

PROBABILISTIC MULTI-STIRLING NUMBERS OF THE SECOND KIND AND PROBABILISTIC MULTI-LAH NUMBERS

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Assume that the moment generating function of the random variable Y exists in a neighborhood of the origin. We introduce the probabilistic multi-Stirling numbers of the second kind associated with Y and the probabilistic multi-Lah numbers associated with Y . The aim of this paper is to study some properties, related identities, recurrence relations and explicit expressions of those probabilistic extension numbers in connection with several other special numbers.

1. INTRODUCTION

Assume that Y is a random variable whose moment generating function exists in a neighborhood of the origin. The aim of this paper is to study the probabilistic multi-Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$ associated with Y and the probabilistic multi-Lah numbers $L_Y^{(k_1, k_2, \dots, k_r)}(n, r)$ associated with Y . We investigate some properties, related identities, recurrence relations and explicit expressions for those numbers in connection with the multi-Bernoulli numbers $B_n^{(k_1, k_2, \dots, k_r)}$, the probabilistic Fubini polynomials of order r , $F_n^{(r, Y)}(y)$, associated with Y , the probabilistic Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}_Y$ associated with Y , the unsigned multi-Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right]$, the unsigned Stirling numbers of first kind $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ and the Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}$.

We remark that $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$ is a probabilistic extension of the multi-Stirling number of the second kind $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}$, which reduces to the Stirling number

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of the second kind $\left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}$, for $(k_1, k_2, \dots, k_r) = (1, 1, \dots, 1)$. We also note that $L_Y^{(k_1, k_2, \dots, k_r)}(n, r)$ is a probabilistic extension of the multi-Lah number $L^{(k_1, \dots, k_r)}(n, r)$, which boils down to the unsigned Lah number $L(n, r)$, for $(k_1, k_2, \dots, k_r) = (1, 1, \dots, 1)$. In addition, $\left[\begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right]$ and $B_n^{(k_1, k_2, \dots, k_r)}$ generalize respectively $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ and $\frac{(-1)^n}{r!} B_n^{(r)}$, for $(k_1, k_2, \dots, k_r) = (1, 1, \dots, 1)$, where $B_n^{(r)}$ are the Bernoulli numbers of order r . The common feature of the six numbers $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$, $L_Y^{(k_1, k_2, \dots, k_r)}(n, r)$, $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}$, $L^{(k_1, k_2, \dots, k_r)}(n, r)$, $\left[\begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right]$ and $B_n^{(k_1, k_2, \dots, k_r)}$ is that they are all defined with the help of the multiple logarithm $\text{Li}_{k_1, k_2, \dots, k_r}(t)$, which reduces to the polylogarithm $\text{Li}_{k_1}(t)$, for $r = 1$ and to $\frac{1}{r!}(-\log(1-t))^r$, for $(k_1, k_2, \dots, k_r) = (1, 1, \dots, 1)$.

The outline of this paper is as follows. First, we recall the multiple logarithms, the multi-Lah numbers and the multi-Bernoulli numbers. We remind the reader of the unsigned Stirling numbers of the first kind and the Stirling numbers of the second kind. We recall the probabilistic Lah numbers $L_Y(n, k)$ associated with Y . We remind the reader of the unsigned multi-Stirling numbers of the first kind and the multi-Stirling numbers of the second kind. Finally, we recall the probabilistic Stirling numbers of the second. Section 2 contains the main results of this paper. We derive first that $\text{Li}_{\underbrace{1, 1, \dots, 1}_{r\text{-times}}}(t) = \frac{1}{r!}(-\log(1-t))^r$. Then we

define the probabilistic multi-Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$ associated with Y , and show $\left\{ \underbrace{1, 1, \dots, 1}_{r\text{-times}} \right\}_Y = \left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}_Y$. In Theorem 2.1, we express

$\left\{ \begin{smallmatrix} n+1 \\ k_1, k_2, \dots, k_{r-1}, 1 \end{smallmatrix} \right\}$ as a finite sum involving $\left\{ \begin{smallmatrix} m \\ k_1, k_2, \dots, k_{r-1} \end{smallmatrix} \right\}$, $(m = r-1, \dots, n)$, which generalizes a well-known identity for the Stirling numbers of the second kind to that for the probabilistic Stirling numbers of the second kind associated with Y . In Theorem 2.2, we express $\sum_{m=0}^n B_m^{(k_1, k_2, \dots, k_r)} \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}_Y$ as a finite sum involving the product $\left\{ \begin{smallmatrix} i \\ j \end{smallmatrix} \right\}_Y \left\{ \begin{smallmatrix} n-i \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$. In Theorem 2.3, we represent $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$ as a finite sum involving the triple product $\left\{ \begin{smallmatrix} l \\ m \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} n \\ l \end{smallmatrix} \right\}_Y \left[\begin{smallmatrix} m \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right]$. We define the probabilistic multi-Lah numbers $L_Y^{(k_1, k_2, \dots, k_r)}(n, r)$ associated with Y . In Theorem 2.4, we show that $L_Y^{(k_1, k_2, \dots, k_r)}(n, r) = \sum_{k=r}^n \left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$. In Theorem 2.5, we express $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$ as a finite sum involving the triple product $\left\{ \begin{smallmatrix} l \\ r \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} n \\ m+l \end{smallmatrix} \right\}_Y B_m^{(k_1, k_2, \dots, k_r)}$. In Theorem 2.6, we show the identity

$$\sum_{k=r}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_Y L^{(k_1, k_2, \dots, k_r)}(k, r) = \sum_{k=r}^n \binom{n}{k} \left\{ \begin{smallmatrix} k \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y F_{n-k}^{(r, Y)}(1),$$

where $F_n^{(r, Y)}(y)$ are the probabilistic Fubini polynomials of order r .

Recently, Lah numbers have found practical applications in telecommunications and optics. One notable use is in steganography, where they are employed for hiding data within images. This method offers a significant advantage due to its lower computational complexity compared to alternative techniques like the Dis-

crete Fourier Transform (DFT) and Discrete Wavelet Transform (DWT). Further details on their use in steganography can be found in [14]. For the rest of this section, we recall the facts that are needed throughout this paper.

For integers k_i ($1 \leq i \leq r$), the multiple logarithm of index $(k_1, k_2, \dots, k_r)$ is given by

$$(1) \quad \text{Li}_{k_1, k_2, \dots, k_r}(t) = \sum_{0 < m_1 < m_2 < \dots < m_r} \frac{t^{m_r}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}}, \quad (|t| < 1). \quad (\text{see [20, 25, 27]}).$$

In particular, for $r = 1$, we have

$$\text{Li}_{k_1}(t) = \sum_{m=1}^{\infty} \frac{t^m}{m^{k_1}}, \quad (\text{see [20, 25, 29]}),$$

where $\text{Li}_{k_1}(t)$ is called the polylogarithm.

Recently, the multi-Lah number of index $(k_1, k_2, \dots, k_r)$ is defined by

$$(2) \quad \frac{\text{Li}_{k_1, k_2, \dots, k_r}(1 - e^{-t})}{(1 - t)^r} = \sum_{n=r}^{\infty} L^{(k_1, \dots, k_r)}(n, r) \frac{t^n}{n!}, \quad (\text{see [21, 26]}).$$

Note that

$$(3) \quad \sum_{n=r}^{\infty} L^{\overbrace{(1, 1, \dots, 1)}^{r\text{-times}}}(n, r) \frac{t^n}{n!} = \frac{\text{Li}_{1, 1, \dots, 1}(1 - e^{-t})}{(1 - t)^r} = \frac{1}{r!} \left(\frac{t}{1 - t} \right)^r = \sum_{n=r}^{\infty} L(n, r) \frac{t^n}{n!}.$$

Here $L(n, r)$ are the unsigned Lah numbers given by

$$(4) \quad \langle x \rangle_n = \sum_{k=0}^n L(n, k)(x)_k, \quad (n \geq 0), \quad (\text{see [12, 18, 19, 21 - 26, 28]}),$$

where $\langle x \rangle_0 = 1$, $\langle x \rangle_n = x(x+1) \cdots (x+n-1)$, ($n \geq 1$), and $(-1)^n \langle -x \rangle_n = (x)_n$.

In [20], the multi-Bernoulli number of index $(k_1, k_2, \dots, k_r)$ is defined by Kim-Kim as

$$(5) \quad \frac{1}{(1 - e^{-t})^r} \text{Li}_{k_1, k_2, \dots, k_r}(1 - e^{-t}) = \sum_{n=0}^{\infty} B_n^{(k_1, k_2, \dots, k_r)} \frac{t^n}{n!}.$$

Note that $B_n^{\overbrace{(1, 1, \dots, 1)}^{r\text{-times}}} = \frac{(-1)^n}{r!} B_n^{(r)}$, where $B_n^{(r)}$ are the Bernoulli numbers of order r , given by

$$\left(\frac{t}{e^t - 1} \right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!}, \quad (\text{see [1 - 13, 15 - 35]}).$$

The unsigned Stirling numbers of the first kind are defined by

$$(6) \quad \langle x \rangle_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad (n, k \geq 0), \quad (\text{see [18, 19, 21]}).$$

The Stirling numbers of the second kind are defined by

$$(7) \quad x^n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (x)_k, \quad (n \geq 0), \quad (\text{see [8, 12, 30]}).$$

Throughout this paper, we assume that Y is random variable such that the moment generating function of Y ,

$$(8) \quad E[e^{tY}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[Y^n], \quad (|t| < r), \quad \text{exists for some } r > 0.$$

Let $(Y_j)_{j \geq 1}$ be a sequence of mutually independent copies of random variable Y , and let

$$S_k = Y_1 + Y_2 + \cdots + Y_k, \quad (k \geq 1), \quad \text{with } S_0 = 0.$$

The probabilistic Lah numbers associated with Y are given by

$$(9) \quad \frac{1}{k!} \left(E \left[\left(\frac{1}{1-t} \right)^Y \right] - 1 \right)^k = \sum_{n=k}^{\infty} L_Y(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [2, 11, 21, 22]}).$$

When $Y = 1$, $L_Y(n, k) = L(n, k)$, $(n \geq k \geq 0)$.

From (9), we note that

$$L_Y(n, k) = \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} E[\langle S_l \rangle_n], \quad (\text{see [28]}).$$

In [25], the unsigned multi-Stirling numbers of the first kind are defined by

$$(10) \quad \text{Li}_{k_1, k_2, \dots, k_r}(t) = \sum_{n=r}^{\infty} \begin{bmatrix} n \\ k_1, k_2, \dots, k_r \end{bmatrix} \frac{t^n}{n!}.$$

From (10), we note that

$$(11) \quad \sum_{n=r}^{\infty} \underbrace{\begin{bmatrix} n \\ 1, 1, \dots, 1 \end{bmatrix}}_{r\text{-times}} \frac{t^n}{n!} = \text{Li}_{\underbrace{1, 1, \dots, 1}_{r\text{-times}}}(t) = \frac{1}{r!} (-\log(1-t))^r = \sum_{n=r}^{\infty} \begin{bmatrix} n \\ r \end{bmatrix} \frac{t^n}{n!}.$$

Thus, by (11), we get

$$\underbrace{\begin{bmatrix} n \\ 1, 1, \dots, 1 \end{bmatrix}}_{r\text{-times}} = \begin{bmatrix} n \\ r \end{bmatrix}, \quad (n \geq r \geq 0).$$

As the inverse relation of (10), the multi-Stirling numbers of the second kind are given by

$$(12) \quad \text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-e^t} \right) = \sum_{n=r}^{\infty} \left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_r \end{matrix} \right\} \frac{t^n}{n!}, \quad (\text{see [25]}).$$

From (12), we note that

$$(13) \quad \sum_{n=r}^{\infty} \left\{ \underbrace{1, 1, \dots, 1}_{r\text{-times}} \right\} \frac{t^n}{n!} = \text{Li}_{\underbrace{1, 1, \dots, 1}_{r\text{-times}}} \left(1 - e^{1-e^t} \right) = \frac{1}{r!} (e^t - 1)^r = \sum_{n=r}^{\infty} \left\{ \begin{matrix} n \\ r \end{matrix} \right\} \frac{t^n}{n!}.$$

Thus, by (13), we get

$$(14) \quad \left\{ \underbrace{1, 1, \dots, 1}_{r\text{-times}} \right\} = \left\{ \begin{matrix} n \\ r \end{matrix} \right\}, \quad (n \geq r \geq 1).$$

By (11) and (12), we get for $n \geq 0$ and $r \in \mathbb{N}$, we have

$$\left\{ \begin{matrix} n+1 \\ k_1, k_2, \dots, k_{r-1}, 1 \end{matrix} \right\} = \sum_{m=r-1}^n \binom{n}{m} \left\{ \begin{matrix} m \\ k_1, k_2, \dots, k_{r-1} \end{matrix} \right\}, \quad (\text{see [25]}).$$

The probabilistic Stirling numbers of the second kind associated with Y are defined by

$$(15) \quad \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} E[S_j^n], \quad (n \geq k \geq 0), \quad (\text{see [21–23, 26, 33]}).$$

When $Y = 1$,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} j^n = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$$

2. PROBABILISTIC MULTI-STIRLING NUMBERS OF THE SECOND KIND AND PROBABILISTIC MULTI-LAH NUMBERS

For $k_i \geq 1$, ($i = 1, 2, \dots, r$), by (1), we get

$$(16) \quad \begin{aligned} \frac{d}{dt} \text{Li}_{k_1, k_2, \dots, k_r}(t) &= \frac{d}{dt} \sum_{0 < m_1 < m_2 < \dots < m_r} \frac{t^{m_r}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}} \\ &= \sum_{0 < m_1 < \dots < m_r} \frac{t^{m_r-1}}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r-1}} = \frac{1}{t} \text{Li}_{k_1, k_2, \dots, k_{r-1}, k_r-1}(t). \end{aligned}$$

Letting $k_r = 1$ in (16), we have

$$\begin{aligned}
 (17) \quad \frac{d}{dt} \text{Li}_{k_1, k_2, \dots, k_{r-1}, 1}(t) &= \sum_{0 < m_1 < m_2 < \dots < m_{r-1}} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_{r-1}^{k_{r-1}}} \sum_{m_r = m_{r-1} + 1}^{\infty} t^{m_r - 1} \\
 &= \frac{1}{1-t} \sum_{0 < m_1 < \dots < m_{r-1}} \frac{t^{m_{r-1}}}{m_1^{k_1} m_2^{k_2} \dots m_{r-1}^{k_{r-1}}} = \frac{1}{1-t} \text{Li}_{k_1, k_2, \dots, k_{r-1}}(t).
 \end{aligned}$$

Thus, by (17), we get

$$\text{Li}_{1,1}(t) = \frac{1}{2!} (-\log(1-t))^2, \quad \text{Li}_{1,1,1}(t) = \frac{1}{3!} (-\log(1-t))^3.$$

Continuing this process, we have

$$(18) \quad \underbrace{\text{Li}_{1,1,\dots,1}}_{r\text{-times}}(t) = \frac{1}{r!} (-\log(1-t))^r, \quad (r \geq 1).$$

Now, we define the *probabilistic multi-Stirling numbers of the second kind associated with Y* as

$$(19) \quad \text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-E[e^{Yt}]} \right) = \sum_{n=r}^{\infty} \left\{ k_1, k_2, \dots, k_r \right\}_Y \frac{t^n}{n!}.$$

When $Y = 1$, we have

$$\left\{ k_1, k_2, \dots, k_r \right\}_Y = \left\{ k_1, k_2, \dots, k_r \right\}.$$

Note that

$$\begin{aligned}
 (20) \quad \sum_{n=r}^{\infty} \underbrace{\left\{ 1, 1, \dots, 1 \right\}}_{r\text{-times}} \frac{t^n}{n!} &= \underbrace{\text{Li}_{1,1,\dots,1}}_{r\text{-times}} \left(1 - e^{1-E[e^{Yt}]} \right) \\
 &= \frac{1}{r!} \left(E[e^{Yt}] - 1 \right)^r = \frac{1}{r!} \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} E[e^{S_j t}] \\
 &= \sum_{n=0}^{\infty} \frac{1}{r!} \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} E[S_j^n] \frac{t^n}{n!}.
 \end{aligned}$$

Thus, by (15) and (20), we get

$$(21) \quad \underbrace{\left\{ 1, 1, \dots, 1 \right\}}_{r\text{-times}} \frac{t^n}{n!} = \left\{ \begin{matrix} n \\ r \end{matrix} \right\}_Y, \quad (n \geq r \geq 1).$$

From (1), we note that

$$\begin{aligned}
 (22) \quad & \frac{d}{dt} \text{Li}_{k_1, k_2, \dots, k_{r-1}, 1} \left(1 - e^{1-E[e^{Yt}]} \right) = \frac{d}{dt} \sum_{0 < m_1 < m_2 < \dots < m_r} \frac{(1 - e^{1-E[e^{Yt}]})^{m_r}}{m_1^{k_1} m_2^{k_2} \dots m_{r-1}^{k_{r-1}} m_r} \\
 &= \sum_{0 < m_1 < \dots < m_{r-1}} \frac{E[Y e^{Yt}] e^{1-E[e^{Yt}]} }{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \sum_{n_r=m_{r-1}+1}^{\infty} \left(1 - e^{1-E[e^{Yt}]} \right)^{m_{r-1}} \\
 &= \sum_{0 < m_1 < \dots < m_{r-1}} \frac{E[Y e^{Yt}] e^{1-E[e^{Yt}]} }{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \frac{(1 - e^{1-E[e^{Yt}]})^{m_{r-1}}}{e^{1-E[e^{Yt}]}} \\
 &= E[Y e^{Yt}] \text{Li}_{k_1, k_2, \dots, k_{r-1}} \left(1 - e^{1-E[e^{Yt}]} \right) \\
 &= \sum_{l=0}^{\infty} E[Y^{l+1}] \frac{t^l}{l!} \sum_{m=r-1}^{\infty} \left\{ \begin{matrix} m \\ k_1, \dots, k_{r-1} \end{matrix} \right\}_Y \frac{t^m}{m!} \\
 &= \sum_{n=r-1}^{\infty} \sum_{m=r-1}^n \binom{n}{m} \left\{ \begin{matrix} m \\ k_1, k_2, \dots, k_{r-1} \end{matrix} \right\}_Y E[Y^{n-m+1}] \frac{t^n}{n!}.
 \end{aligned}$$

On the other hand, by (19), we get

$$\begin{aligned}
 (23) \quad & \frac{d}{dt} \text{Li}_{k_1, k_2, \dots, k_{r-1}, 1} \left(1 - e^{1-E[e^{Yt}]} \right) = \frac{d}{dt} \sum_{n=r}^{\infty} \left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_{r-1}, 1 \end{matrix} \right\}_Y \frac{t^n}{n!} \\
 &= \sum_{n=r}^{\infty} \left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_{r-1}, 1 \end{matrix} \right\}_Y \frac{t^{n-1}}{(n-1)!} \\
 &= \sum_{n=r-1}^{\infty} \left\{ \begin{matrix} n+1 \\ k_1, k_2, \dots, k_{r-1}, 1 \end{matrix} \right\}_Y \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by (21), (22) and (23), we obtain the following theorem,

Theorem 1. For $n, r \in \mathbb{N}$ with $n \geq r-1$, we have

$$\sum_{m=r-1}^n \binom{n}{m} E[Y^{n-m+1}] \left\{ \begin{matrix} m \\ k_1, k_2, \dots, k_{r-1} \end{matrix} \right\}_Y = \left\{ \begin{matrix} n+1 \\ k_1, k_2, \dots, k_{r-1}, 1 \end{matrix} \right\}_Y.$$

In particular, for $n \geq r$, we have

$$\sum_{m=r-1}^{n-1} \binom{n-1}{m} \left\{ \begin{matrix} m \\ r-1 \end{matrix} \right\}_Y E[Y^{n-m}] = \left\{ \begin{matrix} n \\ r \end{matrix} \right\}_Y, \quad \sum_{m=r-1}^{n-1} \binom{n-1}{m} \left\{ \begin{matrix} m \\ r-1 \end{matrix} \right\}_Y = \left\{ \begin{matrix} n \\ r \end{matrix} \right\}_Y.$$

Replacing t by $E[e^{Yt}] - 1$ in (5), on the one hand, we get

$$\begin{aligned}
 (24) \quad & \frac{1}{(1 - e^{1-E[e^{Yt}]})^r} \text{Li}_{k_1, k_2, \dots, k_{r-1}, k_r} \left(1 - e^{1-E[e^{Yt}]} \right) = \sum_{m=0}^{\infty} B_m^{(k_1, \dots, k_r)} \frac{1}{m!} \left(E[e^{Yt}] - 1 \right)^m \\
 & = \sum_{m=0}^{\infty} B_m^{(k_1, k_2, \dots, k_r)} \sum_{n=m}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_Y \frac{t^n}{n!} \\
 & = \sum_{n=0}^{\infty} \sum_{m=0}^n B_m^{(k_1, k_2, \dots, k_r)} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_Y \frac{t^n}{n!}.
 \end{aligned}$$

On the other hand, by (19), we get

$$\begin{aligned}
 (25) \quad & \frac{1}{(1 - e^{1-E[e^{Yt}]})^r} \text{Li}_{k_1, k_2, \dots, k_{r-1}, k_r} \left(1 - e^{1-E[e^{Yt}]} \right) \\
 & = \sum_{l=0}^{\infty} \binom{r+l-1}{l} e^{l(1-E[e^{Yt}])} \sum_{m=r}^{\infty} \left\{ \begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \frac{t^m}{m!} \\
 & = \sum_{l=0}^{\infty} \binom{r+l-1}{l} \sum_{j=0}^{\infty} l^j (-1)^j \frac{1}{j!} \left(E[e^{Yt}] - 1 \right)^j \sum_{m=r}^{\infty} \left\{ \begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \frac{t^m}{m!} \\
 & = \sum_{l=0}^{\infty} \binom{r+l-1}{l} \sum_{j=0}^{\infty} l^j (-1)^j \sum_{i=j}^{\infty} \left\{ \begin{matrix} i \\ j \end{matrix} \right\}_Y \frac{t^i}{i!} \sum_{m=r}^{\infty} \left\{ \begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \frac{t^m}{m!} \\
 & = \sum_{l=0}^{\infty} \binom{r+l-1}{l} \sum_{i=0}^{\infty} \left(\sum_{j=0}^i l^j (-1)^j \left\{ \begin{matrix} i \\ j \end{matrix} \right\}_Y \frac{t^i}{i!} \right) \sum_{m=r}^{\infty} \left\{ \begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \frac{t^m}{m!} \\
 & = \sum_{n=r}^{\infty} \left(\sum_{l=0}^{\infty} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n}{i} \binom{r+l-1}{l} l^j (-1)^j \left\{ \begin{matrix} i \\ j \end{matrix} \right\}_Y \left\{ \begin{matrix} n-i \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by (24) and (25), we obtain the following theorem.

Theorem 2. For $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$\begin{aligned}
 & \sum_{m=0}^n B_m^{(k_1, k_2, \dots, k_r)} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_Y \\
 & = \sum_{l=0}^{\infty} \sum_{i=0}^{n-r} \sum_{j=0}^i \binom{n}{i} \binom{r+l-1}{l} l^j (-1)^j \left\{ \begin{matrix} i \\ j \end{matrix} \right\}_Y \left\{ \begin{matrix} n-i \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y.
 \end{aligned}$$

Replacing t by $1 - e^{1-E[e^{Yt}]}$ in (10), we get

$$\begin{aligned}
 (26) \quad \text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-E[e^{Yt}]} \right) &= \sum_{m=r}^{\infty} \left[\begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right] \frac{1}{m!} \left(1 - e^{1-E[e^{Yt}]} \right)^m \\
 &= \sum_{m=r}^{\infty} \left[\begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right] (-1)^m \sum_{l=m}^{\infty} \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \frac{1}{l!} \left(1 - E[e^{Yt}] \right)^l \\
 &= \sum_{l=r}^{\infty} \sum_{m=r}^l \left[\begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right] (-1)^{m-l} \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \sum_{n=l}^{\infty} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}_Y \frac{t^n}{n!} \\
 &= \sum_{n=r}^{\infty} \sum_{m=r}^l \left[\begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right] (-1)^{m-l} \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \sum_{l=r}^n \left\{ \begin{matrix} n \\ l \end{matrix} \right\}_Y \frac{t^n}{n!} \\
 &= \sum_{n=r}^{\infty} \left(\sum_{l=r}^n \sum_{m=r}^l \left[\begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right] (-1)^{m-l} \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}_Y \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by (19) and (26), we obtain the following theorem.

Theorem 3. For $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$\left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y = \sum_{l=r}^n \sum_{m=r}^l (-1)^{m-l} \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}_Y \left[\begin{matrix} m \\ k_1, k_2, \dots, k_r \end{matrix} \right].$$

Now, we consider the *probabilistic multi-Lah numbers associated with Y* which are given by

$$(27) \quad \text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-E\left[\left(\frac{1}{1-t}\right)^Y\right]} \right) = \sum_{n=r}^{\infty} L_Y^{(k_1, k_2, \dots, k_r)}(n, r) \frac{t^n}{n!}.$$

By (18), we note that

$$\begin{aligned}
 (28) \quad \sum_{n=r}^{\infty} L_Y^{\overbrace{(1, 1, \dots, 1)}^{r\text{-times}}}(n, r) \frac{t^n}{n!} &= \text{Li}_{\underbrace{1, 1, \dots, 1}_{r\text{-times}}} \left(1 - e^{1-E\left[\left(\frac{1}{1-t}\right)^Y\right]} \right) \\
 &= \frac{1}{r!} \left(E \left[\left(\frac{1}{1-t} \right)^Y \right] - 1 \right)^r \\
 &= \sum_{n=r}^{\infty} L_Y(n, r) \frac{t^n}{n!}.
 \end{aligned}$$

Thus, by (28), we get

$$L_Y^{\overbrace{(1, 1, \dots, 1)}^{r\text{-times}}}(n, r) = L_Y(n, r), \quad (n \geq r > 0).$$

Using (27) and (19) with t replaced by $-\log(1-t)$, we get

$$\begin{aligned}
 (29) \quad \sum_{n=r}^{\infty} L_Y^{(k_1, k_2, \dots, k_r)}(n, r) \frac{t^n}{n!} &= \text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-E\left[\left(\frac{1}{1-t}\right)^Y\right]} \right) \\
 &= \sum_{k=r}^{\infty} \left\{ \begin{matrix} k \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \frac{1}{k!} (-\log(1-t))^k \\
 &= \sum_{k=r}^{\infty} \left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{t^n}{n!} \\
 &= \sum_{n=r}^{\infty} \sum_{k=r}^n \left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by comparing the coefficients on both sides of (29), we obtain the following theorem.

Theorem 4. For $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$L_Y^{(k_1, k_2, \dots, k_r)}(n, r) = \sum_{k=r}^n \left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \left[\begin{matrix} n \\ k \end{matrix} \right].$$

Now, from (5) and (19), we observe that

$$\begin{aligned}
 (30) \quad \sum_{n=r}^{\infty} \left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \frac{t^n}{n!} &= \text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-E[e^{Yt}]} \right) \\
 &= \frac{\text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-E[e^{Yt}]} \right)}{\left(1 - e^{1-E[e^{Yt}]} \right)^r} \left(1 - e^{1-E[e^{Yt}]} \right)^r \\
 &= \sum_{m=0}^{\infty} B_m^{(k_1, k_2, \dots, k_r)} \frac{1}{m!} \left(E[e^{Yt}] - 1 \right)^m (-1)^r r! \frac{1}{r!} \left(e^{1-E[e^{Yt}]} - 1 \right)^r \\
 &= \sum_{m=0}^{\infty} B_m^{(k_1, k_2, \dots, k_r)} \frac{1}{m!} \left(E[e^{Yt}] - 1 \right)^m (-1)^r r! \sum_{l=r}^{\infty} \left\{ \begin{matrix} l \\ r \end{matrix} \right\} \frac{1}{l!} \left(1 - E[e^{Yt}] \right)^l \\
 &= \sum_{m=0}^{\infty} \sum_{l=r}^{\infty} B_m^{(k_1, k_2, \dots, k_r)} r! (-1)^{l-r} \left\{ \begin{matrix} l \\ r \end{matrix} \right\} \frac{(m+l)!}{m! l!} \frac{1}{(m+l)!} \left(E[e^{Yt}] - 1 \right)^{m+l} \\
 &= \sum_{m=0}^{\infty} \sum_{l=r}^{\infty} B_m^{(k_1, k_2, \dots, k_r)} r! (-1)^{l-r} \left\{ \begin{matrix} l \\ r \end{matrix} \right\} \binom{m+l}{m} \sum_{n=m+l}^{\infty} \left\{ \begin{matrix} n \\ m+l \end{matrix} \right\}_Y \frac{t^n}{n!} \\
 &= \sum_{n=r}^{\infty} \left(\sum_{m=0}^{n-r} \sum_{l=r}^{n-m} B_m^{(k_1, k_2, \dots, k_r)} r! (-1)^{l-r} \binom{m+l}{m} \left\{ \begin{matrix} l \\ r \end{matrix} \right\} \left\{ \begin{matrix} n \\ m+l \end{matrix} \right\}_Y \right) \frac{t^n}{n!}.
 \end{aligned}$$

By comparing the coefficients on both sides of (30), we obtain the following theorem.

Theorem 5. Let n, r be positive integers with $n \geq r$. Then we have

$$\left\{ \begin{matrix} n \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y = \sum_{m=0}^{n-r} \sum_{l=r}^{n-m} r! (-1)^{l-r} \binom{m+l}{m} \left\{ \begin{matrix} l \\ r \end{matrix} \right\} \left\{ \begin{matrix} n \\ m+l \end{matrix} \right\}_Y B_m^{(k_1, k_2, \dots, k_r)}.$$

In particular, we have

$$\left\{ \begin{matrix} n \\ r \end{matrix} \right\}_Y = \sum_{m=0}^{n-r} \sum_{l=r}^{n-m} (-1)^{m+l-r} \binom{m+l}{m} \left\{ \begin{matrix} l \\ r \end{matrix} \right\} \left\{ \begin{matrix} n \\ m+l \end{matrix} \right\}_Y B_m^{(r)}.$$

For $r \in \mathbb{N}$, the probabilistic Fubini polynomials of order r associated with Y are defined by

$$(31) \quad \frac{1}{(1 - y(E[e^{Yt}] - 1))^r} = \sum_{n=0}^{\infty} F_n^{(r,Y)}(y) \frac{t^n}{n!}, \quad (\text{see [32]}).$$

Replacing t by $E[e^{Yt}] - 1$ in (2), on the one hand, we have

$$(32) \quad \begin{aligned} \frac{1}{(2 - E[e^{Yt}])^r} \text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-E[e^{Yt}]} \right) &= \sum_{k=r}^{\infty} L^{(k_1, k_2, \dots, k_r)}(k, r) \frac{1}{k!} \left(E[e^{Yt}] - 1 \right)^k \\ &= \sum_{k=r}^{\infty} L^{(k_1, k_2, \dots, k_r)}(k, r) \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y \frac{t^n}{n!} \\ &= \sum_{n=r}^{\infty} \sum_{k=r}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y L^{(k_1, k_2, \dots, k_r)}(k, r) \frac{t^n}{n!}. \end{aligned}$$

On the other hand, by (19) and (31), we get

$$(33) \quad \begin{aligned} \frac{1}{(2 - E[e^{Yt}])^r} \text{Li}_{k_1, k_2, \dots, k_r} \left(1 - e^{1-E[e^{Yt}]} \right) &= \sum_{k=r}^{\infty} \left\{ \begin{matrix} k \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \frac{t^k}{k!} \sum_{l=0}^{\infty} F_l^{(r,Y)}(1) \frac{t^l}{l!} \\ &= \sum_{n=r}^{\infty} \sum_{k=r}^n \left\{ \begin{matrix} k \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y \binom{n}{k} F_{n-k}^{(r,Y)}(1) \frac{t^n}{n!}. \end{aligned}$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 6. For $n, r \in \mathbb{N}$ with $n \geq r$, we have

$$\sum_{k=r}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y L^{(k_1, k_2, \dots, k_r)}(k, r) = \sum_{k=r}^n \binom{n}{k} \left\{ \begin{matrix} k \\ k_1, k_2, \dots, k_r \end{matrix} \right\}_Y F_{n-k}^{(r,Y)}(1).$$

3. CONCLUSION

Let Y be a random variable whose moment generating function exists in a neighborhood of the origin. The Stirling numbers of the second $\left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}$ are generalized to the multi-Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}$ and further to the probabilistic multi-Stirling numbers of the second kind associated with Y , $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$. Likewise, the unsigned Lah numbers $L(n, r)$ are generalized to the multi-Lah numbers $L^{(k_1, k_2, \dots, k_r)}(n, r)$ and further to the probabilistic multi-Lah numbers associated with $L_Y^{(k_1, k_2, \dots, k_r)}(n, r)$. In connection with several other special numbers, we investigated some properties, related identities, recurrence relations and explicit expressions of the numbers $\left\{ \begin{smallmatrix} n \\ k_1, k_2, \dots, k_r \end{smallmatrix} \right\}_Y$ and $L_Y^{(k_1, k_2, \dots, k_r)}(n, r)$.

In recent years, degenerate versions, λ -analogues and probabilistic extensions of many special numbers and polynomials have been studied and many remarkable results have been obtained (see [2,3,10,11,17-19,21,22,24,26-29,33] and the references therein). It is one of our future research projects to continue to study degenerate versions, λ -analogues and probabilistic extensions of various special numbers and polynomials and to find their applications to physics, science and engineering as well as to mathematics.

REFERENCES

1. M. ABRAMOWITZ, I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series, U. S. Government Printing Office, Washington, DC, 1964.
2. J. A. ADELL, B. BÉNYI, *Probabilistic stirling numbers and applications*, Aequat. Math. (2024). <https://doi.org/10.1007/s00010-024-01073-1>
3. M. S. AYDIN, M. ACIKGOZ, S. ARACI, *A new construction on the degenerate Hurwitz-zeta function associated with certain applications*, Proc. Jangjeon Math. Soc. **25** (2022), no. 2, 195-203.
4. V. E. BENING, *On the asymptotic deficiency of some statistical estimators based on samples with random size*, Proc. Jangjeon Math. Soc. **21** (2018), no. 2, 185-193.
5. C. R. BLYTH, P. K. PATHAK, Notes: *A note on easy proofs of Stirling's theorem*, Amer. Math. Monthly **93** (1986), no. 5, 376-379.
6. K. BOUBELLOUTA, A. BOUSSAYOUD, S. ARACI, M. KERADA, *Some theorems on generating functions and their applications*, Adv. Stud. Contemp. Math. (Kyungshang) **30**(2020), no. 3, 307-324.
7. K. N. BOYADZHIEV, *Convolutions for Stirling numbers, Lah numbers, and binomial coefficients*, Proc. Jangjeon Math. Soc. **25** (2022), no. 2, 227-244.
8. A. Z. BRODER, *The r -Stirling numbers*, Discrete Math. **49** (1984), 241-259.
9. L. CARLITZ, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51-88.
10. N. P. ČAKIĆ, B. S. EL-DESOUKY, G. V. MILOVANOVIC, *Explicit formulas and combinatorial identities for generalized Stirling numbers*, Mediterr. J. Math. **10** (2013), no. 1, 57-72.

11. L. CHEN, D. V. DOLGY, T. KIM, D. S. KIM, *Probabilistic type 2 Bernoulli and Euler polynomials*, AIMS Math. **9** (2024), no. 6, 14312-14324.
12. L. CHEN, T. KIM, D. S. KIM, H. LEE, S.-H. LEE, *Probabilistic degenerate central Bell polynomials*, Math. Comput. Model. Dyn. Syst. **30** (2024), no. 1, 523-542.
13. L. COMTET, *Advanced Combinatorics. The Art of Finite and Infinite Expansions*, Revised and enlarged edition., D. Reidel Publishing Co., Dordrecht, 1974.
14. S. K. GHOSAL, S. MUKHOPADHYAY, S. HOSSAIN, R. SARKAR, Application of Lah transform for security and privacy of data through information hiding in telecommunication, *T. on Emerg. Telecommun. T.*, **32** (2020), no. 2, e3984. <https://doi.org/10.1002/ett.3984>
15. R. L. GRAHAM, D. E. KNUTH, O. PATASHNIK, *Concrete Mathematics: A Foundation for Computer Science*, 2 Eds., Addison Wesley Publishing Company, Massachusetts, 1994.
16. D. GUN, Y. SIMSEK, *Combinatorial sums involving Stirling, Fubini, Bernoulli numbers and approximate values of Catalan numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **30** (2020), no. 4, 503-513.
17. K.-S. HWANG, *Almost sure convergence of weighted sums for widely negative dependent random variables under sub-linear expectations*, Adv. Stud. Contemp. Math. (Kyungshang) **32** (2022), no. 4, 465-477.
18. D. S. KIM, T. KIM, *Normal ordering associated with λ -Whitney numbers of the first Kind in λ -shift algebra*, Russ. J. Math. Phys. **30** (2023), no. 3, 310-319.
19. D. S. KIM, KIM, *A note on a new type of degenerate Bernoulli numbers*, Russ. J. Math. Phys. **27** (2020), no. 2, 227-235.
20. M.-S. KIM, T. KIM, *An explicit formula on the generalized Bernoulli number with order n* , Indian J. Pure Appl. Math. **31** (2000), no. 11, 1455-1461.
21. T. KIM, D. S. KIM, *Probabilistic Bernoulli and Euler polynomials*, Russ. J. Math. Phys. **31** (2024), no. 1, 94-105.
22. T. KIM, D. S. KIM, *Probabilistic degenerate Bell polynomials associated with random variables*, Russ. J. Math. Phys. **30** (2023), no. 4, 528-542.
23. T. KIM, D. S. KIM, *Generalization of Spivey's recurrence relation*, Russ. J. Math. Phys. **31** (2024), no. 2, 218-226.
24. T. KIM, D. S. KIM, *Some identities involving degenerate Stirling numbers associated with several degenerate polynomials and numbers*, Russ. J. Math. Phys. **30** (2023), no. 1, 62-75.
25. T. KIM, D. S. KIM, H. K. KIM, *Multi-Stirling numbers of the second kind*, Filomat **38** (2024), no.24, 8653-8661.
26. T. KIM, D. S. KIM, J. KWON, *Probabilistic degenerate Stirling polynomials of the second kind and their applications*, Math. Comput. Model. Dyn. Syst. **30** (2024), no. 1, 16-30.
27. T. KIM, D. S. KIM, *Explicit formulas for probabilistic multi-poly-Bernoulli polynomials and numbers*, Russ. J. Math. Phys. **31** (2024), no. 3.
28. L. LUO, T. KIM, D. S. KIM, Y. MA, *Probabilistic degenerate Bernoulli and degenerate Euler polynomials*, Math. Comput. Model. Dyn. Syst. **30** (2024), no. 1, 342-363.

29. Y. MA, T. KIM, D. S. KIM, *Probabilistic Lah numbers and Lah-Bell polynomials*, arXiv:2406.01200
30. Y. MA, T. KIM, D. S. KIM, R. XU, *Recurrence for probabilistic extension of Dowling polynomials*, Open Math. **23** (2025), no. 1, Paper No. 20250150, 7 pp.
31. S. ROMAN, *The umbral calculus*, Pure and Applied Mathematics, 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
32. S. M. ROSS, *Introduction to probability models*, Thirteenth edition, Academic Press, London, 2024.
33. Y. SIMSEK, *Identities and relations related to combinatorial numbers and polynomials*, Proc. Jangjeon Math. Soc. **20** (2017), no. 1, 127-135.
34. R. SONI, P. VELLAISAMY, A. K. PATHAK, *A probabilistic generalization of the Bell polynomials*, J. Anal. **32**, no. 2, 711–732.
35. P. XUE, Y. MA, T. KIM, ; D. S. KIM,; W. ZHANG, *Probabilistic degenerate poly-Bell polynomials associated with random variables*, Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, Paper No. 2497367, 17 pp.

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