

FINDING COUNTEREXAMPLES FOR A CONJECTURE OF AKBARI, ALAZEMI AND ANĐELIĆ

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For a graph G , the energy $\mathcal{E}(G)$ is the sum of absolute values of the eigenvalues of the adjacency matrix of G , $\mu(G)$ is the matching number of G , and Δ is the maximum vertex degree of G . It was recently proved that $\mathcal{E}(G) \leq 2\mu(G)\sqrt{\Delta}$ for connected G with $\Delta \geq 6$, and conjectured that this also holds for $2 \leq \Delta \leq 5$ unless G is a cycle of length three, five or seven. We refute the conjecture by a computer-assisted search for small counterexamples, and use their structural patterns to construct two infinite families of counterexamples.

1. INTRODUCTION

Developments in computer-assisted methods have had a major impact on graph theory, allowing researchers to test conjectures and investigate graph structures computationally before attempting theoretical proofs. Early systems such as GRAPH [7, 8, 9, 10] and the package nauty [16, 17, 18] enabled efficient generation and analysis of graphs, while modern frameworks such as graph6java [11] provide convenient tools for programmatic experimentation on generated graph classes. Optimization-based approaches were further popularized by AutoGraphiX [3, 5, 6], which employed the variable neighborhood search metaheuristic [19] to search for extremal graphs and generate conjectures.

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More recently, reinforcement learning methods have also been applied to problems in extremal combinatorics and graph theory. In his pioneering paper, Wagner [22] used deep reinforcement learning to construct explicit counterexamples to several conjectures by iteratively generating and refining graph constructions.

Here, we combine reinforcement learning and exhaustive search to study a recent conjecture of Akbari, Alazemi and Anđelić [1]. For a graph G with the vertex set V and the edge set E , the adjacency matrix of G is the matrix $A \in \mathbb{R}^{V \times V}$ such that $A_{uv} = 1$ if $uv \in E$, and $A_{uv} = 0$ otherwise. The energy of G is defined as $\mathcal{E}(G) = \sum_{j=1}^n |\lambda_j|$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A ; see [13, 14]. The matching number $\mu(G)$ is the number of edges in a maximum matching of G , while Δ is the maximum vertex degree in G . Akbari, Alazemi and Anđelić recently proved the following theorem.

Theorem 1 ([1, Theorem 18]). *Let G be a connected graph with maximum vertex degree $\Delta \geq 6$. Then $\mathcal{E}(G) \leq 2\mu(G)\sqrt{\Delta}$.*

In the same paper, the following related conjecture was also posed.

Conjecture 2 ([1, Conjecture 23]). *For any (connected) graph $G \not\cong C_3, C_5, C_7$, with maximum vertex degree $\Delta \in \{2, 3, 4, 5\}$, $\mathcal{E}(G) \leq 2\mu(G)\sqrt{\Delta}$.*

In Section 2, we first apply Wagner’s reinforcement learning approach from [22] to find that the potential counterexamples are likely to have $\Delta = 3$. Subsequently, we refute Conjecture 2 by performing an exhaustive search to enumerate the counterexamples among connected graphs with $\Delta \leq 3$ and at most 19 vertices. Finally, in Section 3, we use the structural patterns observed in these counterexamples to construct two families of graphs, each containing infinitely many counterexamples to Conjecture 2.

2. COMPUTER-ASSISTED SEARCH

The reinforcement learning code used by Wagner in [22] was made publicly available in [23]. After adapting the `calcScore` function to our setting, the score assigned to a graph G was $\mathcal{E}(G) - 2\mu(G)\sqrt{\Delta}$ if G was connected, and $-\infty$ otherwise, thereby enforcing connectivity throughout the learning process.

The learning process was performed on graphs with 19 vertices. Each generation consisted of 1,000 graphs, and evaluating one generation required about ten seconds on a standard Intel i5 processor. After 3,200 generations, corresponding to approximately nine hours of computation, the best graph found satisfied

$$\mathcal{E}(G) - 2\mu(G)\sqrt{\Delta} \approx -1.665,$$

so no counterexample to Conjecture 2 was detected; see Fig. 1.

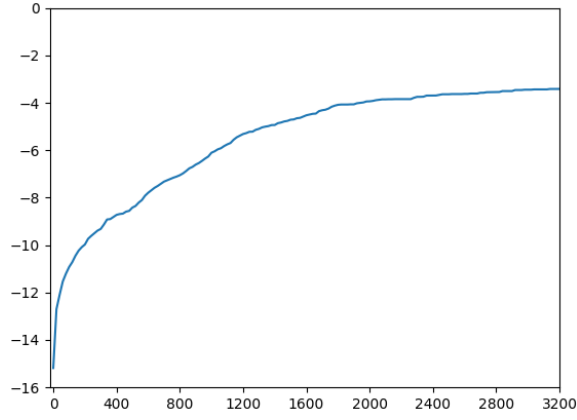


Figure 1: The mean value of $\mathcal{E}(G) - 2\mu(G)\sqrt{\Delta}$ for the 10% of best-performing graphs in each reinforcement learning generation.

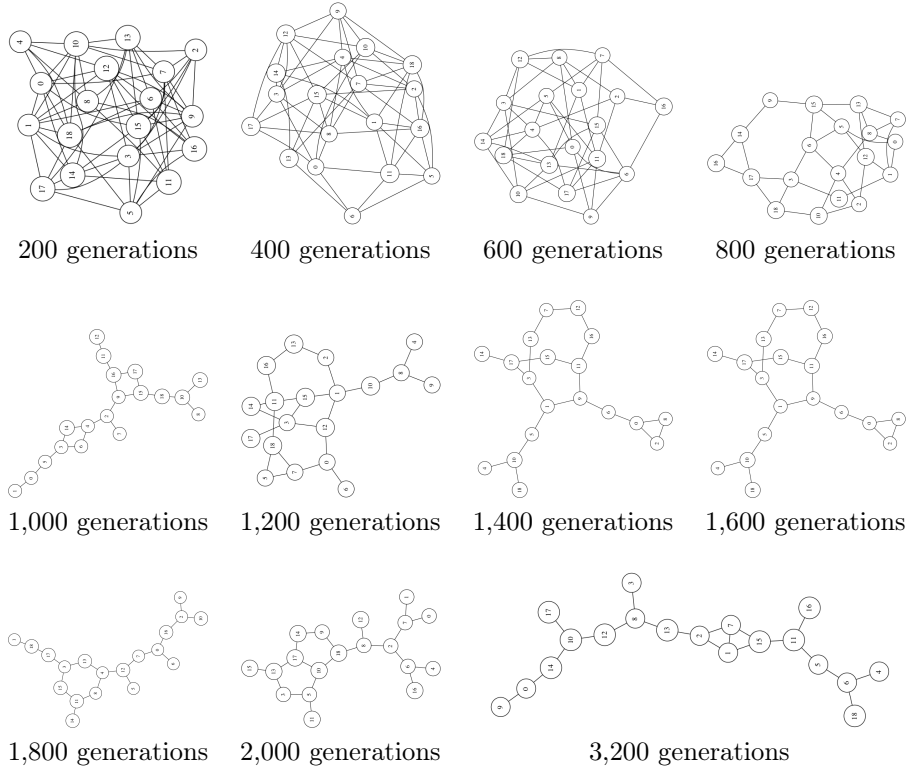


Figure 2: The best-performing graphs with the highest values of $\mathcal{E}(G) - 2\mu(G)\sqrt{\Delta}$ after a given number of generations.

The fact that reinforcement learning was not able to find counterexamples is certainly not an evidence of their absence. As it turns out, the evolution of the best-performing graphs, shown in Fig. 2, indicates that the reinforcement learning process nevertheless identified structural properties of near-extremal graphs. In particular, after roughly 1,400 generations, the best-performing graphs consistently had maximum degree $\Delta = 3$. Motivated by this observation, we proceeded with an exhaustive search among connected graphs with $\Delta \leq 3$ in order to enumerate small counterexamples and study their structure.

For each $6 \leq n \leq 19$, we generated all connected graphs of order n with maximum degree at most three using the **geng** utility from the package **nauty** [17]. We then used the Java framework **graph6java** [11] to test which of these graphs satisfy $\mathcal{E}(G) > 2\mu(G)\sqrt{\Delta} + 10^{-7}$, where the tolerance was introduced to account for numerical inaccuracies. The numbers of counterexamples found for each order n are given in Table 1.

Table 1: The number of counterexamples to Conjecture 2 among connected graphs with $\Delta \leq 3$ and n vertices, with $n \in \{6, 7, \dots, 19\}$. This count excludes C_7 , which is forbidden by the conjecture.

n	6	7	8	9	10	11	12
# counterexamples	1	1	1	3	2	5	3
n	13	14	15	16	17	18	19
# counterexamples	4	8	5	13	16	19	36

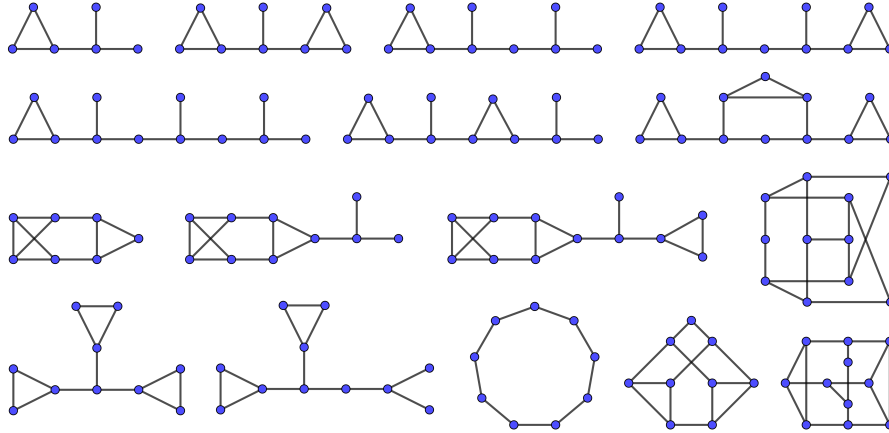


Figure 3: The counterexamples to Conjecture 2 with at most 12 vertices and $\Delta \leq 3$.

The computational results suggest that the number of counterexamples grows steadily for $n \geq 16$, indicating the existence of many additional counterexamples on

larger numbers of vertices. We show the counterexamples with at most 12 vertices in Fig. 3. The complete collection of counterexamples found by the exhaustive search, together with the files required to reproduce the computations, is available in [21].

The drawings in Fig. 3 reveal a common structural feature: many counterexamples can be decomposed into disjoint triangles and 3-stars connected in a tree-like manner. Since every matching contains at most one edge from each such component, this structure strongly restricts the matching number. Among larger counterexamples with between 13 and 19 vertices, this behavior becomes dominant.

For completeness, we also performed exhaustive searches among connected graphs with $\Delta = 4$ on at most 14 vertices, and $\Delta = 5$ on at most 12 vertices. In the former case, we found a unique counterexample with $\Delta = 4$ on 11 vertices, shown in Fig. 4. In the latter case, no counterexamples with $\Delta = 5$ were found.

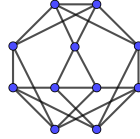


Figure 4: A counterexample to Conjecture 2 with $\Delta = 4$ on 11 vertices.

3. TRIANGULAR COMBS AND CYCLIC TRIANGULAR COMBS

Motivated by the observation that the edge sets of many small counterexamples to Conjecture 2 may be decomposed into tree-like unions of triangles and 3-stars, together with the facts that the path has the largest energy among trees [13, 14] and that the cycle has either the largest or the second largest energy among unicyclic graphs [2, 15], we constructed two infinite families of graphs, illustrated in Fig. 5, each containing infinitely many counterexamples to Conjecture 2. We call these families the *triangular combs* and the *cyclic triangular combs*. The triangular comb TC_k , defined for $k \in \mathbb{N}$, consists of k triangular tree-like gadgets whose bases form a path, as shown in Fig. 5. The cyclic triangular comb CTC_k , defined for $k \geq 2$, is constructed analogously, except that the bases of its k gadgets form a cycle rather than a path. The graph TC_k has $5k + 1$ vertices and matching number $\mu(\text{TC}_k) = 2k$, while CTC_k has $5k$ vertices and matching number $\mu(\text{CTC}_k) = 2k$. Every graph in both families has maximum degree $\Delta = 3$.

In the rest of the paper, we show that infinitely many triangular combs and infinitely many cyclic triangular combs provide counterexamples to Conjecture 2. This is equivalent to proving that $\mathcal{E}(\text{TC}_k) > 4k\sqrt{3}$ and $\mathcal{E}(\text{CTC}_k) > 4k\sqrt{3}$ each hold for infinitely many k . We carry out the proof by showing that $\lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{TC}_k)}{k}$ and $\lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{CTC}_k)}{k}$ both exist and are equal to the same real number that is above $4\sqrt{3}$.

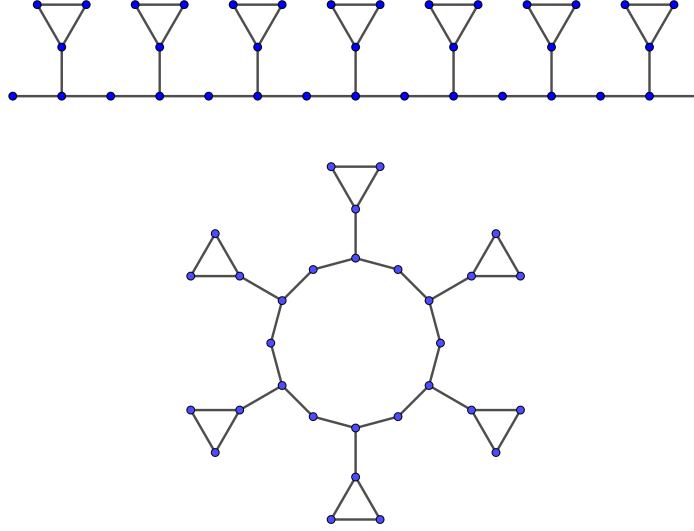


Figure 5: The triangular comb TC_7 (top) and the cyclic triangular comb CTC_6 (bottom).

To begin, for any $t \in [-2, 2]$, we define the auxiliary polynomial $P_t(x) := x^4 - x^3 - (t+5)x^2 + (t+3)x + (2t+4)$. As it turns out, all these polynomials have real roots.

Lemma 3. *Let $t \in [-2, 2]$. Then the four roots of $P_t(x)$ are all real and belong to the intervals $(-\frac{5}{2}, -1)$, $(-1, 0]$, $(\frac{1}{4}, 2)$ and $(2, \frac{5}{2})$.*

Proof. The claim directly follows by observing that $P_t(-\frac{5}{2}) = \frac{319-108t}{16} \geq \frac{103}{16}$, $P_t(-1) = -2$, $P_t(0) = 2t+4 \geq 0$, $P_t(\frac{1}{4}) = \frac{1133+560t}{256} \geq \frac{13}{256}$, $P_t(2) = -2$ and $P_t(\frac{5}{2}) = \frac{59-28t}{16} \geq \frac{3}{16}$. $\square$

With Lemma 3 in mind, for any $t \in [-2, 2]$, let $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ be the real roots of $P_t(x)$ from the intervals $(-\frac{5}{2}, -1)$, $(-1, 0]$, $(\frac{1}{4}, 2)$ and $(2, \frac{5}{2})$, respectively. We proceed by investigating α , β , γ and δ , viewed as functions of t on the interval $[-2, 2]$. To this end, let $f: \mathbb{R} \setminus \{-1, 2\} \rightarrow \mathbb{R}$ be the rational function defined by $f(x) = \frac{x^4 - x^3 - 5x^2 + 3x + 4}{(x+1)(x-2)}$.

Lemma 4. *The functions α , β , γ and δ are continuous. Moreover, α and β are strictly decreasing, while γ and δ are strictly increasing.*

Proof. The continuity of α , β , γ and δ follows from the well-known fact that the roots of polynomials in $\mathbb{C}[x]$ are continuous with respect to the coefficients. Since $P_t(x) = (x^4 - x^3 - 5x^2 + 3x + 4) - t(x+1)(x-2)$, to prove the monotonicity claim,

it suffices to show that f is strictly decreasing on $(-\frac{5}{2}, -1)$ and on $(-1, 0]$, and strictly increasing on $(\frac{1}{4}, 2)$ and on $(2, \frac{5}{2})$. This follows by computing

$$f'(x) = \frac{2(x^5 - 2x^4 - 3x^3 + 4x^2 + 6x - 1)}{(x+1)^2(x-2)^2} \quad (x \in \mathbb{R} \setminus \{-1, 2\})$$

and inspecting the sign of the numerator. Let $Q(x) := x^5 - 2x^4 - 3x^3 + 4x^2 + 6x - 1$. Then $Q(x) = x((x^2 - x - 2)^2 + 2) - 1$, hence $Q(x) < 0$ for $x < 0$. Also, if $x \in (\frac{1}{4}, \frac{1}{2}]$, then $x^2 - x - 2 \in [-\frac{9}{4}, -\frac{35}{16}]$, hence $Q(x) > \frac{1}{4}((\frac{35}{16})^2 + 2) - 1 = \frac{713}{1024} > 0$. Finally, if $x > \frac{1}{2}$, then $Q(x) > \frac{1}{2} \cdot 2 - 1 = 0$. $\square$

We now turn to the triangular combs and the cyclic triangular combs and investigate their spectra. In this context, we consider all spectral properties to correspond to the adjacency matrix, regard the spectra as multisets, and use exponents to denote eigenvalue multiplicities. Moreover, let $\varphi_M(x) := \det(xI - M)$ denote the characteristic polynomial of a square matrix M . We first determine the spectra of cyclic triangular combs using the diagonalization of circulant matrices; see [12, Section 3.1]. The proof of the following proposition follows a similar approach to that used in [20].

Proposition 5. *For any $k \geq 2$, the spectrum of CTC_k is*

$$\{(-1)^k\} \cup \bigcup_{j=0}^{k-1} \left\{ \alpha \left(2 \cos \frac{2j\pi}{k} \right), \beta \left(2 \cos \frac{2j\pi}{k} \right), \gamma \left(2 \cos \frac{2j\pi}{k} \right), \delta \left(2 \cos \frac{2j\pi}{k} \right) \right\}.$$

Proof. Consider the graph CTC_k and let its vertices be $v_j^{(h)}$ with $h \in \{1, 2, \dots, 5\}$ and $j \in \mathbb{Z}_k$, as illustrated in Fig. 6. By arranging the vertices as

$$v_0^{(1)}, \dots, v_{k-1}^{(1)}, v_0^{(2)}, \dots, v_{k-1}^{(2)}, v_0^{(3)}, \dots, v_{k-1}^{(3)}, v_0^{(4)}, \dots, v_{k-1}^{(4)}, v_0^{(5)}, \dots, v_{k-1}^{(5)},$$

the adjacency matrix A of CTC_k gets the block form

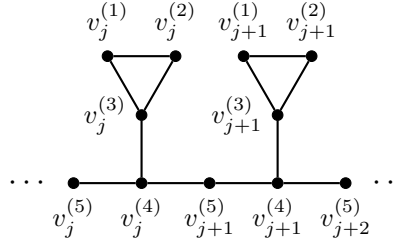
$$A = \begin{bmatrix} O & I_k & I_k & O & O \\ I_k & O & I_k & O & O \\ I_k & I_k & O & I_k & O \\ O & O & I_k & O & C \\ O & O & O & C^\top & O \end{bmatrix},$$

where C is the circulant matrix in $\mathbb{R}^{k \times k}$ with the starting row $[1 \ 1 \ 0 \ \dots \ 0]$.

Now, let $U \in \mathbb{C}^{k \times k}$ be a unitary matrix with the columns

$$\frac{1}{\sqrt{k}} [1 \ \zeta \ \zeta^2 \ \dots \ \zeta^{k-1}]^\top,$$

as ζ ranges over the k -th roots of unity. Recall that all circulant matrices are simultaneously diagonalizable by U , and let $\tilde{U}$ be the block diagonal matrix with

Figure 6: The vertex labeling for CTC_k used in the proof of Proposition 5.

five copies of U . Note that $\tilde{U}$ is a unitary matrix, and observe that $A\tilde{U} = \tilde{U}M$, where

$$M = \begin{bmatrix} O & I_k & I_k & O & O \\ I_k & O & I_k & O & O \\ I_k & I_k & O & I_k & O \\ O & O & I_k & O & D_1 \\ O & O & O & D_2 & O \end{bmatrix},$$

and D_1 (resp. D_2) is a diagonal matrix in $\mathbb{C}^{k \times k}$ with diagonal entries $1 + \zeta$ (resp. $1 + \frac{1}{\zeta}$), as ζ ranges over the k -th roots of unity. Therefore, A and M are similar.

By simultaneously rearranging the rows and columns of M in the order

$$\begin{aligned} &1, k+1, 2k+1, 3k+1, 4k+1, 2, k+2, 2k+2, 3k+2, 4k+2, \\ &3, k+3, 2k+3, 3k+3, 4k+3, \dots, k, 2k, 3k, 4k, 5k, \end{aligned}$$

we obtain the matrix $\bigoplus_{\zeta} M_{\zeta}$, where

$$M_{\zeta} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 + \zeta \\ 0 & 0 & 0 & 1 + \frac{1}{\zeta} & 0 \end{bmatrix},$$

with ζ ranging over the k -th roots of unity. A routine computation gives

$$\begin{aligned} \varphi_{M_{\zeta}}(x) &= x^5 - (6 + \zeta + \frac{1}{\zeta})x^3 - 2x^2 + (7 + 3\zeta + \frac{3}{\zeta})x + (4 + 2\zeta + \frac{2}{\zeta}) \\ &= (x+1) \left(x^4 - x^3 - (5 + \zeta + \frac{1}{\zeta})x^2 + (3 + \zeta + \frac{1}{\zeta})x + (4 + 2\zeta + \frac{2}{\zeta}) \right). \end{aligned}$$

Since $\varphi_A(x) = \prod_{\zeta} \varphi_{M_{\zeta}}(x)$, the result follows by noting that for any k -th root of unity $\zeta = e^{i\frac{2j\pi}{k}}$ with $j \in \{0, 1, 2, \dots, k-1\}$, we have $\zeta + \frac{1}{\zeta} = 2\cos(\frac{2j\pi}{k})$, so that $\varphi_{M_{\zeta}}(x) = (x+1)P_{2\cos(\frac{2j\pi}{k})}(x)$. $\square$

We now determine the spectra of triangular combs from Proposition 5 by adapting the strategy from [4, Subsection 1.4.4].

Proposition 6. *For any $k \in \mathbb{N}$, the spectrum of TC_k is*

$$\{0, (-1)^k\} \cup \bigcup_{j=1}^k \left\{ \alpha \left(2 \cos \frac{j\pi}{k+1} \right), \beta \left(2 \cos \frac{j\pi}{k+1} \right), \gamma \left(2 \cos \frac{j\pi}{k+1} \right), \delta \left(2 \cos \frac{j\pi}{k+1} \right) \right\}.$$

Proof. Consider the graph CTC_{2k+2} and let its vertices be labeled as in the proof of Proposition 5. Also, let the vertex set be V and let the adjacency matrix be A . Now, let W be the vector subspace of $\mathbb{R}^V$ comprising the vectors $\mathbf{z}$ such that $\mathbf{z}(v_{j+k+1}^{(h)}) = \mathbf{z}(v_j^{(h)})$ for every $h \in \{1, 2, \dots, 5\}$ and $j \in \mathbb{Z}_{2k+2}$. Observe that $W^\perp$ comprises the vectors $\mathbf{z}$ such that $\mathbf{z}(v_{j+k+1}^{(h)}) = -\mathbf{z}(v_j^{(h)})$ for every $h \in \{1, 2, \dots, 5\}$ and $j \in \mathbb{Z}_{2k+2}$. Moreover, $\dim(W) = \dim(W^\perp) = 5k + 5$.

It is easy to see that finding the eigenvectors of A belonging to W comes down to finding the eigenvectors of CTC_{k+1} . Since $\dim(W) = 5k + 5$, this means that W has a basis consisting of eigenvectors of A . By Proposition 5, the corresponding eigenvalues are

$$\{(-1)^{k+1}\} \cup \bigcup_{j=0}^k \{\alpha(t_{2j}), \beta(t_{2j}), \gamma(t_{2j}), \delta(t_{2j})\},$$

with $t_j = 2 \cos \frac{j\pi}{k+1}$ for every $j \in \mathbb{Z}$. Furthermore, since W is A -invariant, $W^\perp$ is also A -invariant. Therefore, $W^\perp$ has a basis consisting of eigenvectors of A whose corresponding eigenvalues are precisely the remaining eigenvalues of CTC_{2k+2} . By Proposition 5, these eigenvalues are

$$\{(-1)^{k+1}\} \cup \bigcup_{j=0}^k \{\alpha(t_{2j+1}), \beta(t_{2j+1}), \gamma(t_{2j+1}), \delta(t_{2j+1})\}.$$

Now, let $j \in \{1, 2, \dots, k\}$ and $\lambda \in \{\alpha(t_j), \beta(t_j), \gamma(t_j), \delta(t_j)\}$. Since $t_j = t_{2k+2-j}$, it follows that λ is a double eigenvalue of A with two linearly independent eigenvectors $\mathbf{z}'$ and $\mathbf{z}''$ such that either both $\mathbf{z}', \mathbf{z}'' \in W$ or both $\mathbf{z}', \mathbf{z}'' \in W^\perp$. By taking an appropriate linear combination of $\mathbf{z}'$ and $\mathbf{z}''$, we obtain an eigenvector $\mathbf{z}$ of A for the eigenvalue λ such that $\mathbf{z}(v_0^{(4)}) = \mathbf{z}(v_{k+1}^{(4)}) = 0$. We aim to prove that the restriction of $\mathbf{z}$ to

$$(1) \quad \bigcup_{j=1}^k \bigcup_{h=1}^5 \{v_j^{(h)}\} \cup \{v_{k+1}^{(5)}\}$$

or to

$$(2) \quad \bigcup_{j=k+2}^{2k+1} \bigcup_{h=1}^5 \{v_j^{(h)}\} \cup \{v_0^{(5)}\}$$

is nonzero, since this would imply that λ is an eigenvalue of TC_k .

By way of contradiction, suppose that $\mathbf{z}$ is zero on all the vertices from (1) and (2). Then the only vertices on which $\mathbf{z}$ is potentially not equal to zero are $v_0^{(h)}$ and $v_{k+1}^{(h)}$ with $h \in \{1, 2, 3\}$. It is straightforward to see that $\mathbf{z}(v_0^{(3)}) = \mathbf{z}(v_{k+1}^{(3)}) = 0$, as well as $v_0^{(2)} = -v_0^{(1)}$ and $v_{k+1}^{(2)} = -v_{k+1}^{(1)}$. Since $\lambda \neq -1$, it follows that $\mathbf{z} = \mathbf{0}$, yielding a contradiction.

Since $\alpha(t_j)$, $\beta(t_j)$, $\gamma(t_j)$ and $\delta(t_j)$, with j ranging over $\{1, 2, \dots, k\}$, are all mutually distinct, we have determined $4k$ out of the $5k + 1$ eigenvalues of TC_k . Now, consider the graph TC_k and let its vertices be $v_j^{(h)}$ with $h \in \{1, 2, \dots, 5\}$ and $j \in \{0, 1, 2, \dots, k-1\}$, or $(j, h) = (k, 5)$, as shown in Fig. 7. Let V denote the vertex set of TC_k .

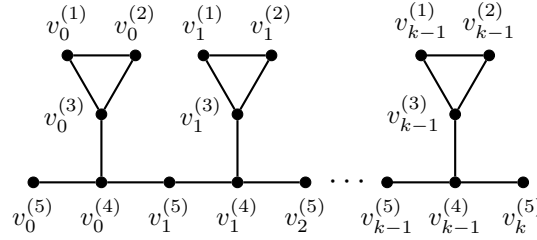


Figure 7: The vertex labeling for TC_k used in the proof of Proposition 6.

For fixed j , observe that the vector $\mathbf{z} \in \mathbb{R}^V$ defined by

$$\mathbf{z}(u) = \begin{cases} 1, & u = v_j^{(1)}, \\ -1, & u = v_j^{(2)}, \\ 0, & \text{otherwise} \end{cases}$$

for each $u \in V$, is an eigenvector of TC_k for the eigenvalue -1 . Since there are k linearly independent such vectors, it follows that -1 is an eigenvalue of TC_k with multiplicity at least k , thus accounting for another k eigenvalues. By Vieta's formulas, we have $\alpha(t) + \beta(t) + \gamma(t) + \delta(t) = 1$ for any $t \in [-2, 2]$. Since the sum of all eigenvalues of TC_k is zero, the remaining eigenvalue must be zero. $\square$

As a direct consequence of Propositions 5 and 6, we obtain the following result.

Corollary 7. *For any $k \geq 2$,*

$$\begin{aligned} \mathcal{E}(\text{TC}_k) = k - \sum_{j=0}^{k-1} \alpha \left(2 \cos \left(\frac{2j\pi}{k} \right) \right) - \sum_{j=0}^{k-1} \beta \left(2 \cos \left(\frac{2j\pi}{k} \right) \right) \\ + \sum_{j=0}^{k-1} \gamma \left(2 \cos \left(\frac{2j\pi}{k} \right) \right) + \sum_{j=0}^{k-1} \delta \left(2 \cos \left(\frac{2j\pi}{k} \right) \right). \end{aligned}$$

Also, for any $k \in \mathbb{N}$,

$$\begin{aligned} \mathcal{E}(\text{TC}_k) &= k - \sum_{j=1}^k \alpha \left(2 \cos \left(\frac{j\pi}{k+1} \right) \right) - \sum_{j=1}^k \beta \left(2 \cos \left(\frac{j\pi}{k+1} \right) \right) \\ &\quad + \sum_{j=1}^k \gamma \left(2 \cos \left(\frac{j\pi}{k+1} \right) \right) + \sum_{j=1}^k \delta \left(2 \cos \left(\frac{j\pi}{k+1} \right) \right). \end{aligned}$$

We now need the following basic result from mathematical analysis whose proof we omit.

Lemma 8. *Let $g: [a, b] \rightarrow [c, d]$ be a strictly monotone and continuous bijection and let g^{-1} be its inverse. Then*

$$\int_a^b g(x) \, dx + \int_{g(a)}^{g(b)} g^{-1}(x) \, dx = b g(b) - a g(a).$$

Recall that $f(x) = \frac{x^4 - x^3 - 5x^2 + 3x + 4}{(x+1)(x-2)}$. The following theorem is the main result of this section.

Theorem 9. *We have*

$$\lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{CTC}_k)}{k} = \lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{TC}_k)}{k} = L,$$

where

$$\begin{aligned} L &= 2 - 2\alpha(-2) + \frac{1}{\pi} \int_{\alpha(2)}^{\alpha(-2)} \arccos \left(\frac{f(x)}{2} \right) \, dx + \frac{1}{\pi} \int_{\beta(2)}^{\beta(-2)} \arccos \left(\frac{f(x)}{2} \right) \, dx \\ &\quad + \frac{1}{\pi} \int_{\gamma(-2)}^{\gamma(2)} \arccos \left(\frac{f(x)}{2} \right) \, dx + \frac{1}{\pi} \int_{\delta(-2)}^{\delta(2)} \arccos \left(\frac{f(x)}{2} \right) \, dx. \end{aligned}$$

Proof. By Corollary 7, we have

$$\begin{aligned} \frac{\mathcal{E}(\text{CTC}_k)}{k} &= 1 - \frac{1}{k} \sum_{j=0}^{k-1} \alpha \left(2 \cos \left(\frac{2j\pi}{k} \right) \right) - \frac{1}{k} \sum_{j=0}^{k-1} \beta \left(2 \cos \left(\frac{2j\pi}{k} \right) \right) \\ (3) \quad &\quad + \frac{1}{k} \sum_{j=0}^{k-1} \gamma \left(2 \cos \left(\frac{2j\pi}{k} \right) \right) + \frac{1}{k} \sum_{j=0}^{k-1} \delta \left(2 \cos \left(\frac{2j\pi}{k} \right) \right). \end{aligned}$$

The functions $x \mapsto \alpha(2 \cos(x))$, $x \mapsto \beta(2 \cos(x))$, $x \mapsto \gamma(2 \cos(x))$ and $x \mapsto \delta(2 \cos(x))$ are Riemann integrable on $[0, 2\pi]$. Each sum from (3) is a Riemann sum. For instance, $\frac{1}{k} \sum_{j=0}^{k-1} \alpha \left(2 \cos \left(\frac{2j\pi}{k} \right) \right) = \frac{1}{2\pi} \sum_{j=0}^{k-1} \frac{2\pi}{k} \alpha \left(2 \cos \left(\frac{2j\pi}{k} \right) \right)$. Hence,

$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} \alpha\left(2 \cos\left(\frac{2j\pi}{k}\right)\right) = \frac{1}{2\pi} \int_0^{2\pi} \alpha(2 \cos(x)) dx$. Applying the same argument to the remaining sums yields

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{CTC}_k)}{k} &= 1 - \frac{1}{2\pi} \int_0^{2\pi} \alpha(2 \cos(x)) dx - \frac{1}{2\pi} \int_0^{2\pi} \beta(2 \cos(x)) dx \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \gamma(2 \cos(x)) dx + \frac{1}{2\pi} \int_0^{2\pi} \delta(2 \cos(x)) dx. \end{aligned}$$

Similarly,

$$(4) \quad \begin{aligned} \lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{TC}_k)}{k} &= 1 - \frac{1}{\pi} \int_0^{\pi} \alpha(2 \cos(x)) dx - \frac{1}{\pi} \int_0^{\pi} \beta(2 \cos(x)) dx \\ &\quad + \frac{1}{\pi} \int_0^{\pi} \gamma(2 \cos(x)) dx + \frac{1}{\pi} \int_0^{\pi} \delta(2 \cos(x)) dx. \end{aligned}$$

Observe that each of the functions $x \mapsto \alpha(2 \cos(x))$, $x \mapsto \beta(2 \cos(x))$, $x \mapsto \gamma(2 \cos(x))$ and $x \mapsto \delta(2 \cos(x))$ is symmetric with respect to π . Thus, the integral of each of these functions over $[0, 2\pi]$ is equal to twice its integral over $[0, \pi]$. Therefore, $\lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{CTC}_k)}{k} = \lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{TC}_k)}{k}$.

By Lemma 8,

$$\int_0^{\pi} \alpha(2 \cos(x)) dx + \int_{\alpha(2)}^{\alpha(-2)} \arccos\left(\frac{f(x)}{2}\right) dx = \pi \alpha(-2),$$

hence

$$\frac{1}{\pi} \int_0^{\pi} \alpha(2 \cos(x)) dx = \alpha(-2) - \frac{1}{\pi} \int_{\alpha(2)}^{\alpha(-2)} \arccos\left(\frac{f(x)}{2}\right) dx.$$

Similar conclusions hold for the functions $x \mapsto \beta(2 \cos(x))$, $x \mapsto \gamma(2 \cos(x))$ and $x \mapsto \delta(2 \cos(x))$. By plugging these formulae into (4), we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\mathcal{E}(\text{TC}_k)}{k} &= 1 - \alpha(-2) + \frac{1}{\pi} \int_{\alpha(2)}^{\alpha(-2)} \arccos\left(\frac{f(x)}{2}\right) dx \\ &\quad - \beta(-2) + \frac{1}{\pi} \int_{\beta(2)}^{\beta(-2)} \arccos\left(\frac{f(x)}{2}\right) dx \\ &\quad + \gamma(-2) - \frac{1}{\pi} \int_{\gamma(2)}^{\gamma(-2)} \arccos\left(\frac{f(x)}{2}\right) dx \\ &\quad + \delta(-2) - \frac{1}{\pi} \int_{\delta(2)}^{\delta(-2)} \arccos\left(\frac{f(x)}{2}\right) dx. \end{aligned}$$

The desired result now follows by observing that $\alpha(-2) + \beta(-2) + \gamma(-2) + \delta(-2) = 1$ and $\beta(-2) = 0$. $\square$

It is straightforward to verify using mathematical software that the limit L from Theorem 9 satisfies $L > 4\sqrt{3}$. For example, using *Mathematica* [24], we evaluate:

```

1 N[2 - 2*Root[x^4 - x^3 - 3 x^2 + x, 1] +
2   1/Pi*Integrate[
3     ArcCos[(x^4 - x^3 - 5 x^2 + 3 x + 4)/(2 x^2 - 2 x - 4)], {x,
4       Root[x^4 - x^3 - 7 x^2 + 5 x + 8, 1],
5       Root[x^4 - x^3 - 3 x^2 + x, 1]}] +
6   1/Pi*Integrate[
7     ArcCos[(x^4 - x^3 - 5 x^2 + 3 x + 4)/(2 x^2 - 2 x - 4)], {x,
8       Root[x^4 - x^3 - 7 x^2 + 5 x + 8, 2],
9       Root[x^4 - x^3 - 3 x^2 + x, 2]}] +
10  1/Pi*Integrate[
11    ArcCos[(x^4 - x^3 - 5 x^2 + 3 x + 4)/(2 x^2 - 2 x - 4)], {x,
12      Root[x^4 - x^3 - 3 x^2 + x, 3],
13      Root[x^4 - x^3 - 7 x^2 + 5 x + 8, 3]}] +
14  1/Pi*Integrate[
15    ArcCos[(x^4 - x^3 - 5 x^2 + 3 x + 4)/(2 x^2 - 2 x - 4)], {x,
16      Root[x^4 - x^3 - 3 x^2 + x, 4],
17      Root[x^4 - x^3 - 7 x^2 + 5 x + 8, 4]}] - 4 Sqrt[3], 100]

```

This yields $L - 4\sqrt{3} \approx 0.03909742778$. Therefore, infinitely many cyclic triangular combs and infinitely many triangular combs provide counterexamples to Conjecture 2.

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