \geq \phi_A(x) > 0 \quad (\text{or } \phi_p(x) \leq \phi_A(x) < 0, \text{ respectively}).$$

Lemma 2 ([18, Theorem 6]). Let $\{\phi_p(x)\}_{p \in \mathbb{P}}$ be a family of functions with $x \in (a, b) \subseteq \mathbb{R}$ and $\mathbb{P} \subseteq \mathbb{R}$ such that $\mathbb{P} \neq \emptyset$. If

- 1) the family of functions $\{\phi_p(x)\}_{p \in \mathbb{P}}$ is increasingly (or decreasingly, respectively) stratified on (a, b) ,
- 2) there exists a continuous and decreasing function $g : (a, b) \rightarrow \mathbb{P}$ satisfying (21);
- 3) there exists the limits $\lim_{x \rightarrow b} g(x) = A$ and $\lim_{x \rightarrow a} g(x) = B$ in $(-\infty, \infty) \cup \{\pm\infty\}$ such that $(A, B) \subseteq \mathbb{P}$,

then the following conclusions are valid:

i) For $p \leq A$ and $x \in (a, b)$, we have

$$\phi_p(x) \leq \phi_A(x) < 0 \quad (\text{or } \phi_p(x) \geq \phi_A(x) > 0, \text{ respectively});$$

ii) For $p \in (A, B)$ and $x \in (a, b)$, the equation $\phi_p(x) = 0$ has a unique solution $x_0^{(p)} \in (a, b)$ such that $\phi_p(x) < 0$ in $x \in (a, x_0^{(p)})$ and $\phi_p(x) > 0$ in $x \in (x_0^{(p)}, b)$ (or such that $\phi_p(x) > 0$ in $x \in (a, x_0^{(p)})$ and $\phi_p(x) < 0$ in $x \in (x_0^{(p)}, b)$, respectively);

iii) For $p \geq B$ and $x \in (a, b)$, we have

$$\phi_p(x) \geq \phi_A(x) > 0 \quad (\text{or } \phi_p(x) \leq \phi_A(x) < 0, \text{ respectively}).$$

Fully considering and applying Lemma 1 to Theorem 3, we can obtain the following generalization.

Theorem 5. If $p \in (-\infty, \frac{2}{5})$, then

$$\sinh(2x) + 2 \tanh x > 3 + \frac{2}{5}x^3 \tanh x > 3 + px^3 \tanh x, \quad x > 0.$$

The constant $\frac{2}{5}$ is the best possible.

If $p \in (\frac{2}{5}, \infty)$, the equation $\phi_p(x) = 0$ has a unique solution $x_0^{(p)}$ such that

$$3 + \frac{2}{5}x^3 \tanh x < \sinh c(2x) + 2 \tanh c x < 3 + px^3 \tanh x, \quad x \in (0, x_0^{(p)})$$

and

$$\sinh c(2x) + 2 \tanh c x > 3 + px^3 \tanh x > 3 + \frac{2}{5}x^3 \tanh x, \quad x \in (x_0^{(p)}, \infty).$$

Proof. Define the family of functions $\{\phi_p(x)\}_{p \in \mathbb{P}}$ by

$$(22) \quad \phi_p(x) = \sinh c(2x) + 2 \tanh c x - 3 - px^3 \tanh x$$

for $x \in (0, \infty)$ and $p \in \mathbb{P} = \mathbb{R}$. A direct differentiation yields

$$\frac{\partial \phi_p(x)}{\partial p} = -x^3 \tanh x < 0$$

for $x > 0$. Therefore, the family $\{\phi_p(x)\}_{p \in \mathbb{P}}$ is decreasingly stratified on the interval $(0, \infty)$ with respect to the parameter $p \in \mathbb{P} = \mathbb{R}$.

Let $g(x) = F(x)$ for $x \in (0, \infty)$, where $F(x)$ is defined by (15). In the proof of Theorem 3, we proved that the function $F(x)$ is increasing on $(0, \infty)$. Therefore, from the limits in (17), it follows that the relation in (21) holds for $p = \frac{2}{5}$.

From (17), we conclude that $A = \frac{2}{5}$ and $B = \infty$ in Lemma 1.

Combining the above discussion and making use of Lemma 1, we derive that, if and only if $p \leq \frac{2}{5}$,

$$\phi_p(x) \geq \phi_{2/5}(x) > 0, \quad x \in (0, \infty).$$

Accordingly, the inequalities formulated in Theorem 5 and the best possibility of the constant $\frac{2}{5}$ are verified. The proof of Theorem 5 is complete. $\square$

Carefully considering and applying Lemma 2 to Theorem 4, we can obtain the following generalization.

Theorem 6. Let $A = 0$ and $B = \frac{4}{15}$.

1) If $p \in (-\infty, A)$, then

$$3 + px^3 \tanh x < 3 + Ax^3 \tanh x < \frac{1}{\sinh c(2x)} + \frac{2}{\tanh c x} < 3 + Bx^3 \tanh x, \quad x > 0.$$

The constant $A = 0$ is the best possible.

2) If $p \in (A, B)$, the equation $\phi_p(x) = 0$ has a unique solution $x_0^{(p)}$ such that

$$3 + px^3 \tanh x < \frac{1}{\sinh c(2x)} + \frac{2}{\tanh c x} < 3 + Bx^3 \tanh x, \quad x \in (0, x_0^{(p)})$$

and

$$3 < \frac{1}{\sinh c(2x)} + \frac{2}{\tanh c x} < 3 + px^3 \tanh x < 3 + Bx^3 \tanh x, \quad x \in \left(x_0^{(p)}, \frac{4}{15}\right).$$

3) If $p \in (B, \infty)$, then

$$3 < \frac{1}{\sinh(2x)} + \frac{2}{\tanh x} < 3 + Bx^3 \tanh x < 3 + px^3 \tanh x, \quad x > 0.$$

The constant $B = \frac{4}{15}$ is the best possible.

Proof. Define the family of functions $\{\phi_p(x)\}_{p \in \mathbb{P}}$ by

$$(23) \quad \phi_p(x) = \frac{1}{\sinh(2x)} + \frac{2}{\tanh x} - 3 - px^3 \tanh x$$

for $x \in (0, \infty)$ and $p \in \mathbb{P} = \mathbb{R}$. A straightforward differentiation gives

$$\frac{\partial \phi_p(x)}{\partial p} = -x^3 \tanh x < 0$$

for $x > 0$. Therefore, the family $\{\phi_p(x)\}_{p \in \mathbb{P}}$ is decreasingly stratified on the interval $(0, \infty)$ with respect to the parameter $p \in \mathbb{P} = \mathbb{R}$.

Let $g(x) = L(x)$ for $x \in (0, \infty)$, where $L(x)$ is defined by (18). In the proof of Theorem 4, we proved that $L(x)$ is decreasing on $(0, \infty)$. Hence, from the limits in (20), it follows that the relation in (21) holds for $p = 0$ and $p = \frac{4}{15}$.

From (20), we conclude that $A = 0$ and $B = \frac{4}{15}$ in Lemma 2.

Combining the above discussion and making use of Lemma 2, we derive that,

1. if and only if $p \geq \frac{4}{15}$,

$$\phi_p(x) \leq \phi_{4/15}(x) < 0, \quad x \in (0, \infty);$$

2. if and only if $p \leq 0$,

$$\phi_p(x) \geq \phi_0(x) > 0, \quad x \in (0, \infty).$$

Accordingly, the inequalities stated in Theorem 6 and the best possibility of the constants $A = 0$ and $B = \frac{4}{15}$ are verified. The proof of Theorem 6 is complete. $\square$

Remark 1. The family of the functions in (22) can be showed by Figure 1, which are plotted for

$$p = -0.2, \quad -0.1, \quad 0, \quad 0.1, \quad 0.2, \quad 0.3, \quad \frac{2}{5}, \quad 0.43, \quad 0.44, \quad 0.45$$

using the software MATHEMATICA 14.0.0.0.

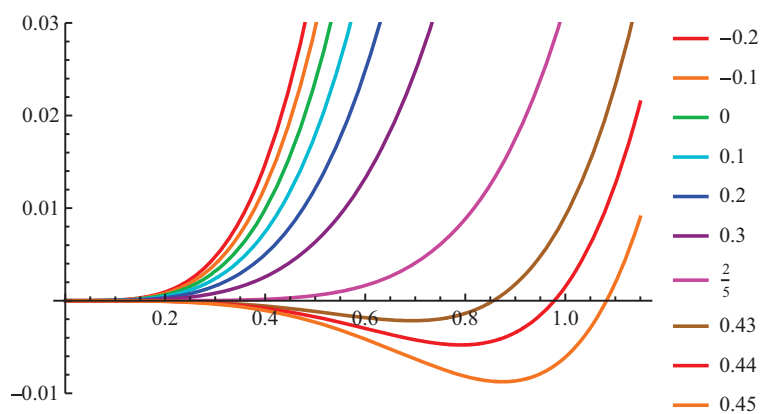


Figure 1: Stratified family of the functions in (22)

The family of the functions in (23) can be showed by Figure 2, which are plotted for

$$p = -\frac{2}{15}, -\frac{1}{15}, 0, \frac{1}{15}, \frac{2}{15}, \frac{1}{5}, \frac{4}{15}, \frac{1}{3}, \frac{2}{5}, 1$$

using the software MATHEMATICA 14.0.0.0.

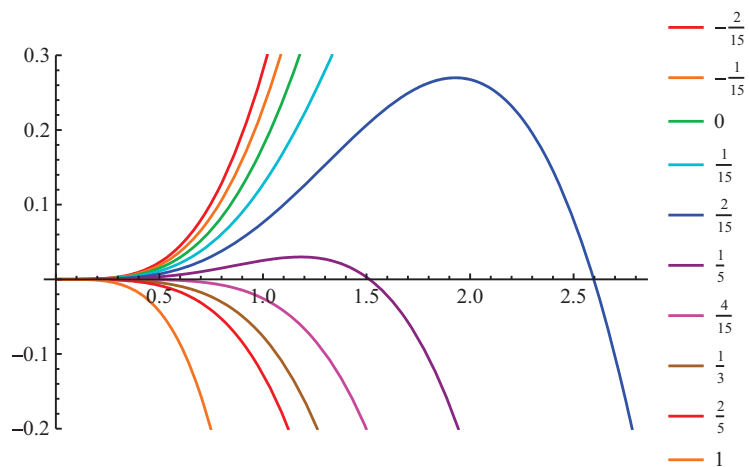


Figure 2: Stratified family of the functions in (23)

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