

HIGHER-ORDER DERIVATIVES OF TWO ARCTANGENT-BASED FUNCTIONS AND THEIR RELATION TO GAUSS HYPERGEOMETRIC FUNCTIONS

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Motivated by a problem on the Maclaurin expansion of $\frac{\arctan \sqrt{2e^{-z}-1}}{\sqrt{2e^{-z}-1}}$, the authors derive higher-order derivatives and series expansions of $\frac{\arctan z}{z}$ and $\frac{\arctan \sqrt{z}}{\sqrt{z}}$, express them via the Gauss hypergeometric functions ${}_2F_1$ and the Chebyshev polynomials $U_n(x)$, and determine their asymptotics as $n \rightarrow \infty$. Several new identities follow, including a trigonometric identity and an infinite series identity from two Fourier–Legendre expansions of the elliptic integral $\mathcal{K}(x)$. Alternative proofs of known ${}_2F_1$ closed forms are also given.

1. PRELIMINARIES

The classical Euler gamma function $\Gamma(z)$ can be defined [21, Chapter 3] by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

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In terms of the Pochhammer symbol, also known as the rising factorial,

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = \begin{cases} \prod_{\ell=0}^{n-1} (z+\ell), & n \in \mathbb{N} = \{1, 2, \dots\} \\ 1, & n = 0 \end{cases}$$

for $z \in \mathbb{C}$, the Gauss hypergeometric function ${}_2F_1$ is defined by

$$(1) \quad {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

for $a, b \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. When $a \in \{0, -1, -2, \dots\}$ or $b \in \{0, -1, -2, \dots\}$, the series in (1) is finite and is a polynomial of $z \in \mathbb{C}$. When $a, b \notin \{0, -1, -2, \dots\}$, the infinite series (1) converges in the unit disc $|z| < 1$ and, on the circle $|z| = 1$, is

1. divergent for $\Re(a+b-c) \geq 1$,
2. absolutely convergent for $\Re(a+b-c) < 0$,
3. conditionally convergent for $0 \leq \Re(a+b-c) < 1$, the point $z = 1$ being excluded.

For $c \in \{0, -1, -2, \dots\}$, if $a \in \{-1, -2, \dots\}$ such that $a > c$ or $b \in \{-1, -2, \dots\}$ such that $b > c$, the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is also defined. The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ can be continued to a single-valued analytic function of $z \in \mathbb{C}$ with the restrictions: $|\arg(-z)| < \pi$ and $a, b \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. For more information on the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$, please refer to [6, Chapter II], [8, p. 1020], [13, Chapter II], [21, Chapter 5], and [22, Chapter 14].

It is well known [8, 15] that the Chebyshev polynomials of the second kind $U_n(x)$ are very important in mathematical sciences and that, in the study of ordinary differential equations [1, Chapter 22], they arise as solutions to the Chebyshev differential equations

$$(1-x^2)y'' - 3xy' + n(n+2)y = 0, \quad n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N};$$

see also [21, Chapter 6]. In [18, p. 1131, Eq. (1.10) and Theorem 1] and [23, p. 433], we find that

$$(2) \quad U_n(x) = \sum_{k=0}^n (-1)^k \binom{n-k}{k} (2x)^{n-2k}$$

and

$$U_n(x) = x^n \sum_{\ell=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2\ell+1} \left(1 - \frac{1}{x^2}\right)^\ell$$

for $|x| < 1$ and $n \in \mathbb{N}_0$, where we used the fact that $\binom{n}{k} = 0$ when $n < k$ for $n, k \in \mathbb{N}_0$ and $\lfloor \lambda \rfloor$ denotes the floor function whose value is the largest integer less than or equal to $\lambda \in \mathbb{R}$; see also [20, Theorem 3.1].

The Hurwitz–Lerch zeta function $\Phi(z, s, a)$ is defined [8, p. 1049] by

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}$$

for $a \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ and $s \in \mathbb{C}$ when $|z| < 1$ or $\Re(s) > 1$ when $|z| = 1$. Let

$$\Omega_a = \begin{cases} \mathbb{C} \setminus [1, \infty), & \Re(a) > 0; \\ \{|z| < 1 : z \in \mathbb{C}\}, & \Re(a) \leq 0. \end{cases}$$

For $|\arg(a)| < \pi$, $s \in \mathbb{C}$, and $z \in \Omega_a$, an asymptotic approximation of the Hurwitz–Lerch zeta function $\Phi(z, s, a)$ for large a and fixed s and z is given [7, Theorem 1] by

$$(3) \quad \Phi(z, s, a) = \frac{1}{1-z} \frac{1}{a^s} + \sum_{n=1}^{N-1} (-1)^n \frac{\text{Li}_{-n}(z)}{n!} \frac{(s)_n}{a^{n+s}} + O\left(\frac{1}{a^{N+s}}\right), \quad N \in \mathbb{N},$$

where $\text{Li}_s(z)$ is the polylogarithm $\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ for all $s \in \mathbb{C}$ and over the open unit disk $|z| < 1$ in the complex plane $\mathbb{C}$. For more information on the polylogarithm $\text{Li}_s(z)$, see [15, p. 611].

Let

$$W(z) = \frac{\arctan \sqrt{2e^{-z}-1}}{\sqrt{2e^{-z}-1}}.$$

In 2009, Herbert S. Wilf (1931–2012) proposed the problem of finding the leading term of the asymptotic behavior of the coefficients a_n in

$$(4) \quad W(z) = \sum_{n=0}^{\infty} a_n z^n.$$

In order to give a general closed-form expression of the coefficients a_n in the power series (4), we need to compute the n th derivatives of the arctangent-based functions $\frac{\arctan z}{z}$ and $\frac{\arctan \sqrt{z}}{\sqrt{z}}$. This computation has been carried out in [17].

In this paper, we further investigate the n th derivatives of the arctangent-based functions $\frac{\arctan z}{z}$ and $\frac{\arctan \sqrt{z}}{\sqrt{z}}$.

2. DERIVATIVES OF $\frac{\arctan z}{z}$ AND RELATED RESULTS

In the proof of [17, p. 9, Theorem 3.1], Qi derived the n th derivative formula

$$(5) \quad \left(\frac{\arctan z}{z} \right)^{(n)} = (-1)^n \frac{n!}{z^n} \left[\frac{\arctan z}{z} + \sum_{k=1}^n \frac{(-1)^k}{2^{k-1}k} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{\ell}{k-\ell-1} \frac{(2z)^{2\ell}}{(1+z^2)^{\ell+1}} \right]$$

for $n \in \mathbb{N}_0$, where an empty sum is conventionally regarded as 0.

In terms of the Chebyshev polynomials of the second kind $U_k(z)$ in (2), we can reformulate the derivative formula (5) as follows.

Theorem 2.1. *For $n \in \mathbb{N}_0$, we have the derivative formula*

$$(6) \quad \left(\frac{\arctan z}{z} \right)^{(n)} = (-1)^n \frac{n!}{z^{n+1}} \left[\arctan z - \frac{1}{\sqrt{1+z^2}} \sum_{k=1}^n \frac{1}{k} \left(\frac{z}{\sqrt{1+z^2}} \right)^k U_{k-1} \left(\frac{z}{\sqrt{1+z^2}} \right) \right],$$

where $U_k(z)$ denotes the Chebyshev polynomials of the second kind.

Proof. Denoting the inner sum in (5) by $S(k-1)$ and replacing $k-\ell-1$ by p in the inner sum in (5) yield

$$S(k-1) = \sum_{p=0}^{k-1} (-1)^{k-1-p} \binom{k-1-p}{p} \frac{(2z)^{2(k-1-p)}}{(1+z^2)^{k-p}}.$$

Accordingly, the sum $S(k)$ can be written as

$$S(k) = (-1)^k \frac{(2z)^k}{(\sqrt{1+z^2})^{k+2}} \sum_{p=0}^{\lfloor k/2 \rfloor} (-1)^p \binom{k-p}{p} \left(\frac{2z}{\sqrt{1+z^2}} \right)^{k-2p}.$$

Employing the formula (2) leads to

$$S(k) = (-1)^k \frac{(2z)^k}{(\sqrt{1+z^2})^{k+2}} U_k \left(\frac{z}{\sqrt{1+z^2}} \right), \quad k \in \mathbb{N}.$$

Substituting this expression into (5) results in the derivative formula (6). The proof of Theorem 2.1 is thus complete. $\square$

Theorem 2.2 ([17, Theorem 3.1]). *The function $\frac{\arctan z}{z}$ can be expanded into the Taylor series*

$$(7) \quad \frac{\arctan z}{z} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{\pi}{4} + T(n) \right] (z-1)^n, \quad |z-1| < \sqrt{2},$$

where

$$T(n) = \sum_{k=1}^n \frac{(-1)^k}{k} \frac{\sin \frac{3k\pi}{4}}{2^{k/2}}, \quad n \in \mathbb{N}_0$$

and an empty sum is conventionally regarded as 0.

Proof. Taking $z = 1$ in (6), using $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, and employing the formula

$$(8) \quad U_n(\cos \theta) = \frac{\sin[(n+1)\theta]}{\sin \theta}$$

in [23, p. 433], we obtain

$$\begin{aligned} \left(\frac{\arctan z}{z} \right)^{(n)} \Big|_{z=1} &= (-1)^n n! \left[\frac{\pi}{4} - \frac{1}{\sqrt{2}} \sum_{k=1}^n \frac{1}{k} \left(\frac{1}{\sqrt{2}} \right)^k U_{k-1} \left(\frac{1}{\sqrt{2}} \right) \right] \\ &= (-1)^n n! \left[\frac{\pi}{4} - \frac{1}{\sqrt{2}} \sum_{k=1}^n \frac{1}{k} \left(\frac{1}{\sqrt{2}} \right)^k U_{k-1} \left(\cos \frac{\pi}{4} \right) \right] \\ &= (-1)^n n! \left[\frac{\pi}{4} - \sum_{k=1}^n \frac{1}{k} \frac{\sin \frac{k\pi}{4}}{2^{k/2}} \right] \\ &= (-1)^n n! \left[\frac{\pi}{4} + \sum_{k=1}^n \frac{(-1)^k \sin \frac{3k\pi}{4}}{k 2^{k/2}} \right] \\ &= (-1)^n n! \left[\frac{\pi}{4} + T(n) \right] \end{aligned}$$

for $n \in \mathbb{N}_0$. The series expansion (7) is thus proved. The proof of Theorem 2.2 is complete. $\square$

Remark 2.1. By the proof of Theorem 2.2, we can express $T(n)$ by

$$T(n) = - \sum_{k=1}^n \frac{1}{k} \frac{\sin \frac{k\pi}{4}}{2^{k/2}}, \quad n \in \mathbb{N}_0.$$

Remark 2.2. When taking $z = \cot \theta$ for $0 < \theta \leq \frac{\pi}{2}$ in the derivative formula (6), by the relation $\frac{z}{\sqrt{1+z^2}} = \cos \theta$ and the formula (8), we derive the derivative formula

$$(9) \quad \left(\frac{\arctan z}{z} \right)^{(n)} \Big|_{z=\cot \theta} = (-1)^n \frac{n!}{\cot^{n+1} \theta} \left[\frac{\pi}{2} - \theta - \sum_{k=1}^n \frac{\cos^k \theta \sin(k\theta)}{k} \right].$$

Theorem 2.3. For $0 < \theta \leq \frac{\pi}{2}$, we have the identity

$$(10) \quad \sum_{k=1}^{\infty} \frac{\cos^k \theta \sin(k\theta)}{k} = \frac{\pi}{2} - \theta.$$

Proof. Direct computation yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\cos^k \theta \sin(k\theta)}{k} &= -\Im [\ln(1 - e^{i\theta} \cos \theta)] \\ &= -\Im [\ln \sin \theta + \ln(\sin \theta - i \cos \theta)] \end{aligned}$$

$$\begin{aligned}
&= -\Im \ln(\sin \theta - i \cos \theta) \\
&= -\Im [\ln |\sin \theta - i \cos \theta| + i \arg(\sin \theta - i \cos \theta)] \\
&= -\arg(\sin \theta - i \cos \theta) \\
&= -\arg[e^{(\theta - \pi/2)i}] \\
&= \frac{\pi}{2} - \theta.
\end{aligned}$$

The proof of Theorem 2.3 is complete. $\square$

Remark 2.3. Taking $\theta = \frac{\pi}{4}$ in (10) results in a BBP-like series

$$\pi = \frac{1}{4} \sum_{p=0}^{\infty} \frac{1}{2^{4p}} \left[\frac{8}{8p+1} + \frac{8}{8p+2} + \frac{4}{8p+3} - \frac{2}{8p+5} - \frac{2}{8p+6} - \frac{1}{8p+7} \right].$$

Setting $\theta = \frac{\pi}{3}$ and $\theta = \frac{\pi}{6}$ in (10), we can deduce the BBP-like representations of the irrational number $\pi\sqrt{3}$. For more information on BBP type sums, please refer to the papers [2, 11] and closely related references therein.

Theorem 2.4. For $0 < \theta < \frac{\pi}{2}$, as $n \rightarrow \infty$, we have the asymptotic approximation

$$(11) \quad \left(\frac{\arctan z}{z} \right)^{(n)} \Big|_{z=\cot \theta} = (-1)^n (n-1)! \sin^n \theta \left[\cos(n\theta) + O\left(\frac{1}{n}\right) \right].$$

Proof. By virtue of the formula (10), we can rewrite the formula (9) as

$$\begin{aligned}
\left(\frac{\arctan z}{z} \right)^{(n)} \Big|_{z=\cot \theta} &= (-1)^n \frac{n!}{\cot^{n+1} \theta} \sum_{k=n+1}^{\infty} \frac{1}{k} \cos^k \theta \sin(k\theta) \\
&= (-1)^n \frac{n! \sin^{n+1} \theta}{\cos^{n+1} \theta} \sum_{k=0}^{\infty} \frac{\cos^{n+k+1} \theta \sin[(n+k+1)\theta]}{n+k+1} \\
&= (-1)^n n! \sin^{n+1} \theta \sum_{k=0}^{\infty} \frac{\cos^k \theta \sin[(n+k+1)\theta]}{n+k+1} \\
&= (-1)^n n! \sin^{n+1} \theta \sum_{k=0}^{\infty} \frac{\cos^k \theta \Im e^{i(n+k+1)\theta}}{n+k+1} \\
&= (-1)^n n! \sin^{n+1} \theta \Im [e^{i(n+1)\theta} \Phi(\cos \theta e^{i\theta}, 1, n+1)].
\end{aligned}$$

Hence, applying the asymptotic approximation (3), we arrive at the asymptotic approximation (11). The proof of Theorem 2.4 is complete. $\square$

3. DERIVATIVES OF $\frac{\arctan \sqrt{z}}{\sqrt{z}}$ AND RELATED RESULTS

Continuing from Section 2, we now discuss the higher derivatives of the function $\frac{\arctan \sqrt{z}}{\sqrt{z}}$ in terms of the Chebyshev polynomials of the second kind $U_n(x)$ with the aid of the Gauss hypergeometric function ${}_2F_1$.

Theorem 3.5. For $n \in \mathbb{N}$, we have the derivative formula

$$(12) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} = (-1)^n \frac{(2n-1)!!}{2^n} \frac{1}{z^n} \left[\frac{\arctan \sqrt{z}}{\sqrt{z}} - \frac{1}{\binom{2n}{n}} \frac{1}{\sqrt{z(1+z)}} \sum_{s=1}^n \frac{2^s}{s} \binom{2n-s}{n} \left(\sqrt{\frac{z}{1+z}} \right)^s U_{s-1} \left(\sqrt{\frac{z}{1+z}} \right) \right].$$

Proof. In [17, pp. 11–12], Qi derived

$$\begin{aligned} & \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} \\ &= \sum_{k=0}^n (-1)^{n+k} \left(\frac{\arctan w}{w} \right)^{(k)} \Big|_{w=\sqrt{z}} z^{k/2-n} \frac{[2(n-k)-1]!!}{2^n} \binom{2n-k-1}{2(n-k)}, \end{aligned}$$

where $w = \sqrt{z}$. This expression can be written as

$$(13) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} = \frac{(-1)^n}{2^{2n} z^n} \sum_{k=1}^n (-1)^k 2^k z^{k/2} \frac{(2n-k-1)!}{(n-k)!(k-1)!} \left(\frac{\arctan w}{w} \right)^{(k)} \Big|_{w=\sqrt{z}}$$

and, in view of (6),

$$(14) \quad \begin{aligned} & \left(\frac{\arctan w}{w} \right)^{(k)} \Big|_{w=\sqrt{z}} \\ &= \frac{(-1)^k k!}{z^{(k+1)/2}} \left[\arctan \sqrt{z} - \frac{1}{\sqrt{1+z}} \sum_{s=1}^k \frac{1}{s} \left(\sqrt{\frac{z}{1+z}} \right)^s U_{s-1} \left(\sqrt{\frac{z}{1+z}} \right) \right]. \end{aligned}$$

In [17, p. 14], Qi proved that

$$(15) \quad \frac{(n-1)!}{2^n} \sum_{k=1}^n \binom{2n-k-1}{n-1} 2^k k = \frac{(n-1)!}{2^n} \binom{2n}{n} n = (2n-1)!!, \quad n \in \mathbb{N},$$

which can be rearranged as

$$(16) \quad \sum_{k=0}^n 2^k k \frac{(2n-k-1)!}{(n-k)!} = (n-1)! \sum_{k=0}^n 2^k k \binom{2n-k-1}{n-1} = \frac{(2n)!}{n!}, \quad n \in \mathbb{N}.$$

Accordingly, we obtain

$$\begin{aligned} & -\frac{1}{\sqrt{1+z}} \sum_{k=1}^n 2^k k \frac{(2n-k-1)!}{(n-k)!} \sum_{s=1}^k \frac{1}{s} \left(\sqrt{\frac{z}{1+z}} \right)^s U_{s-1} \left(\sqrt{\frac{z}{1+z}} \right) \\ &= -\frac{1}{\sqrt{1+z}} \sum_{s=1}^n \frac{1}{s} \left(\sqrt{\frac{z}{1+z}} \right)^s U_{s-1} \left(\sqrt{\frac{z}{1+z}} \right) \sum_{k=s}^n 2^k k \frac{(2n-k-1)!}{(n-k)!} \end{aligned}$$

$$= -\frac{1}{\sqrt{1+z}} \sum_{s=1}^n \frac{2^s}{s} \frac{(2n-s)!}{(n-s)!} \left(\sqrt{\frac{z}{1+z}} \right)^s U_{s-1} \left(\sqrt{\frac{z}{1+z}} \right),$$

where we used the identity

$$\sum_{k=s}^n 2^k k \binom{2n-k-1}{n-1} = 2^s n \binom{2n-s}{n}$$

for $n \in \mathbb{N}$ and $1 \leq s \leq n$, which can be found in [17, p. 6, Lemma 2.1]. Consequently, combining the last result and (16) with (13) and (14) leads to the derivative formula (12). The proof of Theorem 3.5 is complete. $\square$

Remark 3.4. Letting $z = \cot^2 \theta$ in (12) and using (8) yield

$$\begin{aligned} & \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} \Big|_{z=\cot^2 \theta} \\ &= (-1)^n \frac{n!}{2^{2n}} \tan^{2n+1} \theta \left[\binom{2n}{n} \left(\frac{\pi}{2} - \theta \right) - \sum_{s=1}^n \frac{1}{s} \binom{2n-s}{n} (2 \cos \theta)^s \sin(s\theta) \right]. \end{aligned}$$

Remark 3.5. For $z = 1$ in (12), in view of the identity (15), we recover a result

$$(17) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} \Big|_{z=1} = (-1)^n \frac{(2n-1)!!}{2^n} \left[\frac{\pi}{4} - \frac{1}{\binom{2n}{n}} \sum_{s=1}^n \frac{2^{s/2}}{s} \binom{2n-s}{n} \sin \frac{s\pi}{4} \right]$$

given in the proof of [17, pp. 11–12, Theorem 3.2].

Theorem 3.6. For each fixed z with $|z| < 1$, we have the asymptotic approximation

$$(18) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} \sim (-1)^n \frac{(n-1)!}{2(1+z)^n}, \quad n \rightarrow \infty.$$

Proof. In [21, p. 109, Example 5.1], we find

$$(19) \quad \frac{\arctan \sqrt{z}}{\sqrt{z}} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2n+1} = {}_2F_1 \left(\frac{1}{2}, 1; \frac{3}{2}; -z \right).$$

In [4, p. 80] and [13, p. 41], we find the differentiation formula

$$(20) \quad \frac{d^n}{dz^n} {}_2F_1(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} {}_2F_1(a+n, b+n; c+n; z).$$

Then we obtain

$$\begin{aligned} (21) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} &= (-1)^n \frac{\left(\frac{1}{2}\right)_n (1)_n}{\left(\frac{3}{2}\right)_n} {}_2F_1 \left(n + \frac{1}{2}, n+1; n + \frac{3}{2}; -z \right) \\ &= (-1)^n \frac{n!}{2n+1} {}_2F_1 \left(n + \frac{1}{2}, n+1; n + \frac{3}{2}; -z \right). \end{aligned}$$

Applying the Euler transform

$${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z)$$

in [1, p. 559, Entry 15.3.3] to (21), we obtain

$$(22) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} = (-1)^n \frac{n!}{2n+1} \frac{1}{(1+z)^n} {}_2F_1\left(\frac{1}{2}, 1; n + \frac{3}{2}; -z\right).$$

In [15, p. 397, Entry 15.12.3], we find the asymptotic formula

$$(23) \quad {}_2F_1(a, b; c + \lambda; z) \sim \frac{\Gamma(c + \lambda)}{\Gamma(c - b + \lambda)} \sum_{s=0}^{\infty} \frac{q_s(z)(b)_s}{\lambda^{s+b}}, \quad \lambda \rightarrow \infty$$

where $q_0(z) = 1$ and $q_s(z)$ for $s \in \mathbb{N}$ are defined by the generating function

$$\left(\frac{e^t - 1}{t} \right)^{b-1} e^{t(1-c)} \left(1 - z + \frac{z}{e^t} \right)^{-a} = \sum_{s=0}^{\infty} q_s(z) t^s.$$

If $\arg(1 - z) < \pi$, then (23) applies when $|\arg \lambda| < \frac{\pi}{2} - \delta$. If $\Re(z) \leq \frac{1}{2}$, then (23) applies when $|\arg \lambda| \leq \pi - \delta$.

Applying (23) to (22), we obtain (18). The proof of Theorem 3.6 is complete. $\square$

Theorem 3.7. *For $n \in \mathbb{N}$, we have*

$$(24) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} = \frac{(-1)^n (2n)!}{2^{2n} n!} \frac{1}{z^n} \left[\frac{\arctan \sqrt{z}}{\sqrt{z}} - \frac{1}{1+z} \sum_{k=0}^{n-1} \frac{2^{2k}}{(2k+1) \binom{2k}{k}} \left(\frac{z}{1+z} \right)^k \right].$$

Proof. The derivative formula (12) can be written as

$$(25) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} = \frac{(-1)^n (2n)!}{2^{2n} n!} \frac{1}{z^n} \left[\frac{\arctan \sqrt{z}}{\sqrt{z}} - \frac{1}{1+z} A_n \left(\frac{z}{1+z} \right) \right],$$

where

$$(26) \quad A_n(x) = \frac{2}{\binom{2n}{n}} \sum_{s=0}^{n-1} \frac{1}{s+1} \binom{2n-s-1}{n} (2\sqrt{x})^s U_s(\sqrt{x}), \quad n \in \mathbb{N}.$$

Straightforward computation gives

$$A_{n+1}(x) - A_n(x) = \frac{2}{\binom{2n+2}{n+1}} \frac{1}{n+1} (2\sqrt{x})^n U_n(\sqrt{x})$$

$$\begin{aligned}
& + 2 \sum_{s=0}^{n-1} \frac{1}{s+1} \left[\frac{\binom{2n-s+1}{n+1}}{\binom{2n+2}{n+1}} - \frac{\binom{2n-s-1}{n}}{\binom{2n}{n}} \right] (2\sqrt{x})^s U_s(\sqrt{x}) \\
& = \frac{2}{\binom{2n+2}{n+1}} \frac{1}{n+1} (2\sqrt{x})^n U_n(\sqrt{x}) \\
& \quad + \frac{(n-1)!n!}{(2n+1)!} \sum_{s=0}^{n-1} s \binom{2n-s-1}{n-1} (2\sqrt{x})^s U_s(\sqrt{x}) \\
& = \frac{(n-1)!n!}{(2n+1)!} \sum_{s=1}^n s \binom{2n-s-1}{n-1} (2\sqrt{x})^s U_s(\sqrt{x}) \\
& = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} x^n,
\end{aligned}$$

where we used in the last step the identity

$$\sum_{k=1}^n k \binom{2n-k-1}{n-1} (2x)^k U_k(x) = n(2x)^{2n}, \quad n \in \mathbb{N}$$

in [18, Theorem 2]. Considering the initial value $A_0(x) = 0$, we derive

$$(27) \quad A_n(x) = \sum_{k=0}^{n-1} \frac{2^{2k}}{(2k+1)\binom{2k}{k}} x^k, \quad n \in \mathbb{N}_0.$$

Substituting this expression into (25) gives (24). The proof of Theorem 3.7 is thus complete. $\square$

Remark 3.6. From (27), it is easy to see that each term of the summation $A_n(x)$ defined in (26) is even with respect to $\sqrt{x}$, and hence $A_n(x)$ is a polynomial of degree $n-1$ in x .

Remark 3.7. In [12, Theorem 3], Li and Qi established

$$\begin{aligned}
(28) \quad {}_2F_1\left(n + \frac{1}{2}, n+1; n + \frac{3}{2}; -z^2\right) \\
= \begin{cases} \frac{(2n+1)!!}{(2n)!!} \frac{1}{z^{2n}} \left[\frac{\arctan z}{z} - \frac{\mathcal{Q}_{n-1}(z^2)}{(1+z^2)^n} \right], & z \neq 0 \\ 1, & z = 0 \end{cases}
\end{aligned}$$

for $|z| < 1$ and $n \in \mathbb{N}_0$, where $\mathcal{Q}_{-1}(z) = 0$ and

$$\mathcal{Q}_n(z) = \sum_{k=0}^n \left[\sum_{j=0}^k \frac{(-1)^j}{1+2j} \binom{1+n}{k-j} \right] z^k, \quad n \in \mathbb{N}_0.$$

Substituting (28) into (21) yields the derivative formula

$$\left(\frac{\arctan \sqrt{z}}{\sqrt{z}} \right)^{(n)} = (-1)^n \frac{(2n-1)!!}{(2z)^n} \left[\frac{\arctan \sqrt{z}}{\sqrt{z}} - \frac{\mathcal{Q}_{n-1}(z)}{(1+z)^n} \right].$$

Theorem 3.8. For $n \in \mathbb{N}_0$ and $\left|\frac{z}{1+z}\right| < 1$, we have the derivative formula

$$(29) \quad \left(\frac{\arctan \sqrt{z}}{\sqrt{z}}\right)^{(n)} = \begin{cases} (-1)^n \frac{n!}{2n+1}, & z = 0; \\ \frac{(-1)^n (2n)!}{2^{2n} n!} \frac{1}{z^n(1+z)} \sum_{k=n}^{\infty} \frac{2^{2k}}{(2k+1) \binom{2k}{k}} \left(\frac{z}{1+z}\right)^k, & z \neq 0. \end{cases}$$

First proof of Theorem 3.8. Applying Euler's hypergeometric transform

$$(30) \quad {}_2F_1(a, b; c; z) = \frac{1}{(1-z)^b} {}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right)$$

in [1, p. 559, Entry 15.3.5] to the expression (21) gives

$$\left(\frac{\arctan \sqrt{z}}{\sqrt{z}}\right)^{(n)} = (-1)^n \frac{n!}{2n+1} \frac{1}{(1+z)^{n+1}} {}_2F_1\left(1, n+1; n+\frac{3}{2}; \frac{z}{1+z}\right).$$

Then, using the property

$$(31) \quad (x+n)_k = \frac{(x)_{n+k}}{(x)_n}$$

in [3, p. 17], we arrive at

$$\begin{aligned} \left(\frac{\arctan \sqrt{z}}{\sqrt{z}}\right)^{(n)} &= (-1)^n \frac{n!}{2n+1} \frac{1}{(1+z)^{n+1}} \sum_{k=0}^{\infty} \frac{(n+1)_k}{\left(n+\frac{3}{2}\right)_k} \left(\frac{z}{1+z}\right)^k \\ &= (-1)^n \frac{n!}{2n+1} \frac{1}{(1+z)^{n+1}} \frac{\left(\frac{3}{2}\right)_n}{(1)_n} \sum_{k=0}^{\infty} \frac{(1)_{n+k}}{\left(\frac{3}{2}\right)_{n+k}} \left(\frac{z}{1+z}\right)^k \\ &= (-1)^n \frac{\Gamma\left(\frac{3}{2}+n\right)}{2n+1} \frac{1}{(1+z)^{n+1}} \sum_{k=0}^{\infty} \frac{k!}{\Gamma\left(\frac{3}{2}+k\right)} \left(\frac{z}{1+z}\right)^k \end{aligned}$$

for $n \in \mathbb{N}$. Using the recursive relation $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ in [21, pp. 42 and 46], we derive

$$\Gamma\left(\frac{3}{2}+p\right) = \Gamma\left(\frac{1}{2}\right) \prod_{\ell=0}^p \left(\frac{1}{2}+\ell\right) = \frac{\sqrt{\pi}}{2^{2p+1}} \frac{(2p+1)!}{p!}, \quad p \in \mathbb{N}_0,$$

and then the equation (29) follows.

With $n = 0$, we obtain the hypergeometric Euler transform for the function $\frac{\arctan \sqrt{z}}{\sqrt{z}}$. Substituting this case into (24) yields (29). The first proof of Theorem 3.8 is complete. $\square$

Second proof of Theorem 3.8. By virtue of the differentiation formula (20), we can write the n th derivative of the function ${}_2F_1(a, b; a + 1; z)$ as

$$[{}_2F_1(a, b; a + 1; z)]^{(n)} = \frac{(a)_n(b)_n}{(a + 1)_n} {}_2F_1(a + n, b + n; a + 1 + n; z), \quad n \in \mathbb{N}_0.$$

Applying (30) gives

$$\begin{aligned} [{}_2F_1(a, b; a + 1; z)]^{(n)} &= \frac{(a)_n(b)_n}{(a + 1)_n} \frac{1}{(1 - z)^{b+n}} {}_2F_1\left(1, b + n; a + 1 + n; \frac{z}{z - 1}\right) \\ &= \frac{(a)_n(b)_n}{(a + 1)_n} \frac{1}{(1 - z)^{b+n}} \sum_{k=0}^{\infty} \frac{(b + n)_k}{(a + 1 + n)_k} \left(\frac{z}{z - 1}\right)^k. \end{aligned}$$

Using the identity (31), we can write

$$\begin{aligned} [{}_2F_1(a, b; a + 1; z)]^{(n)} &= \frac{(a)_n}{(1 - z)^{b+n}} \sum_{k=0}^{\infty} \frac{(b)_{n+k}}{(a + 1)_{n+k}} \left(\frac{z}{z - 1}\right)^k \\ &= \frac{(a)_n}{(1 - z)^{b+n}} \left(\frac{z}{z - 1}\right)^{-n} \sum_{k=n}^{\infty} \frac{(b)_k}{(a + 1)_k} \left(\frac{z}{z - 1}\right)^k \\ &= \frac{(a)_n}{(-z)^n(1 - z)^b} \sum_{k=n}^{\infty} \frac{(b)_k}{(a + 1)_k} \left(\frac{z}{z - 1}\right)^k. \end{aligned}$$

Further taking $(a, b; c) = (\frac{1}{2}, 1; \frac{3}{2})$, replacing z by $-z$, and employing (19) lead to (29). The second proof of Theorem 3.8 is complete. $\square$

4. APPLICATIONS OF FOURIER-LEGENDRE EXPANSIONS

Because the Legendre polynomials $P_n(x)$ form a complete orthogonal system over $[-1, 1]$ with respect to the unit weight $w(x) = 1$, any function $f(x) \in L^2(-1, 1)$ may be expanded as

$$(32) \quad f(x) = \sum_{n=0}^{\infty} a_n P_n(x);$$

see [10, pp. 508–512, Section 7.14]. Define the generating function $F(x)$ associated with the Legendre coefficients a_n by

$$(33) \quad F(x) = \sum_{n=0}^{\infty} a_n x^n,$$

which is analytic in a domain containing the relevant points $x + \sqrt{x^2 - 1} \cos \phi$ and the segment/path used for its Taylor expansion about u .

Theorem 4.9. *Let the series (32) be a Fourier–Legendre expansion, and suppose that the function $F(x)$ defined in (33) is analytic in a domain containing the set*

$$\{x + \sqrt{x^2 - 1} \cos \phi : 0 \leq \phi \leq \pi\}$$

and a neighborhood of the line segment joining u to these points. Then, whenever the Taylor expansion of F about u converges uniformly on this set,

$$(34) \quad f(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(u)}{n!} (u^2 - 2ux + 1)^{n/2} P_n \left(\frac{x - u}{\sqrt{u^2 - 2ux + 1}} \right).$$

Proof. In [22, p. 313], we find the integral representation

$$(35) \quad P_n(z) = \frac{1}{\pi} \int_0^\pi (z + \sqrt{z^2 - 1} \cos \phi)^n d\phi$$

for all $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$. Since the integrand in (35) is unchanged under $\phi \rightarrow \pi - \phi$, the choice of the branch of the two-valued function $\sqrt{z^2 - 1}$ is clearly immaterial. Expanding F in its Taylor series about u yields

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{a_n}{\pi} \int_0^\pi (x + \sqrt{x^2 - 1} \cos \phi)^n d\phi \\ &= \frac{1}{\pi} \int_0^\pi F(x + \sqrt{x^2 - 1} \cos \phi) d\phi \\ &= \frac{1}{\pi} \int_0^\pi \left[\sum_{n=0}^{\infty} \frac{F^{(n)}(u)}{n!} (x - u + \sqrt{x^2 - 1} \cos \phi)^n \right] d\phi \\ &= \sum_{n=0}^{\infty} \frac{F^{(n)}(u)}{n!} (u^2 - 2ux + 1)^{n/2} P_n \left(\frac{x - u}{\sqrt{u^2 - 2ux + 1}} \right), \end{aligned}$$

where, by the assumed uniform convergence of the power series for F on the relevant compact set, termwise integration is justified. The proof of Theorem 4.9 is thus complete. $\square$

Remark 4.8. By choosing $u = 1, x, 2x, \frac{1}{x}, \dots$ in (34), one can obtain more interesting expansions. For example, choosing $u = 1$ leads to

$$(36) \quad f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{F^{(n)}(u)|_{u=1}}{n!} \left(2\sqrt{\frac{1-x}{2}} \right)^n P_n \left(\sqrt{\frac{1-x}{2}} \right).$$

Theorem 4.10. *The Legendre polynomials $P_n(x)$ satisfy*

$$(37) \quad \sum_{n=0}^{\infty} \left[\sum_{s=1}^n \frac{2^{s/2}}{s} \binom{2n-s}{n} \sin \frac{s\pi}{4} \right] \left(\frac{x}{2} \right)^n P_n(x) = 0, \quad |x| < \frac{1}{2}.$$

Proof. The complete elliptic integral of the first kind

$$\mathcal{K}(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2 \sin^2 \theta}}, \quad |x| < 1$$

has a classical Fourier–Legendre expansion

$$(38) \quad \mathcal{K}(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{2}{2n+1} P_n(2x-1), \quad x \in (0, 1);$$

see [5, Eq. (8)].

Using the parity property

$$(39) \quad P_n(z) = (-1)^n P_n(-z), \quad n \in \mathbb{N}_0$$

in [15, p. 444, Table 18.6.1], we obtain

$$F(x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^n = \frac{2 \arctan \sqrt{x}}{\sqrt{x}}.$$

When $u = 1$, then the expansion (36) becomes

$$\mathcal{K}(x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\arctan \sqrt{u}}{\sqrt{u}} \right)^{(n)} \Big|_{u=1} (2x)^n P_n(x).$$

Due to the asymptotic behavior (18) in Theorem 3.6 and

$$|P_n(x)| = O\left(\frac{1}{\sqrt{n}}\right), \quad n \rightarrow \infty$$

for fixed $x \in (-1, 1)$, see [8, p. 996, Entry 8.918], utilizing (17), we obtain

$$(40) \quad \mathcal{K}(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{x}{2}\right)^n P_n(x) \left[\frac{\pi}{2} - \frac{2}{\binom{2n}{n}} \sum_{s=1}^n \frac{1}{s} \binom{2n-s}{n} 2^{s/2} \sin \frac{s\pi}{4} \right].$$

On the other hand, the Legendre polynomials $P_n(x)$ can be generated [8, p. 997, Entry 8.921] by

$$(41) \quad \frac{1}{\sqrt{1-2tz+t^2}} = \sum_{n=0}^{\infty} P_n(z) t^n, \quad |t| < \min |z \pm \sqrt{z^2-1}|.$$

Setting $z = x$ and $t = 2x \sin^2 \theta$ and $|x| < \frac{1}{2}$ in (41), we obtain

$$\frac{1}{\sqrt{1-x^2 \sin^2(2\theta)}} = \sum_{n=0}^{\infty} (2x)^n P_n(x) \sin^{2n} \theta.$$

Integrating this equality between $\theta = 0$ and $\theta = \frac{\pi}{2}$ results in

$$(42) \quad \mathcal{K}(x) = \frac{\pi}{2} \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{x}{2}\right)^n P_n(x), \quad |x| < \frac{1}{2}.$$

Comparing (40) with (42) gives (37). The proof of Theorem 4.10 is complete. $\square$

Remark 4.9. Replacing x by x^2 in (38) and using the parity property (39), we obtain

$$\mathcal{K}(x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(1-2x^2), \quad 0 \leq x < \frac{1}{2}.$$

Comparing this with (42) yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(1-2x^2) = \frac{\pi}{4} \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{x}{2}\right)^n P_n(x).$$

Theorem 4.11 ([9, Chapter 33]). *For $n \in \mathbb{N}$, we have*

$$(43) \quad \sum_{k=1}^n N(n, k) = C_n,$$

where

$$N(n, k+1) = \frac{1}{n} \binom{n}{k} \binom{n}{k+1} \quad \text{and} \quad C_n = \frac{1}{n+1} \binom{2n}{n}$$

are respectively called the Narayana numbers and the Catalan numbers.

Proof. In [15, p. 442, Entry 18.5.7], we find

$$P_n^{(a,b)}(z) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)\Gamma(a+1)} {}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-z}{2}\right).$$

Taking $(a, b) = (1, -1-2n)$ and $z = 1-2x$ yields

$${}_2F_1(1-n, -n; 2; x) = \frac{1}{n+1} P_n^{(1, -1-2n)}(1-2x).$$

Since

$$P_n^{(a,b)}(-1) = \frac{\Gamma(-b)}{\Gamma(-b-n)n!} = (-1)^n \binom{n+b}{n},$$

see [13, p. 210], we obtain

$$(44) \quad {}_2F_1(1-n, -n; 2; 1) = \frac{1}{n+1} \binom{2n}{n} = C_n;$$

see also [19, Eq. (1)].

In view of the property $(-x)_k = (-1)^k (x - k + 1)_k$ in [21, p. 72, Eq (2)], we arrive at

$${}_2F_1(1-n, -n; 2; z) = \sum_{k=0}^n \frac{(1-n)_k (-n)_k}{(2)_k} \frac{z^k}{k!} = \sum_{k=0}^{n-1} \frac{(n-k)_k (n+1-k)_k}{k!(k+1)!} z^k$$

and

$$\begin{aligned} \frac{(n-k)_k (n+1-k)_k}{k!(k+1)!} &= \frac{(n-1)!n!}{(n-1-k)!(n-k)!} \frac{1}{k!(k+1)!} \\ &= \frac{1}{n} \binom{n}{k} \binom{n}{k+1} = N(n, k+1). \end{aligned}$$

This sequence satisfies the symmetry $N(n, k+1) = N(n, n-k)$. As a result, we obtain

$${}_2F_1(1-n, -n; 2; z) = \sum_{k=0}^{n-1} N(n, k+1) z^k.$$

Combining this with (44) yields (43). The proof of Theorem 4.11 is complete. $\square$

5. MORE REMARKS

In this section, we provide alternative proofs of [14, Theorems 1.1, 1.2, and 1.3] in the form of remarks.

Remark 5.10 (Alternative proof of [14, Theorems 1.1 and 1.3]). Let

$$f(z) = {}_2F_1(a, b; a+1; z).$$

As in the second proof of Theorem 3.8, we have

$$[f(z)]^{(n)} = \frac{(a)_n (b)_n}{(a+1)_n} {}_2F_1(n+a, n+b; n+a+1; z).$$

Employing Euler's transformation (30) gives

$$\begin{aligned} [f(z)]^{(n)} &= \frac{(a)_n (b)_n}{(a+1)_n} \frac{1}{(1-z)^{b+n}} {}_2F_1\left(1, n+b; n+a+1; \frac{z}{z-1}\right) \\ &= \frac{(a)_n (b)_n}{(a+1)_n} \frac{1}{(1-z)^{b+n}} \sum_{k=0}^{\infty} \frac{(n+b)_k}{(n+a+1)_k} \left(\frac{z}{z-1}\right)^k. \end{aligned}$$

Utilizing the property (31) and shifting the summation index lead to

$$[f(z)]^{(n)} = (a)_n \frac{1}{(-z)^n} \frac{1}{(1-z)^b} \sum_{k=n}^{\infty} \frac{(b)_k}{(a+1)_k} \left(\frac{z}{z-1}\right)^k$$

$$\begin{aligned}
&= (a)_n \frac{1}{(-z)^n} \frac{1}{(1-z)^b} \left[{}_2F_1\left(1, b; a+1; \frac{z}{z-1}\right) - \sum_{k=0}^{n-1} \frac{(b)_k}{(a+1)_k} \left(\frac{z}{z-1}\right)^k \right] \\
&= (a)_n \frac{1}{(-z)^n} \left[f(z) - \frac{1}{(1-z)^b} \sum_{k=0}^{n-1} \frac{(b)_k}{(a+1)_k} \left(\frac{z}{z-1}\right)^k \right].
\end{aligned}$$

In terms of the Gauss hypergeometric function ${}_2F_1$, the last expression can be written as

$$\begin{aligned}
&{}_2F_1(n+a, n+b; n+a+1; z) \\
&= \frac{(a+1)_n}{(b)_n} \frac{1}{(-z)^n} \left[f(z) - \frac{1}{(1-z)^b} \sum_{k=0}^{n-1} \frac{(b)_k}{(a+1)_k} \left(\frac{z}{z-1}\right)^k \right].
\end{aligned}$$

Setting $a = b = \frac{1}{2}$ and replacing z with $-z^2$, one obtains

$$\begin{aligned}
(45) \quad &{}_2F_1\left(n + \frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; -z^2\right) \\
&= \frac{2n+1}{z^{2n}} \left[\frac{\operatorname{arcsinh}(z)}{z} - \frac{1}{\sqrt{1+z^2}} \sum_{k=0}^{n-1} \frac{1}{2k+1} \left(\frac{z^2}{1+z^2}\right)^k \right].
\end{aligned}$$

This yields an alternative proof of [14, Theorems 1.1 and 1.3].

Remark 5.11 (Alternative proof of [14, Theorems 1.2 and 1.3]). For the Gauss hypergeometric function ${}_2F_1(1, n+a; n+b; z)$, a direct operation gives

$$\begin{aligned}
{}_2F_1(1, n+a; n+b; z) &= \sum_{k=0}^{\infty} \frac{(a+n)_k}{(b+n)_k} z^k = \frac{(b)_n}{(a)_n} \sum_{k=0}^{\infty} \frac{(a)_{n+k}}{(b)_{n+k}} z^k \\
&= \frac{(b)_n}{(a)_n} \frac{1}{z^n} \sum_{k=n}^{\infty} \frac{(a)_k}{(b)_k} z^k = \frac{(b)_n}{(a)_n} \frac{1}{z^n} \left[{}_2F_1(1, a; b; z) - \sum_{k=0}^{n-1} \frac{(a)_k}{(b)_k} z^k \right].
\end{aligned}$$

Taking $(a, b) = (\frac{1}{2}, \frac{3}{2})$ and replacing z by z^2 , we obtain

$$(46) \quad {}_2F_1\left(1, n + \frac{1}{2}; n + \frac{3}{2}; z^2\right) = \frac{2n+1}{z^{2n}} \left[\frac{\operatorname{arctanh} z}{z} - \sum_{k=0}^{n-1} \frac{z^{2k}}{2k+1} \right].$$

This is an alternative proof of [14, Theorems 1.2 and 1.3].

Remark 5.12. The technique utilizing the formula (30) can be applied to any hypergeometric function of the form

$$f(z) = {}_2F_1(a, b; a+1; z) = \frac{a}{z^a} B_z(a, 1-b),$$

see [16, p. 455, Entry 28], where

$$B_z(a, b) = \int_0^z t^{a-1} (1-t)^{b-1} dt, \quad \Re(a), \Re(b) > 0, \quad z \in [0, 1]$$

denotes the incomplete beta function [1, p. 263, Entry 6.6.1]. In this case,

$$\begin{aligned} \frac{d^n f(z)}{dz^n} &= \frac{(a)_n}{(-z)^n} \frac{1}{(1-z)^b} \sum_{k=n}^{\infty} \frac{(b)_k}{(a+1)_k} \left(\frac{z}{z-1} \right)^k \\ &= \frac{(a)_n}{(-z)^n} \frac{1}{(1-z)^b} \left[{}_2F_1 \left(1, b; a+1; \frac{z}{z-1} \right) - \sum_{k=0}^{n-1} \frac{(b)_k}{(a+1)_k} \left(\frac{z}{z-1} \right)^k \right] \\ &= \frac{(a)_n}{(-z)^n} \left[{}_2F_1(a, b; a+1; z) - \frac{1}{(1-z)^b} \sum_{k=0}^{n-1} \frac{(b)_k}{(a+1)_k} \left(\frac{z}{z-1} \right)^k \right]. \end{aligned}$$

6. CONCLUSIONS

In order to obtain an explicit expression of the coefficients a_n in the power series (4), we need to calculate the n th derivatives of the functions $\frac{\arctan z}{z}$ and $\frac{\arctan \sqrt{z}}{\sqrt{z}}$. Among many other results, the n th derivatives of the functions $\frac{\arctan z}{z}$ and $\frac{\arctan \sqrt{z}}{\sqrt{z}}$ (as well as their representations in terms of the Gauss hypergeometric functions ${}_2F_1$) were evaluated in two papers [12, 17].

In Theorem 2.1, we reformulated the n th derivative of $\frac{\arctan z}{z}$ obtained in [17] in terms of the Chebyshev polynomials of the second kind $U_n(z)$. In Theorem 2.2 and its proof, we presented an elegant derivation of the Taylor expansion of $\frac{\arctan z}{z}$ at $z = 1$, originally established in [17, Theorem 3.1]. In Theorem 2.3, we derived the trigonometric identity (10), and in Theorem 2.4, we obtained an asymptotic approximation for $\left(\frac{\arctan z}{z}\right)^{(n)}$ as $n \rightarrow \infty$.

In Theorem 3.5, building on the proof of [17, Theorem 3.2] and using Theorem 2.1, we expressed the n th derivative of $\frac{\arctan \sqrt{z}}{\sqrt{z}}$ in terms of the Chebyshev polynomials $U_n(z)$. In Theorem 3.6, we derived an asymptotic approximation for $\left(\frac{\arctan \sqrt{z}}{\sqrt{z}}\right)^{(n)}$ as $n \rightarrow \infty$. Furthermore, in Theorems 3.7 and 3.8, we used the Euler transforms of sequences and power series to obtain alternative representations for these derivatives.

In Theorem 4.9, we investigated the Fourier–Legendre expansion of a general function $f(x)$. In Theorem 4.10, by comparing two forms of the Fourier–Legendre expansion of the complete elliptic integral of the first kind $\mathcal{K}(x)$, we derived the infinite series identity (37). In Remark 4.9, we further obtained two infinite series identities involving the Legendre polynomials $P_n(x)$. In Theorem 4.11, by exploiting the relationship between the Jacobi polynomials $P_n^{(a,b)}(x)$ and the Gauss hypergeometric functions ${}_2F_1$, we recovered the classical identity connecting the Catalan numbers C_n and the Narayana numbers $N(n, k)$.

In Remarks 5.10 and 5.11, we provided alternative proofs of the closed-form expressions (45) and (46), originally established in [14].

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