

DECOMPOSITIONS OF GRAPHS WITH DEGREE CONSTRAINTS RELATIVE TO PRESCRIBED SUBGRAPHS

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Let $\mathcal{H}$ consist of the graphs obtained by joining a new vertex to exactly two vertices of $K_4 - e$. For a graph G , set $h(u) = 1$ if u belongs to a subgraph isomorphic to a member of $\mathcal{H}$, and $h(u) = 0$ otherwise. For $|V(G)| \geq 5$ and non-negative integer functions a, b on $V(G)$, we prove that $d_G(u) \geq a(u) + b(u) + h(u)$ and $\min\{a(u), b(u)\} \geq 1 - h(u)$ for every u guarantee a partition into two non-empty parts with internal degrees at least a and b , respectively. For $\mathcal{H} = \{K_{2,3}\}$, the same holds with $\min\{a(u), b(u)\} \geq 2 - h(u)$.

1. INTRODUCTION

All graphs considered in this paper are finite simple and undirected. For such a graph G , we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For a vertex $u \in V(G)$, let $N_G(u)$ and $d_G(u)$ denote the neighbourhood and degree of u , respectively; thus $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and $d_G(u) = |N_G(u)|$. The minimum degree of G is written as $\delta(G)$. For a vertex subset X and a vertex u of G , we define $d_X(u) = |N_G(u) \cap X|$ no matter $u \in X$ or not. Moreover, $G[X]$ denotes the subgraph of G induced by X . For a graph H , we write $H \subseteq G$ if H is a

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subgraph of G . For a fixed graph H , G is said to be H -free if it does not contain H as a subgraph (not necessarily induced). For terminology and notation not explicitly defined in this paper, we refer the reader to [2].

Many fundamental and applied problems in graph theory revolve around the concept of graph partitioning, where the vertex set of a graph is divided into parts satisfying prescribed structural or combinatorial properties; among these, one of the most extensively studied directions involves partitions subject to minimum degree constraints. To provide a concise overview of results relevant to this study, we first introduce the following notions.

Let $\mathbb{N}$ denote the set of non-negative integers. For a function $f: V(G) \rightarrow \mathbb{N}$ defined on the vertex set of an arbitrary graph G , a subset $X \subseteq V(G)$ is said to be f -feasible if $d_X(u) \geq f(u)$ for every $u \in X$. Similarly, given two functions $a, b: V(G) \rightarrow \mathbb{N}$, a pair (A, B) of non-empty disjoint subsets of $V(G)$ is called (a, b) -feasible if A is a -feasible and B is b -feasible. An (a, b) -feasible pair satisfying $A \cup B = V(G)$ is referred to as an (a, b) -feasible partition.

Throughout this paper, $K_4 - e$ denotes the graph obtained by deleting a single edge from the complete graph K_4 , while $K_{2,3}$ denotes the complete bipartite graph with partite sets of cardinalities 2 and 3, respectively. Let $\mathcal{F}$ be the class of graphs obtained by joining one new vertex to exactly two vertices of $K_4 - e$. It is easy to see that $\mathcal{F} = \{K_{2,3}^+, H_1, H_2\}$ (see Fig.1), where $K_{2,3}^+$ denotes the graph obtained from $K_{2,3}$ by adding one edge inside the partite set of cardinality 2.

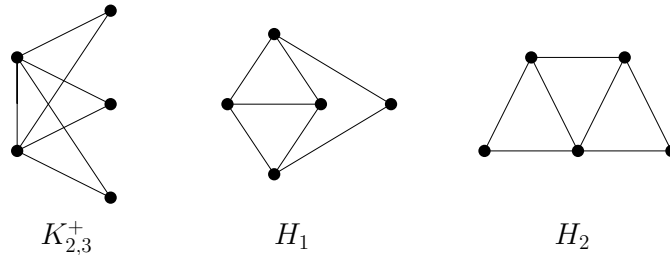


Figure 1: Graphs obtained by joining one new vertex to exactly two vertices of $K_4 - e$

For positive integers s and t , let $g(s, t)$ denote the minimum integer such that every graph G with minimum vertex degree $\delta(G) \geq g(s, t)$ admits a vertex partition (S, T) satisfying $\delta(G[S]) \geq s$ and $\delta(G[T]) \geq t$. When a graph class is specified below, the definition of $g(s, t)$ is restricted to graphs in that class. In 1983, Thomassen [11] conjectured that

$$g(s, t) = s + t + 1,$$

and thirteen years later this conjecture was confirmed by Stiebitz [10].

Theorem A ([10]). *For a graph G and two functions $a, b: V(G) \rightarrow \mathbb{N}$, if $d_G(u) \geq a(u) + b(u) + 1$ holds for every $u \in V(G)$, then G has an (a, b) -feasible partition.*

In subsequent work, a series of Stiebitz-type results has appeared. The degree condition in Theorem A can be relaxed when we impose restrictions on the (local) structure of the target graphs. For instance, Kaneko [7] showed that $g(s, t) \leq s + t$ holds whenever G is triangle-free whenever $\min\{s, t\} \geq 1$. Next, Hou et al. [6] established the following results.

Theorem B ([6]). *For a $\{K_{2,3}^+, H_1, H_2\}$ -free graph G with at least five vertices and two functions $a, b: V(G) \rightarrow \mathbb{N} \setminus \{0\}$, if $d_G(u) \geq a(u) + b(u)$ holds for every $u \in V(G)$, then G admits an (a, b) -feasible partition.*

Theorem C ([6]). *For a $K_{2,3}$ -free graph G and two functions $a, b: V(G) \rightarrow \mathbb{N} \setminus \{0, 1\}$, if $d_G(u) \geq a(u) + b(u)$ holds for every $u \in V(G)$, then G admits an (a, b) -feasible partition.*

These results are followed by additional findings that, while less central to the focus of the present paper, provide useful insights into related problems; in the following, we highlight the following.

In 2000, Diwan [3] proved that $g(s, t) \leq s + t - 1$ holds for graphs with girth at least five whenever $\min\{s, t\} \geq 2$. Gerber and Kobler [5] generalized Diwan's result to a specific function-type setting, not elaborated here. Another noteworthy contribution is due to Bazgan, Tuza and Vanderpooten [1], who presented polynomial-time algorithms to find (a, b) -feasible partitions satisfying the results of Stiebitz, Kaneko and Diwan, respectively. Liu and Xu [8] proved the following result, which generalizes Kaneko's theorem: if G is a K_4^- -free graph with at least four vertices and $a, b: V(G) \rightarrow \mathbb{N} \setminus \{0\}$ are two functions such that $d_G(u) \geq a(u) + b(u)$ for every $u \in V(G)$, then G admits an (a, b) -feasible partition. Ma and Yang [9] further extended Diwan's result by showing that $g(s, t) \leq s + t - 1$ holds for C_4 -free graphs whenever $\min\{s, t\} \geq 2$. Recently, Zeng and Zu [12] proved that a graph in which quadrangles do not share edges with triangles and other quadrangles admits an (a, b) -feasible partition whenever $d_G(u) \geq a(u) + b(u) - 1$ holds for every $u \in V(G)$, where $a, b: V(G) \rightarrow \mathbb{N} \setminus \{0, 1\}$ are two functions.

We proceed with a slightly extended approach introduced by Furuya and Maezawa [4]. Let $J(G)$ be the set of pairs $(u_1 u_2 \dots u_s, v_1 v_2 \dots v_t)$ of vertex sequences of G such that (a) $\{u_i : 1 \leq i \leq s\} \neq \{v_i : 1 \leq i \leq t\}$, (b) $s \in \{3, 4\}$ and $t \in \{3, 4\}$, (c) $(u_1, u_2) = (v_1, v_2)$ and (d) each sequence of $\{u_1, u_2, \dots, u_s\}$ and $\{v_1, v_2, \dots, v_t\}$ induces a cycle. Let $S_1(G) = \{u_1, u_2 : (u_1 u_2 \dots u_m, v_1 v_2 \dots v_n) \in J(G)\}$ and $S_0(G) = V(G) \setminus S_1(G)$.

Theorem D ([4]). *For a graph G and two functions $a, b: V(G) \rightarrow \mathbb{N}$, if $d_G(u) \geq a(u) + b(u) - 1 + 2h(u)$ and $\min\{a(u), b(u)\} \geq 2(1 - h(u))$ for $u \in S_{h(u)}(G)$, with $h(u) \in \{0, 1\}$, then G admits an (a, b) -feasible partition.*

By analyzing the preceding theorem, one may expect that a graph G admits an (a, b) -feasible partition whenever the vertices associated with a given subgraph have comparatively large degrees. This observation motivates the main result of the present work.

The following notation will be used throughout the remainder of the paper without further explicit reference. Let $\mathcal{H}$ be a class of fixed graphs, let $T_1(G)$ denote the set of all vertices of G belonging to at least one subgraph of G isomorphic to some graph of $\mathcal{H}$. Set $T_0(G) = V(G) \setminus T_1(G)$.

Theorem 1.1. *For a graph G with at least five vertices and two functions $a, b: V(G) \rightarrow \mathbb{N}$, if $d_G(u) \geq a(u) + b(u) + h(u)$ holds for $u \in T_{h(u)}(G)$, with $h(u) \in \{0, 1\}$, then G admits an (a, b) -feasible partition whenever one of the following holds:*

- (i) $\min\{a(u), b(u)\} \geq 1 - h(u)$ for $u \in T_{h(u)}(G)$ and $\mathcal{H} = \{K_{2,3}^+, H_1, H_2\}$;
- (ii) $\min\{a(u), b(u)\} \geq 2 - h(u)$ for $u \in T_{h(u)}(G)$ and $\mathcal{H} = \{K_{2,3}\}$.

One can easily check that Theorem 1.1 broadens these results of Theorems B and C by relaxing the degree conditions and allowing specific subgraph configurations that were not covered in previous work. Theorem 1.1(i) looks similar with Theorem D, and however, they are incomparable. Firstly, they concern with different graph categories. Secondly, it requires the same condition $d_G(u) \geq a(u) + b(u) + 1$ and $\min\{a(u), b(u)\} \geq 0$ for $u \in S_1(G)$ of Theorem D and $u \in T_1(G)$ of Theorem 1.1(i). However, Theorem D needs $d_G(u) \geq a(u) + b(u) - 1$ and $\min\{a(u), b(u)\} \geq 2$ for $u \in S_0(G)$, while Theorem 1.1(i) requires $d_G(u) \geq a(u) + b(u)$ and $\min\{a(u), b(u)\} \geq 1$ for $u \in T_0(G)$. Thus, it seems interesting for us to find a (stronger) unified result for both referred results.

The initial framework, together with relevant results, notions and notation, is presented in Section 2. The proof of Theorem 1.1 will be provided in Section 3.

2. PREPARATION

Everything stated in this section will be used in the next one. In particular, the initial assumptions given after the next lemma remain in force throughout the remainder of the paper.

Lemma 2.1 ([10]). *For a graph G and two functions $a, b: V(G) \rightarrow \mathbb{N}$, if $d_G(v) \geq a(v) + b(v) - 1$ holds for every vertex v of G , then G admits an (a, b) -feasible partition if and only if G admits an (a, b) -feasible pair.*

Henceforth, we assume that G has no (a, b) -feasible partition, and therefore G has no (a, b) -feasible pair, by the previous lemma. Next, we choose a and b so that

$$\sum_{u \in V(G)} (a(u) + b(u))$$

is as large as possible, subject to the hypotheses of the fixed alternative of Theorem 1.1 and the non-existence of (a, b) -feasible pairs. For the sake of simplicity, for $h \in \{0, 1\}$, we set $T_h := T_h(G)$.

We shall frequently use the following elementary observation concerning the case $\mathcal{H} = \{K_{2,3}^+, H_1, H_2\}$.

Lemma 2.2. Let $\mathcal{H} = \{K_{2,3}^+, H_1, H_2\}$, $u \in T_0(G)$ and let $S \subseteq V(G)$. If $u \in S$ with $|S| = 5$, then $G[S]$ contains no subgraph isomorphic to any graph of $\{K_{2,3}^+, H_1, H_2\}$.

Proof. Suppose to the contrary that $G[S]$ contains a subgraph H isomorphic to some graph of $\{K_{2,3}^+, H_1, H_2\}$. Note that each graph of $\{K_{2,3}^+, H_1, H_2\}$ has exactly five vertices, so H itself must be isomorphic to $K_{2,3}^+$ or H_1 or H_2 . Since $u \in V(H) = S$, we have $u \in T_1(G) \cap T_0(G) = \emptyset$, a contradiction. $\square$

We proceed with the following notation and notions.

- For $X \subseteq V(G)$, let

$$B_1(X) = \{u \in X : d_X(u) \leq a(u)\} \quad \text{and} \\ B_2(X) = \{u \in X : d_X(u) \leq b(u) + h(u) - 1\}.$$

- For $i \in \{1, 2\}$, a non-empty subset X of $V(G)$ is called *i-good* if $B_i(X) = \emptyset$.
- For $i \in \{1, 2\}$, a non-empty subset X of $V(G)$ is called *i-meager* if $B_i(X') \neq \emptyset$ holds for every non-empty subset X' of X .
- A partition (X_1, X_2) of $V(G)$ is called *degenerate* if X_i is *i-meager* for each $i \in \{1, 2\}$.
- A non-empty subset X of $V(G)$ is called *a-degenerate* if for every non-empty subset X' of X there exists a vertex $x \in X'$ such that $d_{X'}(x) \leq a(x)$.
- A non-empty subset X of $V(G)$ is called *a-nice* if $d_X(x) \geq a(x)$ holds for every $x \in X$.

The subsets that are *b-degenerate* or *b-nice* are defined analogously. Clearly, if X is 1-good, then X is *a-nice*. Similarly, if X is 2-good, then X is *b-nice*.

For two vertices $u, v \in V(G)$, let $e(u, v)$ denote the number of edges joining u and v . Clearly, $e(u, v) \in \{0, 1\}$. For $X \subseteq V(G)$, $e(X)$ denotes the number of edges with each end in X . For a vertex u and a subset $X \subseteq V(G)$, we write $e(u, X) = |N_G(u) \cap X|$; thus $e(u, X) = d_X(u)$.

For a partition (X_1, X_2) of $V(G)$, the *weight* $\omega(X_1, X_2)$ of (X_1, X_2) is defined as

$$\omega(X_1, X_2) = e(X_1) + e(X_2) + \sum_{u \in X_1} b(u) + \sum_{v \in X_2} a(v).$$

For $u \in B_1(X_1)$ and $v \in B_2(X_2)$, a straightforward algebraic calculation shows that

$$(1) \quad \begin{aligned} \omega(X_1 \setminus \{u\}, X_2 \cup \{u\}) - \omega(X_1, X_2) &= d_{X_2}(u) - d_{X_1}(u) + a(u) - b(u) \\ &\geq (b(u) + h(u)) - a(u) + a(u) - b(u) \\ &= h(u), \end{aligned}$$

and

$$\begin{aligned}
 \omega(X_1 \cup \{v\}, X_2 \setminus \{v\}) - \omega(X_1, X_2) &= d_{X_1}(v) - d_{X_2}(v) + b(v) - a(v) \\
 (2) \qquad \qquad \qquad &\geq (a(v) + 1) - (b(v) + h(v) - 1) + b(v) - a(v) \\
 &= 2 - h(v).
 \end{aligned}$$

The exchange of u and v yields a new partition: $X'_1 = X_1 \cup \{v\} \setminus \{u\}$, $X'_2 = X_2 \cup \{u\} \setminus \{v\}$. It follows from (1) and (2) that

$$\begin{aligned}
 \omega(X'_1, X'_2) - \omega(X_1, X_2) &= (d_{X_2}(u) - d_{X_1}(u) + a(u) - b(u)) \\
 &\quad + (d_{X_1}(v) - d_{X_2}(v) - a(v) + b(v)) - 2e(u, v) \\
 (3) \qquad \qquad \qquad &\geq 2 - h(v) + h(u) - 2e(u, v).
 \end{aligned}$$

Note that in the previous computation, the value of $h(u)$ is predetermined and depends solely on whether u belongs to T_1 ; that is, it is independent of whether u lies in X_1 or X_2 . Indeed, (T_0, T_1) is the partition of $V(G)$ fixed in Section 1, that is, $h(u) = 1$ for $u \in T_1$, and $h(u) = 0$ for $u \in T_0$. In part (i) of Theorem 1.1 the set T_1 includes all these vertices belong to a subgraph of G isomorphic to some graph of $\{K_{2,3}^+, H_1, H_2\}$; in part (ii) the set T_1 refers to these vertices belong to a subgraph of G isomorphic to $K_{2,3}$.

3. PROOF

In light of Theorems A, B and C, we may suppose that $T_0 \neq \emptyset$ and $T_1 \neq \emptyset$.

In what follows, we prove a sequence of claims, each under the assumptions introduced after Lemma 2.1. Taken together, these establish the proof of Theorem 1.1.

The first three claims examine the vertices of G .

Claim 3.1. *For every vertex $u \in V(G)$, $d_G(u) = a(u) + b(u) + h(u)$.*

Proof. Suppose that $d_G(u) \neq a(u) + b(u) + h(u)$ for some $u \in V(G)$. Then, $d_G(u) \geq a(u) + b(u) + h(u) + 1$. Let $a': V(G) \rightarrow \mathbb{N}$ be a function satisfying

$$a'(v) = \begin{cases} a(v) + 1, & v = u \\ a(v), & v \neq u. \end{cases}$$

From $d_G(u) \geq a(u) + b(u) + h(u) + 1 = a'(u) + b(u) + h(u)$, we deduce that a' and b satisfy

$$\sum_{v \in V(G)} (a'(v) + b(v)) > \sum_{v \in V(G)} (a(v) + b(v)).$$

Since every (a', b) -feasible pair is (a, b) -feasible, (a', b) admits no feasible pair by our hypothesis before. Moreover, a' still satisfies the relevant hypothesis of Theorem 1.1, contrary to the maximality of $\sum_{v \in V(G)} (a(v) + b(v))$. $\square$

Claim 3.2. *For every vertex $u \in V(G)$, $\min\{a(u), b(u)\} \geq 1$.*

Proof. If $\min\{a(u), b(u)\} \geq 2 - h(u)$, then this claim holds. Otherwise, $\min\{a(u), b(u)\} \geq 1 - h(u)$. By symmetry, suppose that $a(u) \leq 0$. Hence, $1 - h(u) \leq a(u) \leq 0$, and so $h(u) \geq 1$. This, together with definition of $h(u)$, forces $h(u) = 1$ and $a(u) = 0$.

Let $X_1 = \{u\}$ and $X_2 = V(G) \setminus \{u\}$. Then

$$d_{X_2}(v) \geq a(v) + b(v) + h(v) - e(u, v) \geq 1 - h(v) + b(v) + h(v) - 1 = b(v)$$

holds for every vertex $v \in X_2$. Therefore, (X_1, X_2) is an (a, b) -feasible partition of G , which contradicts the non-existence of such a partition (assumed in the previous section). $\square$

Claim 3.3. *For every vertex $u \in V(G)$, $V(G) \setminus \{u\}$ is 2-good.*

Proof. For every vertex $v \in V(G) \setminus \{u\}$, Claim 3.2 ensures

$$d_{V(G) \setminus \{u\}}(v) \geq (a(v) + b(v) + h(v)) - e(v, u) \geq b(v) + 1 + h(v) - 1 = b(v) + h(v).$$

Since v is arbitrary, $V(G) \setminus \{u\}$ is 2-good, as desired. $\square$

We now show the existence of a degenerate partition.

Claim 3.4. *There exists a degenerate partition of $V(G)$.*

Proof. By Claim 3.3, there exists a proper subset of $V(G)$ that is 2-good. Let X_2 be a 2-good subset of $V(G)$ with the minimum cardinality. By the minimality of X_2 , every proper non-empty subset of X_2 is 2-meager. Let $X_1 = V(G) \setminus X_2$. Since X_2 is a proper subset of $V(G)$, $X_1 \neq \emptyset$. If there exists a 1-good subset X'_1 of X_1 , then (X'_1, X_2) is an (a, b) -feasible pair of G , which is impossible. Hence, X_1 is 1-meager.

The minimality of X_2 yields the existence of a vertex $u \in X_2$ with $d_{X_2}(u) = b(u) + h(u)$. Indeed, if $d_{X_2}(w) \geq b(w) + h(w) + 1$ holds for every vertex $w \in X_2$, then $X_2 \setminus \{z\}$ would also be a 2-good subset of $V(G)$ for every vertex $z \in X_2$, contradicting the minimality of X_2 .

Let $u \in X_2$ be a vertex with $d_{X_2}(u) = b(u) + h(u)$, and set $X''_1 = X_1 \cup \{u\}$ and $X''_2 = X_2 \setminus \{u\}$. Obviously, (X''_1, X''_2) is a partition of $V(G)$, and to conclude the proof it is sufficient to show that X''_i is i -meager for $i \in \{1, 2\}$. Since

$$|X_2| \geq d_{X_2}(u) + 1 = b(u) + h(u) + 1 \geq 1 - h(u) + h(u) + 1 = 2,$$

X''_2 is non-empty. Hence, X''_2 is 2-meager. On the other hand, $d_{X''_1}(u) = d_{X_1}(u) = a(u)$ by Claim 3.1. Since X_1 is 1-meager, X''_1 is also 1-meager, and we are done. $\square$

Let $\mathcal{P}$ denote the family of degenerate partitions (X_1, X_2) of $V(G)$ satisfying the following two conditions:

- $\omega(X_1, X_2)$ is maximum;

- $|X_1|$ is minimum, subject to the previous item.

By Claim 3.4, $\mathcal{P} \neq \emptyset$. For every $(X_1, X_2) \in \mathcal{P}$, since X_i is i -meager, we have $B_i(X_i) \neq \emptyset$ for $i \in \{1, 2\}$.

Claim 3.5. *If $(X_1, X_2) \in \mathcal{P}$, then $\min\{|X_1|, |X_2|\} \geq 2$.*

Proof. Let $u \in B_2(X_2)$. By Claims 3.1 and 3.2, we have

$$|X_1| \geq d_{X_1}(u) \geq d_G(u) - (b(u) + h(u) - 1) \geq a(u) + 1 \geq 2.$$

To complete the argument it suffices to establish the same inequality for $|X_2|$. Assume that $|X_2| = 1$, i.e. $X_2 = \{u\}$. We begin by deriving a collection of structural constraints. Using these constraints, we then construct one or more (a, b) -feasible partitions which contradict the assumption, thereby excluding the case $|X_2| = 1$.

If $\min\{a(v), b(v)\} \geq 2 - h(v)$ holds for every $v \in X_1$, then

$$d_{X_1}(v) \geq a(v) + b(v) + h(v) - e(u, v) \geq a(v) + 2 - h(v) + h(v) - e(u, v) \geq a(v) + 1,$$

for every vertex $v \in X_1$. However, this contradicts the fact that $B_1(X_1) \neq \emptyset$. Therefore, we are in case (i) of Theorem 1.1, with $\mathcal{H} = \mathcal{F}$.

Since X_1 is 1-meager, there is a vertex $x \in B_1(X_1)$. Since $d_{X_2}(x) \geq b(x) + h(x) \geq 1$, by Claim 3.2, we have $ux \in E(G)$. Since x was chosen arbitrarily, $ux \in E(G)$ for every $x \in B_1(X_1)$.

If $(X_1 \setminus \{x\}, X_2 \cup \{x\})$ is a degenerate partition, then the inequality (1) implies

$$\omega(X_1 \setminus \{x\}, X_2 \cup \{x\}) - \omega(X_1, X_2) \geq h(x) \geq 0,$$

which contradicts the minimality of $|X_1|$. Thus, the partition is not degenerate, which implies that $X_2 \cup \{x\}$ is 2-good, and so $1 = d_{\{u, x\}}(u) \geq b(u) + h(u)$ and $1 = d_{\{u, x\}}(x) \geq b(x) + h(x)$. Hence, $b(u) = b(x) = 1$ and $h(u) = h(x) = 0$, by Claim 3.2. This, in particular, means that $x, u \in T_0$. In addition, for every $x \in B_1(X_1)$, from $ux \in E(G)$ and $d_{\{u, x\}}(x) = b(x) = 1$, by employing Claim 3.1 we obtain $d_{X_1}(x) = a(x)$.

From $x \in T_0 \cap B_1(X_1)$ and (again) $ux \in E(G)$, we have

$$|N_{X_1}(x) \cap B_1(X_1)| \leq 2,$$

as $|N_{X_1}(x) \cap B_1(X_1)| \geq 3$ would imply $x \in T_1(G)$.

If $|N_{X_1}(x_0) \cap B_1(X_1)| = 0$ for some $x_0 \in B_1(X_1)$, then $(X_1 \setminus \{x_0\}, X_2 \cup \{x_0\})$ is an (a, b) -feasible partition, which is impossible. We select $y_0 \in N_{X_1}(x_0) \cap B_1(X_1)$. By the symmetry of x_0 and y_0 , we have $1 \leq |N_{X_1}(y_0) \cap B_1(X_1)| \leq 2$.

We claim that $|N_{X_1}(y_0) \cap B_1(X_1)| = |N_{X_1}(x_0) \cap B_1(X_1)| = 1$. The other possibility for these cardinalities is 2. Without loss of generality, suppose that $N_{X_1}(y_0) \cap B_1(X_1) = \{x_0, w_0\}$. Since $|V(G)| \geq 5$, we have $X_1 \neq \{w_0, y_0, x_0\}$. Let $x \in X_1 \setminus \{w_0, y_0, x_0\}$. If $x \in B_1(X_1)$, then $N_G(x) \cap \{w_0, x_0, y_0\} = \emptyset$, otherwise $K_{2,3}^+ \subseteq G[\{w_0, y_0, u, x_0, x\}]$ or $H_2 \subseteq G[\{w_0, y_0, x_0, u, x\}]$ and $u \in T_0$, contradicting

Lemma 2.2. If $x \notin B_1(X_1)$, then $|N_G(x) \cap \{w_0, y_0, x_0\}| \leq 1$ and $d_{X_1}(x) \geq a(x) + 1$, by the same argument. Let $X_1'' = X_1 \setminus \{w_0, y_0, x_0\}$ and $X_2'' = X_2 \cup \{w_0, y_0, x_0\}$. Irrespective of whether $x \in B_1(X_1)$, we have $d_{X_1''}(x) \geq a(x)$ and $d_{X_2''}(y) \geq 2 > b(y)$ for every $x \in X_1''$ and $y \in X_2''$. Thus, (X_1'', X_2'') is an (a, b) -feasible partition, which is impossible. This confirms our claim.

On the basis of $|V(G)| \geq 5$, let $x \in X_1 \setminus \{x_0, y_0\} \neq \emptyset$. If $|N(x) \cap \{x_0, y_0\}| \leq 1$ holds for every $x \in X_1 \setminus \{x_0, y_0\}$, then $(X_1 \setminus \{x_0, y_0\}, X_2 \cup \{x_0, y_0\})$ is an (a, b) -feasible partition. Thus, there exists a vertex $x \in X_1 \setminus B_1(X_1)$ such that $|N(x) \cap \{x_0, y_0\}| = 2$. By Lemma 2.2 and $u \in T_0$, x features as the unique vertex of $X_1 \setminus B_1(X_1)$ with $|N(x) \cap \{x_0, y_0\}| = 2$.

Finally, if $d_{X_1}(x) \geq a(x) + 2$, then $(X_1 \setminus \{x_0, y_0\}, X_2 \cup \{x_0, y_0\})$ is an (a, b) -feasible partition. Hence, $d_{X_1}(x) = a(x) + 1$. From $X_1 \setminus \{x, y_0, x_0\} \neq \emptyset$ and Lemma 2.2, we have $|N(v) \cap \{x, y_0, x_0\}| \leq 1$ for every $v \in X_1 \setminus \{x, y_0, x_0\}$, as $u \in T_0$. Once again, by Lemma 2.2 and $u \in T_0$, if $|N(v) \cap \{x, y_0, x_0\}| = 1$, then $v \notin B_1(X_1)$. Hence, $X_1 \setminus \{x, y_0, x_0\}$ is a -nice. Since $b(x) + h(x) = 1 + e(u, x) \leq 2$ and x has the neighbours x_0, y_0 in the second part, we deduce that $(X_1 \setminus \{x, y_0, x_0\}, X_2 \cup \{x, y_0, x_0\})$ features as an (a, b) -feasible partition, the final contradiction. $\square$

We proceed by proving the existence of a particular good subset.

Claim 3.6. *Let $(X_1, X_2) \in \mathcal{P}$. For $u_i \in B_i(X_i)$, there exists a $(3-i)$ -good subset of $X_{3-i} \cup \{u_i\}$ containing u_i , where $i \in \{1, 2\}$.*

Proof. It follows from (1) and (2) that

$$\omega(X_1 \setminus \{u_1\}, X_2 \cup \{u_1\}) - \omega(X_1, X_2) \geq h(u_1) \geq 0,$$

and

$$\omega(X_1 \cup \{u_2\}, X_2 \setminus \{u_2\}) - \omega(X_1, X_2) \geq 2 - h(u_2) \geq 1.$$

This, together with the maximality of $\omega(X_1, X_2)$ and the minimality of $|X_1|$, implies that the partitions $(X_1 \setminus \{u_1\}, X_2 \cup \{u_1\})$ and $(X_1 \cup \{u_2\}, X_2 \setminus \{u_2\})$ are not degenerate.

Since $|X_1| \geq 2$, we have $X_1 \setminus \{u_1\} \neq \emptyset$. Observing that X_1 is 1-meager, we find that the same holds for $X_1 \setminus \{u_1\}$, and so $X_2 \cup \{u_1\}$ is not 2-meager. Hence, there exists a 2-good subset X_2' of $X_2 \cup \{u_1\}$. Since X_2 is 2-meager, we have $u_1 \in X_2'$, and so the desired result holds for $i = 1$. The case $i = 2$ is considered analogously. $\square$

Within the same degenerate partition, we now establish a structural characterisation of the vertices of $B_1(X_1)$, formulated in relation to the vertices of $B_2(X_2)$.

Claim 3.7. *For $(X_1, X_2) \in \mathcal{P}$, every vertex in $B_1(X_1)$ is adjacent to all vertices of $B_2(X_2)$.*

Proof. By Claim 3.5, we have $\min\{|X_1|, |X_2|\} \geq 2$. Suppose that there exists $u_1 \in B_1(X_1)$ and $u_2 \in B_2(X_2)$ such that $u_1 u_2 \notin E(G)$. By Claim 3.6, there

exists a 2-good subset B of $X_2 \cup \{u_1\}$ with $u_1 \in B$. Since $u_1 u_2 \notin E(G)$ and $d_{X_2}(u_2) \leq b(u_2) + h(u_2) - 1$, we have $u_2 \notin B$.

Analogously, by considering $(X_1 \cup \{u_2\}, X_2 \setminus \{u_2\})$, we deduce the existence of a 1-good subset A of $X_1 \cup \{u_2\}$, with $u_2 \in A$ and $u_1 \notin A$.

Thus, (A, B) features as an (a, b) -feasible pair of G , and we are done. $\square$

We next examine a perturbation of the initial degenerate partition, achieved through the exchange of two designated vertices.

Claim 3.8. *Let $(X_1, X_2) \in \mathcal{P}$ and $u_i \in B_i(X_i)$ for $i \in \{1, 2\}$. If $h(u_1) \geq h(u_2)$, then $((X_1 \setminus \{u_1\}) \cup \{u_2\}, (X_2 \setminus \{u_2\}) \cup \{u_1\}) \in \mathcal{P}$, $u_1 u_2 \in E(G)$ and $h(u_1) = h(u_2)$.*

Proof. Let $X'_i = (X_i \setminus \{u_i\}) \cup \{u_{3-i}\}$. We first prove that (X'_1, X'_2) is a degenerate partition. It suffices to show that X'_i is i -meager for each $i \in \{1, 2\}$.

We show that X'_2 is 2-meager. For the sake of contradiction, we assume the opposite. Since X_2 is 2-meager, there exists a 2-good subset B of X'_2 with $u_1 \in B$. It follows from Claim 3.6 that there exists a 1-good subset A of $X_1 \cup \{u_2\}$ containing u_2 . If $u_1 \notin A$, then (A, B) is an (a, b) -feasible pair of G . Otherwise,

$$a(u_1) + 1 \leq d_{X_1 \cup \{u_2\}}(u_1) = d_{X_1}(u_1) + e(u_1, u_2) \leq a(u_1) + 1$$

forces $d_{X_1}(u_1) = a(u_1)$ and $u_1 u_2 \in E(G)$. Hence, $d_{X_2}(u_1) = b(u_1) + h(u_1)$ and $d_{X_2 \setminus \{u_2\}}(u_1) = b(u_1) + h(u_1) - 1$, but this violates the assumption $u_1 \in B$.

We now show that X'_1 is 1-meager. Again we assume the opposite. Since X_1 is 1-meager, there exists a 1-good subset A of X'_1 with $u_2 \in A$. It follows from Claim 3.6 that there exists a 2-good subset B of $X_2 \cup \{u_1\}$ containing u_1 . The case $u_2 \notin B$ would imply that (A, B) is an (a, b) -feasible pair of G . Therefore, we have

$$\begin{aligned} b(u_2) + h(u_2) &\leq d_{X_2 \cup \{u_1\}}(u_2) = d_{X_2}(u_2) + e(u_1, u_2) \\ &\leq b(u_2) + h(u_2) - 1 + 1 = b(u_2) + h(u_2), \end{aligned}$$

along with $d_{X_2}(u_2) = b(u_2) + h(u_2) - 1$ and $u_1 u_2 \in E(G)$. Hence, $d_{X_1}(u_2) = a(u_2) + 1$ and $d_{X'_1}(u_2) = d_{X_1 \setminus \{u_1\}}(u_2) = a(u_2)$, violating $u_2 \in A$.

In conclusion, (X'_1, X'_2) is a degenerate partition of $V(G)$. Since $u_i \in B_i(X_i)$ and $h(u_1) \geq h(u_2)$, inequality (3) yields

$$\omega(X'_1, X'_2) - \omega(X_1, X_2) \geq 2 - h(u_2) + h(u_1) - 2e(u_1, u_2) \geq h(u_1) - h(u_2) \geq 0.$$

By the maximality of $\omega(X_1, X_2)$, we have $\omega(X_1, X_2) = \omega(X'_1, X'_2)$ and forces that $e(u_1, u_2) = 1$ and $h(u_1) = h(u_2)$ in (3), which is what gives the final assertion. $\square$

Claim 3.9. *Let $(X_1, X_2) \in \mathcal{P}$. If $u \in B_1(X_1)$ and $v \in B_2(X_2)$, then $d_{X_1}(u) = a(u)$ and $d_{X_2}(v) = b(v) + h(v) - 1$, this also implies that, X_1 is a -nice.*

Proof. By Claim 3.5, we have $\min\{|X_2|, |X_1|\} \geq 2$. By Claim 3.6, there exists a subset $A \subseteq X_1$ such that $A \cup \{v\}$ is 1-good, which implies that A is a -nice.

We prove that X_1 is a -nice. Otherwise, there exists a vertex $x \in X_1 \setminus A$ such that $d_{X_1}(x) \leq a(x) - 1$. Let $X_1'' = X_1 \setminus \{x\}$ and $X_2'' = X_2 \cup \{x\}$. By employing (1), we find

$$\begin{aligned} \omega(X_1'', X_2'') - \omega(X_1, X_2) &= d_{X_2}(x) - d_{X_1}(x) + a(x) - b(x) \\ &\geq (b(x) + h(x) + 1) - (a(x) - 1) + a(x) - b(x) \\ &= 2 + h(x) \geq 2. \end{aligned}$$

Since X_1'' is 1-meager, there exists a 2-good subset $B \subseteq X_2''$ such that $x \in B$. Observing that $x \notin A$, we deduce that (A, B) is an (a, b) -feasible pair, a contradiction. Hence, X_1 is a -nice, and so $d_{X_1}(u) = a(u)$.

Claim 3.1, together with $d_{X_1}(u) = a(u)$, yields $d_{X_2}(u) = b(u) + h(u)$. By Claim 3.6, let B be a 2-good subset of $X_2 \cup \{u\}$ with $u \in B$. We claim that $v \in B$ for every $v \in B_2(X_2)$. Otherwise, suppose $v \notin B$ and take, by Claim 3.6, a 1-good $A \subseteq X_1 \cup \{v\}$ with $v \in A$. Then A , or $A \setminus \{u\}$ if $u \in A$, is non-empty, a -nice (since 1-goodness gives $d_A(x) \geq a(x) + 1$) and disjoint from B . Either way this gives an (a, b) -feasible pair, a contradiction. This proves that $v \in B$, which implies $d_{X_2}(v) = b(v) + h(v) - 1$. $\square$

Collecting all preceding observations, we arrive at the final step and thus complete the proof.

Proof of Theorem 1.1. If $B_2(X_2) \subseteq T_1$, then (X_1, X_2) is an (a, b) -feasible partition by Claim 3.9. Therefore, $B_2(X_2) \cap T_0 \neq \emptyset$. In what follows, we always suppose that

$$v_0 \in B_2(X_2) \cap T_0 \quad \text{and} \quad u_0 \in B_1(X_1).$$

If $u_0 \in B_1(X_1) \cap T_1$, then by Claim 3.8 we have $(X_1 \setminus \{u_0\} \cup \{v_0\}, X_2 \setminus \{v_0\} \cup \{u_0\}) \in \mathcal{P}$ and $v_0 \in T_1$, contradicting $v_0 \in B_2(X_2) \cap T_0$. Therefore, $B_1(X_1) \subseteq T_0$.

If $B_1(X_1) \cap N_{X_1}(u_0) = \emptyset$, then there exists $x \in (X_1 \setminus B_1(X_1)) \cap N_{X_1}(u_0)$ such that $d_{X_1}(x) \geq a(x) + 1$, as $d_{X_1}(u_0) = a(u_0) \geq 1$. Since $N_{X_1}(u_0) \subseteq X_1 \setminus B_1(X_1)$, $X_1 \setminus \{u_0\}$ is a -nice by Claim 3.9. Let $X_1' = X_1 \setminus \{u_0\}$ and $X_2' = X_2 \cup \{u_0\}$. By Claim 3.6, there exists a 2-good subset $B \subseteq X_2'$ such that $u_0 \in B$. Therefore, (X_1', B) is an (a, b) -feasible pair, a contradiction. Thus, $B_1(X_1) \cap N_{X_1}(u_0) \neq \emptyset$.

Gathering the previous conclusions, we arrive at the following setting:

$$(4) \quad v_0 \in B_2(X_2) \cap T_0 \neq \emptyset, \quad u_0 \in B_1(X_1) \subseteq T_0 \quad \text{and} \quad B_1(X_1) \cap N_{X_1}(u_0) \neq \emptyset.$$

We now consider separately the two cases arising from parts (i) and (ii) of the theorem.

- *Case 1:* $H \in \mathcal{H} = \{K_{2,3}^+, H_1, H_2\}$.

We first claim that $|N_{X_1}(u_0) \cap B_1(X_1)| \leq 2$. Otherwise, suppose that $|N_{X_1}(u_0) \cap B_1(X_1)| \geq 3$ and choose three vertices $u_1, u_2, u_3 \in N_{X_1}(u_0) \cap B_1(X_1)$. By (4) we may take $v_0 \in B_2(X_2)$ and by Claim 3.7, the vertex $v_0 \in B_2(X_2)$ is adjacent to every vertex of $\{u_0, u_1, u_2, u_3\}$. Hence $K_{2,3}^+ \subseteq G[\{v_0, u_0, u_1, u_2, u_3\}]$ and

$u_0 \in T_0$, contrary with Lemma 2.2. Thus, by (4), we have $1 \leq |N_{X_1}(u_0) \cap B_1(X_1)| \leq 2$. We choose $y_0 \in N_{X_1}(u_0) \cap B_1(X_1)$. By the symmetry of u_0 and y_0 , we have $1 \leq |N_{X_1}(y_0) \cap B_1(X_1)| \leq 2$.

To facilitate reading, we briefly outline the remainder of this part of the proof:

- (a) we first establish the forthcoming equalities (5),
- (b) we then consider the case $X_1 = \{u_0, y_0\}$,
- (c) and we finally exclude the case $X_1 \neq \{u_0, y_0\}$,

showing in each instance that an appropriate (a, b) -feasible pair exists, thereby eliminating the case.

We proceed with (a), i.e., we claim that

$$(5) \quad |N_{X_1}(y_0) \cap B_1(X_1)| = |N_{X_1}(u_0) \cap B_1(X_1)| = 1.$$

Otherwise, assume that $\max\{|N_{X_1}(y_0) \cap B_1(X_1)|, |N_{X_1}(u_0) \cap B_1(X_1)|\} = 2$. Without loss of generality, suppose that $N_{X_1}(y_0) \cap B_1(X_1) = \{u_0, w_0\}$. If $|B_2(X_2)| \geq 2$, then let $v_0, v_1 \in B_2(X_2)$. By Claim 3.7, $H_1 \subseteq G[\{v_0, v_1, u_0, y_0, w_0\}]$ and $u_0, y_0, w_0 \in T_0$, contrary to Lemma 2.2. Thus, $|B_2(X_2)| = 1$. From our hypothesis on (4), we have $B_2(X_2) = \{v_0\}$.

Assuming that $X_1 = \{w_0, y_0, u_0\}$, we let $X'_1 = \{w_0, y_0, v_0, u_0\}$ and $X'_2 = V(G) \setminus X'_1$. Since $K_4 - e \subseteq G[\{w_0, y_0, v_0, u_0\}]$, every vertex $y \in X_2 \setminus \{v_0\}$ is adjacent to at most one vertex of $\{w_0, y_0, v_0, u_0\}$. This implies

$$d_{X'_2}(y) \geq a(y) + b(y) + h(y) - 1 \geq b(y) + 1 + h(y) - 1 = b(y) + h(y).$$

By Claim 3.6, there exists a 1-good subset of X'_1 . Thus, (X'_1, X'_2) contains an (a, b) -feasible pair, it is impossible. Therefore, $X_1 \neq \{w_0, y_0, u_0\}$. Let $x \in X_1 \setminus \{w_0, y_0, u_0\}$. If $x \in B_1(X_1)$, then $N_G(x) \cap \{w_0, y_0, u_0\} = \emptyset$ (otherwise, by Claim 3.7, either $K_{2,3}^+ \subseteq G[\{w_0, y_0, v_0, u_0, x\}]$ or $H_2 \subseteq G[\{w_0, y_0, v_0, u_0, x\}]$, contrary to Lemma 2.2 and $u_0 \in T_0$). If $x \notin B_1(X_1)$, then $|N_G(x) \cap \{w_0, y_0, u_0\}| \leq 1$ (otherwise, $H_1 \subseteq G[\{w_0, y_0, v_0, u_0, x\}]$ or $H_2 \subseteq G[\{w_0, y_0, v_0, u_0, x\}]$, contrary to Lemma 2.2 and $u_0 \in T_0$) and $d_{X_1}(x) \geq a(x) + 1$. Let $X''_1 = X_1 \setminus \{w_0, y_0, u_0\}$ and $X''_2 = X_2 \cup \{w_0, y_0, u_0\}$. Irrespective of whether $x \in B_1(X_1)$ or not, we have $d_{X''_1}(x) \geq a(x)$. By Claim 3.6, there exists a 2-good subset of X''_2 . Thus, (X''_1, X''_2) contains an (a, b) -feasible pair, a contradiction. This implies that (5) holds.

With (5), we first consider the case (b): $X_1 = \{u_0, y_0\}$. Note that $|V(G)| \geq 5$, which shows $|X_2| \geq 3$. If $|N_G(v) \cap \{v_0, y_0, u_0\}| \leq 1$ for every $v \in X_2 \setminus \{v_0\}$, then we set $X'_1 = \{v_0, y_0, u_0\}$ and $X'_2 = X_2 \setminus \{v_0\}$. In this setting, X'_1 contains an a -nice set by Claim 3.6, which in light of Claim 3.1 implies

$$d_{X'_2}(v) \geq d_G(v) - d_{X'_1}(v) \geq a(v) + b(v) + h(v) - 1 \geq 1 + b(v) + h(v) - 1 = b(v) + h(v),$$

for $v \in X'_2$. Hence, (X'_1, X'_2) contains an (a, b) -feasible pair, a contradiction.

Thus, there exists a vertex $v \in X_2 \setminus \{v_0\}$ such that $|N_G(v) \cap \{v_0, y_0, u_0\}| \geq 2$. Since $u_0 \in T_0$ by (4), there is exactly one vertex, say v_1 , in $X_2 \setminus \{v_0\}$ such that $|N_G(v_1) \cap \{v_0, y_0, u_0\}| \geq 2$ and $|N(v_2) \cap \{v_0, v_1, y_0, u_0\}| \leq 1$ holds for every $v_2 \in X_2 \setminus \{v_0, v_1\}$. Since

$$\begin{aligned} d_{X_2 \setminus \{v_0, v_1\}}(w) &= d_G(w) - d_{X_1 \cup \{v_0, v_1\}}(w) \geq a(w) + b(w) + h(w) - 1 \\ &\geq 1 + b(w) + h(w) - 1 = b(w) + h(w) \end{aligned}$$

for $w \in X_2 \setminus \{v_0, v_1\}$, Claim 3.6 ensures that $(\{u_0, y_0, v_0, v_1\}, X_2 \setminus \{v_0, v_1\})$ contains an (a, b) -feasible pair, a contradiction.

We now consider the case (c) in which $\{u_0, y_0\}$ is a proper subset of X_1 , and set $x \in X_1 \setminus \{u_0, y_0\}$. We claim that there exists $x_0 \in X_1 \setminus \{u_0, y_0\}$ such that $x_0 \notin B_1(X_1)$ and $|N(x_0) \cap \{u_0, y_0\}| = 2$. Otherwise, assume that $|N(x) \cap \{u_0, y_0\}| \leq 1$ holds for every $x \in X_1 \setminus B_1(X_1)$. If $y \in B_1(X_1) \setminus \{u_0, y_0\}$, then $|N(y) \cap \{u_0, y_0\}| = 0$, as $|N_{X_1}(y_0) \cap B_1(X_1)| = |N_{X_1}(u_0) \cap B_1(X_1)| = 1$. Combining this with the previous inequality, we deduce that $X_1 \setminus \{u_0, y_0\}$ and $X_2 \cup \{u_0, y_0\}$ create an (a, b) -feasible pair, by Claim 3.6, a contradiction. This confirms the claim.

Also, x_0 is the unique vertex in $X_1 \setminus B_1(X_1)$ such that $|N(x_0) \cap \{u_0, y_0\}| = 2$ (otherwise there exists a vertex $x_1 \in X_1 \setminus B_1(X_1)$ with $x_1 \neq x_0$ such that $K_{2,3}^+ \subseteq G[\{y_0, v_0, u_0, x_0, x_1\}]$ and $u_0 \in T_0$, contrary to Lemma 2.2). The possibility $d_{X_1}(x_0) \geq a(x_0) + 2$ is eliminated by the existence of an (a, b) -feasible pair $(X_1 \setminus \{u_0, y_0\}, X_2 \cup \{u_0, y_0\})$. Hence, $d_{X_1}(x_0) = a(x_0) + 1$.

If $X_1 \setminus \{u_0, y_0, x_0\} \neq \emptyset$, then by Lemma 2.2, we have $|N(v) \cap \{u_0, y_0, x_0\}| \leq 1$ for $v \in X_1 \setminus \{u_0, y_0, x_0\}$. In addition $N_G(x_0) \cap B_1(X_1) = \{u_0, y_0\}$ (otherwise there exists a vertex $w_0 \in B_1(X_1)$ such that $w_0, u_0, y_0 \in N_G(x_0)$. By Claim 3.7, $H_1 \subseteq G[\{y_0, v_0, u_0, x_0, w_0\}]$ and $u_0 \in T_0$, contrary to Lemma 2.2). Hence, $X_1 \setminus \{u_0, y_0, x_0\}$ is a -nice, and Claim 3.6 ensures that $(X_1 \setminus \{u_0, y_0, x_0\}, X_2 \cup \{u_0, y_0, x_0\})$ contains an (a, b) -feasible pair, a contradiction. Thus, $X_1 = \{u_0, y_0, x_0\}$.

Since $u_0 \in T_0$, we have $B_2(X_2) = \{v_0\}$ and $|N(v) \cap \{v_0, y_0, x_0, u_0\}| \leq 1$ holds for every $v \in X_2 \setminus \{v_0\}$. We set $X'_1 = X_1 \cup \{v_0\}$ and $X'_2 = X_2 \setminus \{v_0\}$. In this context $d_{X'_2}(v) \geq a(v) + b(v) + h(v) - 1 \geq 1 + b(v) + h(v) - 1 = b(v) + h(v)$ holds for every $v \in X'_2$, which, together with Claim 3.6, implies that (X'_1, X'_2) contains an (a, b) -feasible pair, a contradiction. This completes the last possibility for X_1 , thereby finalizes the proof of Case 1.

• *Case 2: $H = K_{2,3}$.*

If $|B_1(X_1)| = 1$, then $B_1(X_1) = \{u_0\}$. For every $x \in X_1 \setminus B_1(X_1)$, since $d_{X_1}(x) \geq a(x) + 1$, $X_1 \setminus \{u_0\}$ is a -nice. Combining this with Claim 3.6, we find that $(X_1 \setminus \{u_0\}, X_2 \cup \{u_0\})$ contains an (a, b) -feasible pair. Thus, $|B_1(X_1)| \geq 2$.

Since $H = K_{2,3}$ and $u_0 \in T_0$ by (4), we have $|B_2(X_2)| \leq 2$ by Claim 3.7. We first claim that $|B_2(X_2)| = 1$. Otherwise, suppose that $|B_2(X_2)| = 2$. Since $u_0 \in T_0$, we also have $|B_1(X_1)| = 2$. From $d_{X_1}(u_0) = a(u_0) \geq 2 - h(u_0) = 2$, we have $X_1 \setminus B_1(X_1) \neq \emptyset$. Now, $x \in X_1 \setminus B_1(X_1)$ is adjacent to at most one vertex of $B_1(X_1)$, as $u_0 \in T_0$. This implies that $X_1 \setminus B_1(X_1)$ is a -nice. By Claim 3.6, $(X_1 \setminus B_1(X_1), X_2 \cup B_1(X_1))$ contains an (a, b) -feasible pair, which confirms the claim.

Therefore, $|B_2(X_2)| = 1$, say $B_2(X_2) = \{v_0\}$. By (4), we also have $v_0 \in T_0$. By (4) and Claim 3.9, we also have $v_0 \in T_0$ and $d_{X_2}(v_0) \leq b(v_0) - 1$, as G does not admit an (a, b) -feasible pair.

We now claim the existence of a vertex $v \in T_0 \cap X_2$ such that $v_0v \in E(G)$ and $d_{X_2}(v) = b(v)$. Let $X'_1 = X_1 \cup \{v_0\}$ and $X'_2 = X_2 \setminus \{v_0\}$. By Claim 3.6, there exists a 1-good subset A of X'_1 containing v_0 . Since G does not admit an (a, b) -feasible pair, there must exist a vertex $v \in X'_2$ such that $d_{X'_2}(v) \leq b(v) - 1$. Hence, we have

$$b(v) + h(v) \leq d_{X_2}(v) = d_{X'_2}(v) + e(v_0, v) \leq b(v) - 1 + 1 = b(v),$$

as $v \notin B_2(X_2)$. This forces that $e(v_0, v) = 1$, $h(v) = 0$ and $d_{X_2}(v) = b(v)$, as desired.

If there exists a vertex x_0 of $B_1(X_1)$ such that $x_0v \notin E(G)$, then $(X_1 \setminus \{x_0\} \cup \{v_0\}, X_2 \setminus \{v_0\} \cup \{x_0\}) \in \mathcal{P}$ by Claim 3.8, as $v_0 \in T_0$. Since $d_{X_2}(v) = b(v)$ and $x_0v \notin E(G)$, we have $x_0, v \in B_2(X_2 \setminus \{v_0\} \cup \{x_0\})$ by Claim 3.9. The remainder follows the line of the previous part of the proof eliminating $|B_2(X_2)| \geq 2$. Thus, v is adjacent to every vertex of $B_1(X_1)$.

Since $v_0 \in T_0$, we have

$$|X_1| \geq d_{X_1}(v_0) = a(v_0) + b(v_0) + h(v_0) - (b(v_0) - 1) \geq 2 - h(v_0) + h(v_0) + 1 = 3.$$

Since $u_0 \in T_0$ and v is adjacent to every vertex of $B_1(X_1)$, we have $|B_1(X_1)| = 2$ and every $x \in X_1 \setminus B_1(X_1)$ is adjacent to at most one vertex of $B_1(X_1)$. This implies that $(X_1 \setminus B_1(X_1), X_2 \cup B_1(X_1))$ contains an (a, b) -feasible pair. This was the final contradiction, completing Case 2 and the entire proof. $\square$

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