

NECESSARY AND SUFFICIENT CONDITIONS OF CUSA–HUYGENS TYPE INEQUALITIES INVOLVING GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Tiehong Zhao 

This paper investigates Cusa–Huygens type inequalities for generalized trigonometric and hyperbolic functions. Using absolute monotonicity and NP/PN-type power series, we establish necessary and sufficient conditions for these inequalities and their reverses for $p \in (1, +\infty)$. We extend the range of validity of a closely related Cusa–Huygens type inequality from $p \in [2, +\infty)$ to $p \in (1, +\infty)$, prove a long-standing conjecture for $p \in [\sqrt{2} + 1, +\infty)$, and derive refined inequalities with explicit remainder terms.

1. INTRODUCTION

The classical Cusa–Huygens inequality is a cornerstone of trigonometric functional inequalities. Originally formulated by Nicolaus de Cusa (1401–1464) and rigorously established by Huygens [11], it asserts that for $x \in (0, \pi/2)$,

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3}.$$

Over the years, this elementary inequality has been refined and extended through various approaches (see, e.g., [3, 5, 7, 19]). In recent decades, the emergence of generalized trigonometric and hyperbolic functions has opened new avenues

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for generalizing this inequality. These generalized functions play a critical role in nonlinear differential equations involving the p -Laplacian [9], approximation theory [10, 16], and eigenvalue problems for nonstandard operators [6]. Consequently, their inequality properties have become a focal point in modern special function research.

Let $p \in (1, +\infty)$ and define a differentiable function $F_p : [0, 1) \mapsto (0, +\infty)$ by

$$(1) \quad F_p(x) = \int_0^x (1 - t^p)^{-1/p} dt.$$

This function is interpreted as the generalized inverse sine function, i.e., $F_p = \arcsin_p$, since it reduces to the classical inverse sine when $p = 2$ ($F_2(x) = \arcsin x$). As F_p is strictly increasing, it admits an inverse, denoted by $\sin_p x$ (the generalized sine function), whose domain and periodicity are determined by p . For $p \in (1, +\infty)$, $\sin_p x$ is initially defined on $[0, \pi_p/2]$, where

$$\pi_p = 2 \int_0^1 (1 - t^p)^{-1/p} dt = \frac{2\pi}{p \sin(\pi/p)}.$$

It is then extended to the entire real line $\mathbb{R}$ by odd symmetry and $2\pi_p$ -periodicity.

The generalized cosine function $\cos_p x$ is defined as the derivative of $\sin_p x$, i.e., $\cos_p x = \frac{d}{dx} \sin_p x$. It is even, $2\pi_p$ -periodic, and satisfies the key identity

$$(2) \quad \cos_p x = (1 - \sin_p^p x)^{1/p}, \quad x \in [0, \pi_p/2],$$

which in turn yields the generalized Pythagorean identity

$$(3) \quad \sin_p^p x + \cos_p^p x = 1.$$

For $p = 2$, these functions reduce to their classical counterparts, as expected.

Analogously, the generalized hyperbolic functions are defined as follows:

- The generalized inverse hyperbolic sine $\operatorname{arcsinh}_p x$ is given for $x \in (0, +\infty)$ by

$$(4) \quad \operatorname{arcsinh}_p x = \int_0^x (1 + t^p)^{-1/p} dt.$$

- The generalized inverse hyperbolic tangent $\operatorname{arctanh}_p x$ is given for $x \in (0, 1)$ by

$$(5) \quad \operatorname{arctanh}_p x = \int_0^x (1 - t^p)^{-1/p} dt.$$

The generalized hyperbolic sine $\sinh_p x$ is the inverse of $\operatorname{arcsinh}_p x$. The generalized hyperbolic cosine and tangent are then defined by

$$\cosh_p x = \frac{d}{dx} \sinh_p x, \quad \tanh_p x = \frac{\sinh_p x}{\cosh_p x}.$$

These functions are extended to negative x by parity: $\sinh_p(-x) = -\sinh_p x$ and $\cosh_p(-x) = \cosh_p x$. Mirroring the classical identity $\cosh^2 x - \sinh^2 x = 1$, the generalized hyperbolic functions satisfy

$$(6) \quad \cosh_p^p x - \sinh_p^p x = 1.$$

The study of these functions and their generalizations was initiated by Lindqvist [17, 18] (see also [14] and the references therein).

Early efforts to extend the Cusa–Huygens inequality to p -trigonometric and hyperbolic functions, while yielding important results, were largely confined to restricted parameter ranges. In 2014, Klén, Vuorinen, and Zhang [15] established the following Cusa–Huygens type inequalities for $p \in (1, 2]$:

- For $x \in (0, \pi_p/2]$,

$$(7) \quad \frac{\sin_p x}{x} < \frac{\cos_p x + p}{1 + p} \leq \frac{\cos_p x + 2}{3}.$$

- For $x \in (0, +\infty)$,

$$(8) \quad \frac{\sinh_p x}{x} < \frac{\cosh_p x + p}{1 + p}.$$

For $p \in [2, +\infty)$, they further proved a reverse trigonometric inequality

$$(9) \quad \frac{\sin_p x}{x} > \frac{p - 1 + \cos_p x}{p},$$

together with a mixed-type comparison

$$(10) \quad \frac{\sin_p x}{x} < \frac{x}{\sinh_p x}.$$

The interplay between (7) and (10) naturally suggests a hyperbolic-trigonometric counterpart inequality. As a first step in this direction, Klén et al. proved the weak bound, for $p \in [2, +\infty)$,

$$\frac{\sinh_p x}{x} < \frac{3}{2 + \cos_p x}, \quad x \in (0, \pi_p/2),$$

and proposed the following stronger conjecture.

CONJECTURE 1. ([15, Conjecture 3.29]) *For $p \in (2, +\infty)$ and $x \in (0, \pi_p/2)$,*

$$(11) \quad \frac{\sinh_p x}{x} < \frac{p + 1}{p + \cos_p x}.$$

This conjecture was later explicitly listed as an open problem in the survey [24], and has remained unresolved for most $p \in (2, +\infty)$.

Subsequently, Huang, Yin, et al. [12, 23] refined (7) by deriving optimal constants for double inequalities. For $p \in (1, 2]$ and $x \in (0, \pi_p/2)$, they showed:

$$\left(\frac{\cos_p x + 2}{3}\right)^{\alpha_1} < \frac{\sin_p x}{x} < \left(\frac{\cos_p x + 2}{3}\right)^{\beta_1}$$

with optimal constants $\alpha_1 = \frac{\ln(2/\pi_p)}{\ln(2/3)}$ and $\beta_1 = \frac{3}{p+1}$, and

$$\left(\frac{\cos_p x + p}{p+1}\right)^{\alpha_2} < \frac{\sin_p x}{x} < \left(\frac{\cos_p x + p}{p+1}\right)^{\beta_2}$$

with optimal constants $\alpha_2 = \frac{\ln(2/\pi_p)}{\ln(p/(p+1))}$ and $\beta_2 = 1$.

Despite these advances, three critical gaps remain:

- *Incomplete parameter coverage:* No unified necessary and sufficient conditions exist for Cusa–Huygens type inequalities (and their reverses) across the full range $p \in (1, +\infty)$; previous results are restricted to $p \in (1, 2]$ or $[2, +\infty)$.
- *Unresolved conjecture:* Conjecture 1 remains unproven for most $p \in (2, +\infty)$, limiting the theoretical connection between generalized hyperbolic and trigonometric functions.
- *Lack of refined bounds:* Existing inequalities lack explicit remainder terms, failing to quantify approximation errors—a critical issue for high-precision applications.

The present paper aims to address these gaps and advance the theory of Cusa–Huygens type inequalities for generalized trigonometric and hyperbolic functions. Our core objectives are as follows:

- (i) Establish necessary and sufficient conditions for the Cusa–Huygens type inequalities (7), (8), and their reverses to hold for $p \in (1, +\infty)$;
- (ii) Extend the valid range of the reverse inequality (9) from $p \in [2, +\infty)$ to $p \in (1, +\infty)$;
- (iii) Resolve Conjecture 1 for a substantial subset of p ;
- (iv) Derive refined inequalities with explicit remainder terms to quantify approximation errors.

2. PRELIMINARIES

To prove our main results, we first recall some necessary preliminaries and analytical tools, and then establish several auxiliary lemmas concerning absolute monotonicity and sequence monotonicity.

2.1 Basic knowledge

For real numbers a, b, c with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function (denoted by $F(a, b; c; x)$, or ${}_2F_1(a, b; c; x)$ in standard notation) is defined by the power series [2]:

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad \text{for } |x| < 1,$$

where $(\cdot)_n$ denotes the Pochhammer symbol (or shifted factorial), given by

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (n \in \mathbb{N}).$$

Here, $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ denotes the classical Euler gamma function, which extends the factorial to non-integer arguments [25, 27].

We only retain properties critical to subsequent proofs:

- **Derivative formula:**

The hypergeometric function has a simple derivative

$$(12) \quad \frac{d}{dx} F(a, b; c; x) = \frac{ab}{c} F(a+1, b+1; c+1; x).$$

- **Behavior near $x = 1$:**

To analyze limits of generalized functions (e.g., at $x \rightarrow \pi_p/2$ for $\sin_p x$), we require the following asymptotic behavior of $F(a, b; c; x)$ as $x \rightarrow 1^-$:

- If $c > a + b$, then

$$F(a, b; c; 1^-) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

(cf. [20, p.49]).

- If $c = a + b$, Ramanujan's asymptotic formula gives

$$(13) \quad B(a, b)F(a, b; c; x) + \log(1-x) = R(a, b) + O((1-x)\log(1-x)),$$

where $B(a, b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ denotes the beta function, $R(a, b) := -2\gamma - \psi(a) - \psi(b)$ is the Ramanujan constant, $\psi := \Gamma'/\Gamma$ is the digamma function, and γ is the Euler-Mascheroni constant (cf. [1, 15.3.10]).

- If $c < a + b$, the desired behavior follows by applying the transformation identity (cf. [1, 15.3.3])

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x).$$

The inverse of generalized trigonometric and hyperbolic functions can be expressed by $F(a, b; c; x)$, forming a direct link between hypergeometric series and our target functions [4, Lemma 1]:

$$(14) \quad \arcsin_p x = xF\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x^p\right),$$

$$(15) \quad \operatorname{arctanh}_p x = xF\left(1, \frac{1}{p}; 1 + \frac{1}{p}; x^p\right).$$

2.2 Tools

This section presents two key analytical tools—absolute monotonicity and NP/PN-type power series—that are essential for establishing the sign and monotonicity of functions arising in generalized trigonometric and hyperbolic inequalities.

A real function φ is called *absolutely monotonic* on an interval I if its k -th derivative $\varphi^{(k)}(x)$ exists and is nonnegative for all $k \geq 0$ and $x \in I$. For functions representable by power series, this condition is equivalent to all coefficients of the series being nonnegative. To establish the absolute monotonicity of ratios of two power series, we employ the following criterion due to Jurkat [13] (Proposition 1).

Proposition 1. *Let $p(x) = \sum_{n=0}^{\infty} p_n x^n$ and $q(x) = \sum_{n=0}^{\infty} q_n x^n$ be power series converging on $(0, r)$ with $p_n > 0$ for all $n \geq 0$. Suppose that the sequence $\{p_{n+1}/p_n\}_{n \geq 0}$ is increasing¹. Then:*

- *If $\{q_n/p_n\}_{n \geq 0}$ is increasing, $(q/p)'$ is absolutely monotonic on $(0, r)$;*
- *If $\{q_n/p_n\}_{n \geq 0}$ is decreasing, $-(q/p)'$ is absolutely monotonic on $(0, r)$.*

To determine the sign of infinite series that arise in approximations of generalized functions, we focus on power series of Negative-Positive (NP)-type and Positive-Negative (PN)-type, defined as follows:

- **NP-type Series:** A power series $S(x)$ is said to be of NP-type if it can be written as

$$S(x) = -\sum_{k=0}^m a_k x^k + \sum_{k=m+1}^{\infty} a_k x^k,$$

where $a_k \geq 0$ for all $k \geq 0$, and there exist indices $0 \leq k_1 \leq m$ and $k_2 \geq m+1$ such that $a_{k_1} > 0$ and $a_{k_2} > 0$.

- **PN-type Series:** A power series $S(x)$ is of PN-type if $-S(x)$ is of NP-type.

The term “Negative-Positive type” was first introduced in [30], although series of this form have been extensively studied in [8, 21, 26, 29, 31].

The following properties (Proposition 2; see also [26, 28]) determine the sign of NP/PN-type series on their interval of convergence $(0, r)$, where $r > 0$ denotes the radius of convergence.

¹As in Theorem 1, Jurkat (1954), p. 965, the terms “increasing” and “decreasing” are used in the non-strict sense (i.e., equality allowed).

Proposition 2. (Sign of NP/PN-type Series)

- Let $S(x)$ be an NP-type series with radius of convergence $r > 0$. Then:
 - If $S(r^-) \leq 0$, then $S(x) < 0$ for all $x \in (0, r)$;
 - If $S(r^-) > 0$, then there exists a unique $\tilde{x} \in (0, r)$ such that $S(x) < 0$ for $x \in (0, \tilde{x})$ and $S(x) > 0$ for $x \in (\tilde{x}, r)$.
- If $S(x)$ is of PN-type, the same conclusions hold with all inequalities reversed.

2.3 Lemmas

We now establish several auxiliary results that will be used repeatedly in the proofs of the main theorems.

Lemma 1. Define the function $\vartheta(p) = \frac{\pi}{\sin(\pi/p)} - p - 1$ for $p \in (1, +\infty)$. Then $\vartheta(p)$ is strictly decreasing on $(1, +\infty)$. Consequently, there exists a unique root $p_* \in (1, +\infty)$, with $p_* = 2.16398 \cdots^2$, such that $\vartheta(p) \geq 0$ for $p \in (1, p_*]$ and $\vartheta(p) < 0$ for $p \in (p_*, +\infty)$.

Proof. Consider the auxiliary function $\theta(s) = \frac{\pi}{\sin s} - \frac{\pi}{s} - 1$ for $s \in (0, \pi)$. Using the power series expansion of $1/\sin s$ (refer to [1, 4.3.68]), we express $\theta(s)$ as

$$\theta(s) = -1 + \pi \sum_{n=1}^{\infty} \frac{2(2^{2n-1} - 1)|B_{2n}|}{(2n)!} s^{2n-1},$$

where B_{2n} denotes the $2n$ -th Bernoulli number.

All coefficients of the power series are positive, implying $\theta'(s) > 0$ for all $s \in (0, \pi)$. Substituting $s = \pi/p$ (s decreases as p increases on $(1, +\infty)$) shows $\vartheta(p)$ is strictly decreasing on $(1, +\infty)$. Moreover,

$$\lim_{p \rightarrow 1^+} \vartheta(p) = +\infty, \quad \lim_{p \rightarrow +\infty} \vartheta(p) = -1.$$

By the Intermediate Value Theorem, $\vartheta(p) = 0$ has a unique root $p_* \in (1, +\infty)$. This establishes existence and uniqueness, and the sign partition follows. $\square$

Lemma 2. For $p \in (1, +\infty)$ and $\tau \in (0, +\infty)$, define the function ϕ_τ on $(0, 1)$ by

$$\phi_\tau(x) = (1-x)^{1/p} F\left(\tau, \frac{1}{p}; 1 + \frac{1}{p}; x\right).$$

Then $-\phi'_\tau(x)$ is absolutely monotonic on $(0, 1)$ if and only if $\tau \leq 1 + 1/p$.

²This numerical value is obtained by solving $\vartheta(p) = 0$ using a root-finding algorithm in MATHEMATICA.

Proof. Rewrite ϕ_τ as the ratio of two power series:

$$\phi_\tau(x) = \frac{F(\tau, \frac{1}{p}, 1 + \frac{1}{p}; x)}{(1-x)^{-1/p}} = \frac{\sum_{n=0}^{\infty} A_n x^n}{\sum_{n=0}^{\infty} B_n x^n},$$

where

$$A_n = \frac{(\tau)_n (1/p)_n}{(1 + 1/p)_n n!} \quad \text{and} \quad B_n = \frac{(1/p)_n}{n!}.$$

First, suppose $-\phi'_\tau(x)$ is absolutely monotonic on $(0, 1)$. Then, by definition, all derivatives of $-\phi'_\tau(x)$ are nonnegative on $(0, 1)$; in particular, the zeroth derivative gives $\phi'_\tau(x) \leq 0$ for all $x \in (0, 1)$. To derive a necessary condition on τ , we evaluate the right-hand derivative of $\phi_\tau(x)$ at $x = 0^+$:

$$\phi'_\tau(0^+) = \frac{A_1 B_0 - A_0 B_1}{B_0^2} = \frac{p\tau - (p+1)}{p(p+1)}.$$

Since $\phi'_\tau(x) \leq 0$ for all $x \in (0, 1)$, we must have $\phi'_\tau(0^+) \leq 0$, which yields $p\tau - (p+1) \leq 0$, i.e., $\tau \leq 1 + 1/p$.

Conversely, assume $\tau \leq 1 + 1/p$. We apply Proposition 1 to establish absolute monotonicity.

- For $\sum B_n x^n$: The ratio $\frac{B_{n+1}}{B_n} = 1 - \frac{p-1}{p(n+1)}$ is strictly increasing in n (since $\frac{p-1}{p(n+1)}$ decreases with n).
- For $\{A_n/B_n\}$: Simplify the difference: $\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{[p\tau - (p+1)](\tau)_n}{(pn + p + 1)(1 + 1/p)_n} \leq 0$, as $p\tau - (p+1) \leq 0$ (from $\tau \leq 1 + 1/p$) and all other terms are positive. Thus, $\{A_n/B_n\}_{n \geq 0}$ is decreasing.

By Proposition 1, it follows that $-\phi'_\tau(x)$ is absolutely monotonic on $(0, 1)$.

The necessity and sufficiency have thus been established, and the proof is complete. $\square$

Lemma 3. For $p \in (1, +\infty)$, define two power series, both convergent on $(0, 1)$, by

$$(1-x)^{1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x\right) = \sum_{n=0}^{\infty} u_n x^n \quad \text{and} \\ (1-x)^{1/p} F\left(1, \frac{1}{p}; 1 + \frac{1}{p}; x\right) = \sum_{n=0}^{\infty} \bar{u}_n x^n.$$

Then the following statements hold:

- (i) $u_n < 0$ and $\bar{u}_n < 0$ for all $n \geq 1$;
- (ii) For $n \geq 1$, the coefficients satisfy the recurrence relations

$$(16) \quad u_{n+1} = \alpha_n u_n - \beta_n u_{n-1},$$

$$(17) \quad \bar{u}_{n+1} = \bar{\alpha}_n \bar{u}_n - \bar{\beta}_n \bar{u}_{n-1}$$

with initial conditions $u_0 = \bar{u}_0 = 1$, $u_1 = -1/(p+1)$, $\bar{u}_1 = -1/[p(p+1)]$, and the coefficients

$$(18) \quad \alpha_n = \frac{2pn^2 + n - 1}{(n+1)(pn+p+1)}, \quad \beta_n = \frac{p(n-1)^2}{(n+1)(pn+p+1)},$$

$$(19) \quad \bar{\alpha}_n = \frac{2p^2n^2 + p^2n - 1}{p(n+1)(pn+p+1)}, \quad \bar{\beta}_n = \frac{(pn-1)(n-1)}{(n+1)(pn+p+1)}.$$

(iii) $\alpha_n > 0$, $\bar{\alpha}_n > 0$ and $\beta_n \geq 0$, $\bar{\beta}_n \geq 0$ for all $n \geq 1$.

Proof. By Lemma 2, the functions $-\phi'_{1/p}(x)$ and $-\phi'_1(x)$ (where ϕ_τ is defined as in Lemma 2) are absolutely monotonic on $(0, 1)$. This implies $u_n < 0$ and $\bar{u}_n < 0$ for $n \geq 1$.

The recurrence relations (16) and (17) follow as a special case of [22, Corollary 2.2]; we therefore omit the detailed derivation and refer the reader to that result.

For $n \geq 1$ and $p > 1$, the numerators and denominators of $\alpha_n, \bar{\alpha}_n$ are all positive (e.g., $2pn^2 + n - 1 > 0$ for $p > 1, n \geq 1$), while the numerators of $\beta_n, \bar{\beta}_n$ are nonnegative (e.g., $p(n-1)^2 \geq 0$), so $\alpha_n > 0$, $\bar{\alpha}_n > 0$, $\beta_n \geq 0$, $\bar{\beta}_n \geq 0$. $\square$

Lemma 4. For $p \in (1, +\infty)$, let u_n be defined as in Lemma 3 and $v_n = \frac{p(1/p)_n}{(1+pn)n!}$.

Then the sequence $\{u_n/v_n\}_{n \geq 1}$ satisfies the following monotonicity properties:

- (i) If $p \in [\sqrt{2} + 1, +\infty)$, then $\{u_n/v_n\}_{n \geq 1}$ is decreasing;
- (ii) If $p \in (2, \sqrt{2} + 1)$, then either $\{u_n/v_n\}_{n \geq 1}$ is increasing, or there exists $n_0 \geq 2$ such that $\{u_n/v_n\}$ is increasing for $1 \leq n \leq n_0$ and decreasing for $n \geq n_0$;
- (iii) If $p \in (1, 2]$, then $\{u_n/v_n\}_{n \geq 1}$ is increasing.

Proof. Define

$$D_n = u_{n+1} - \frac{v_{n+1}}{v_n} u_n, \quad n \geq 1.$$

Since $v_{n+1} > 0$ and

$$\frac{u_{n+1}}{v_{n+1}} - \frac{u_n}{v_n} = \frac{D_n}{v_{n+1}},$$

the sign of D_n determines the monotonicity of the sequence $\{u_n/v_n\}$.

Using the recurrence relation $u_{n+1} = \alpha_n u_n - \beta_n u_{n-1}$ (cf. Lemma 3, Eq. (16)), we obtain

$$(20) \quad D_n = \alpha_n^* u_n - \beta_n u_{n-1},$$

where

$$\alpha_n^* = \alpha_n - \frac{v_{n+1}}{v_n} = \frac{p^2 n^2 - pn - p - 1}{p(n+1)(1+p+np)} > 0 \quad (n \geq 2).$$

Furthermore, a direct computation yields the recurrence

$$(21) \quad D_{n+1} = \alpha_{n+1}^* D_n + \left[\alpha_{n+1}^* (\alpha_n - \alpha_n^*) - \beta_{n+1} \right] u_n,$$

and for $p \geq 2$,

$$\alpha_{n+1}^* (\alpha_n - \alpha_n^*) - \beta_{n+1} = \frac{\left[\begin{array}{l} 1 + (p-2)(1+3p+3p^2) + p^2(p-2)(n-1)^2 \\ + 2p(p-2)(1+2p)(n-1) + p(n-1) \end{array} \right]}{p^2(n+1)(n+2)(1+p+np)(1+2p+np)} > 0.$$

Moreover,

$$(22) \quad D_1 = \frac{(1 + \sqrt{2} - p)(p - 1 + \sqrt{2})}{2p(1+p)(1+2p)}.$$

We now examine the sign of D_n in three cases:

Case 1 $p \in [\sqrt{2} + 1, +\infty)$.

From (22), we have $D_1 \leq 0$. Since $\alpha_{n+1}^* > 0$ and, by Lemma 3, $u_n < 0$ for $n \geq 1$, the recurrence for D_n implies by induction that $D_n \leq 0$ for all $n \geq 1$. Hence $\{u_n/v_n\}_{n \geq 1}$ is decreasing.

Case 2 $p \in (2, \sqrt{2} + 1)$.

Here $D_1 > 0$ by (22). Either $D_n \geq 0$ for all $n \geq 1$, in which case the sequence is increasing; or there exists a first index $n_0 > 1$ with $D_{n_0} \leq 0$, and then the same inductive argument gives $D_n \leq 0$ for all $n \geq n_0$. Thus the sequence increases initially and then decreases.

Case 3 $p \in (1, 2]$.

From the definition of v_n , we compute

$$\frac{v_{n+1}}{v_n} = \frac{(n+1/p)^2}{(n+1)(n+1+1/p)},$$

so that

$$D_n = \frac{(n+1)(n+1+\frac{1}{p})u_{n+1} - (n+\frac{1}{p})^2 u_n}{(n+1)(n+1+\frac{1}{p})}.$$

It suffices to show that the numerator is positive for all $n \geq 1$.

Let $f(x) = (1-x)^{1/p} F(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; x) = \sum_{n=0}^{\infty} u_n x^n$. Then

$$x f'(x) + \frac{1}{p} f(x) = \sum_{n=0}^{\infty} \left[\left(n + \frac{1}{p} \right) u_n \right] x^n.$$

A direct computation gives

$$(23) \quad \begin{aligned} & (1-x) \left[x f'(x) + \frac{1}{p} f(x) \right]' - \frac{1}{p} \left[x f'(x) + \frac{1}{p} f(x) \right] \\ &= \sum_{n=0}^{\infty} \left[(n+1) \left(n+1+\frac{1}{p} \right) u_{n+1} - \left(n+\frac{1}{p} \right)^2 u_n \right] x^n. \end{aligned}$$

On the other hand, using the defining relation for $\arcsin_p x$ (Eq. (1)) and differentiating the series expansion (14), we obtain

$$\frac{d}{dx} \left[F \left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x \right) \right] = \frac{(1-x)^{-1/p} - F \left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x \right)}{px}.$$

Consequently,

$$\begin{aligned} (24) \quad & (1-x) \left[x f'(x) + \frac{1}{p} f(x) \right]' - \frac{1}{p} \left[x f'(x) + \frac{1}{p} f(x) \right] \\ &= -\frac{2}{p^2} + \frac{1}{p^2} (1-p+x) (1-x)^{\frac{1}{p}-1} F \left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x \right) \\ &= -\frac{2}{p^2} + \frac{1}{p^2} (1-p+x) \frac{f(x)}{1-x}. \end{aligned}$$

Writing $s_n = \sum_{k=0}^n u_k$ (partial sum of $f(x)$), we have

$$\frac{f(x)}{1-x} = \sum_{n=0}^{\infty} s_n x^n,$$

and hence

$$-\frac{2}{p^2} + \frac{1}{p^2} (1-p+x) \frac{f(x)}{1-x} = -\frac{p+1}{p^2} + \frac{1}{p^2} \sum_{n=1}^{\infty} [(1-p)s_n + s_{n-1}] x^n.$$

By Lemma 3, $u_n < 0$ for $n \geq 1$, so $\{s_n\}_{n \geq 0}$ is decreasing and

$$s_n \geq \lim_{n \rightarrow \infty} s_n = \sum_{n=0}^{\infty} u_n = f(1^-) = 0.$$

Thus

$$(1-p)s_n + s_{n-1} \geq s_{n-1} - s_n = -u_n > 0.$$

Therefore all coefficients in the series are positive, and from (23) and (24) it follows that

$$(n+1) \left(n + 1 + \frac{1}{p} \right) u_{n+1} - \left(n + \frac{1}{p} \right)^2 u_n > 0 \quad \text{for all } n \geq 1,$$

i.e., $D_n > 0$. Hence $\{u_n/v_n\}_{n \geq 1}$ is increasing.

This completes the proof. $\square$

Remark 1. The sign of D_n for $n \geq 1$ in the range $p \in (1, 2]$ is of particular importance, as it constitutes an indispensable component in the refined theory of Cusa–Huygens type inequalities. Although the essential necessary and sufficient conditions for these inequalities can still be established using existing results such

as (7). the refined estimate (26) for $p \in (1, 2.142]$, which in particular covers the interval $(1, 2]$, would remain inaccessible without this detailed monotonicity result. A key difficulty arises from the fact that, as p approaches 2, the term $\alpha_{n+1}^*(\alpha_n - \alpha_n^*) - \beta_{n+1}$ no longer preserves a uniform sign for all n , rendering the standard inductive sign-checking framework—effective for $p \in [\sqrt{2} + 1, +\infty)$ or $p \in (2, \sqrt{2} + 1)$ —inapplicable in the present setting. To overcome this obstruction, we adopt a nonstandard strategy. Rather than pursuing a direct recurrence-based sign check, we introduce a novel auxiliary function and investigate the sign pattern of its power-series coefficients. This approach not only establishes the monotonicity of $\{u_n/v_n\}$, which is foundational for Lemma 5, but also facilitates a quantitative control of the remainder term of $\mathcal{F}_p(x)$, thereby serving as a crucial step toward the proof of (26).

Lemma 5. *For $p \in (1, +\infty)$, define $\mathcal{F}_p : (0, 1) \mapsto \mathbb{R}$ by*

$$\mathcal{F}_p(x) = (1-x)^{1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x\right) + p F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x\right) - (p+1).$$

Let $p_* = 2.16398 \dots$ denote the unique root of $\vartheta(p) = \frac{\pi}{\sin(\pi/p)} - p - 1$ (see Lemma 1). Then the following statements hold:

- (i) $\mathcal{F}_p(x) > 0$ for all $x \in (0, 1)$ if and only if $p \in (1, p_*]$. Moreover, in the subcase $p \in (1, 2]$, $\mathcal{F}_p(x)$ is absolutely monotonic on $(0, 1)$.
- (ii) If $p \in (p_*, \sqrt{2} + 1)$, there exists $x_0 \in (0, 1)$ such that $\mathcal{F}_p(x) > 0$ on $(0, x_0)$ and $\mathcal{F}_p(x) < 0$ on $(x_0, 1)$;
- (iii) $\mathcal{F}_p(x) < 0$ for all $x \in (0, 1)$ if and only if $p \in [\sqrt{2} + 1, +\infty)$. In this case, $-\mathcal{F}_p(x)$ is absolutely monotonic on $(0, 1)$.

Proof. First, we express $\mathcal{F}_p(x)$ as a power series. With u_n and v_n as defined in Lemmas 3 and 4, respectively, we have

$$(25) \quad \mathcal{F}_p(x) = 1 + \sum_{n=1}^{\infty} u_n x^n + p + \sum_{n=1}^{\infty} \frac{p(1/p)_n^2}{(1+1/p)_n n!} x^n - (p+1) = \sum_{n=1}^{\infty} (u_n + v_n) x^n.$$

Thus the sign analysis of $\mathcal{F}_p$ reduces to that of the coefficients $u_n + v_n = v_n(1 + u_n/v_n)$.

The proof proceeds by cases, exploiting the monotonicity properties of $\{u_n/v_n\}$ established in 4, together with the theory of NP/PN-type series developed in Section 2.1.

(i) $p \in (1, p_*]$.

◦ Subcase $p \in (1, 2]$.

By Lemma 4(iii), the sequence $\{u_n/v_n\}_{n \geq 1}$ is increasing. Hence

$$u_n + v_n = v_n(u_n/v_n + 1) \geq v_n(u_1/v_1 + 1) = 0,$$

since $u_1 + v_1 = 0$. Consequently, by (25), all coefficients in the series expansion of $\mathcal{F}_p(x)$ are nonnegative, so $\mathcal{F}_p(x)$ is absolutely monotonic on $(0, 1)$, and in particular strictly positive.

◦ *Subcase $p \in (2, p_*)$.*

Lemma 4(ii) asserts that $\{u_n/v_n\}$ either is increasing for all $n \geq 1$, or unimodal in the sense that it increases up to some index n_0 and decreases thereafter. In the former case, $\mathcal{F}_p$ is absolutely monotonic and hence positive. In the latter case, the coefficient sequence $\{u_n + v_n\}$ is of PN-type (see Section 2.1). Since $\mathcal{F}_p(1^-) = \vartheta(p) \geq 0$ by Lemma 1, Proposition 2 for PN-type series implies that $\mathcal{F}_p(x) > 0$ for all $x \in (0, 1)$.

(ii) $p \in (p_*, \sqrt{2} + 1)$.

In this range, Lemma 1 gives $\mathcal{F}_p(1^-) = \vartheta(p) < 0$. Since the series converges at $x = 1$, this forces the existence of some index N such that $u_N + v_N < 0$. Indeed, if $u_n + v_n \geq 0$ for all $n \geq 1$, then $\mathcal{F}_p(1^-) = \sum_{n=1}^{\infty} (u_n + v_n) \geq 0$, contradicting the negativity of $\vartheta(p)$. Moreover, because $u_1 + v_1 = 0$, we may take $N > 2$. By Lemma 4(ii), the sequence $\{u_n/v_n\}$ is unimodal, which implies that $\{u_n + v_n\}$ is of PN-type. Applying Proposition 2 together with $\mathcal{F}_p(1^-) < 0$, we conclude that there exists $x_0 \in (0, 1)$ such that $\mathcal{F}_p(x) > 0$ on $(0, x_0)$ and $\mathcal{F}_p(x) < 0$ on $(x_0, 1)$.

(iii) $p \in [\sqrt{2} + 1, \infty)$.

Lemma 4(i) shows that $\{u_n/v_n\}_{n \geq 1}$ is decreasing. Therefore

$$u_n + v_n = v_n(u_n/v_n + 1) \leq v_n(u_1/v_1 + 1) = 0.$$

Thus all coefficients of $\mathcal{F}_p(x)$ are nonpositive, so $-\mathcal{F}_p(x)$ is absolutely monotonic on $(0, 1)$ by (25), and consequently $\mathcal{F}_p(x) < 0$ for all $x \in (0, 1)$.

This completes the proof. $\square$

Corollary 1. *Let $p \in (1, +\infty)$ and $\mathcal{F}_p$ be defined as in Lemma 5. Then for all $x \in (0, 1)$, the following hold:*

(1) *If $p \in (1, 2.142]$, then*

$$(26) \quad \mathcal{F}_p(x) > \frac{(1 + \sqrt{2} - p)(p - 1 + \sqrt{2})}{2p(1 + p)(1 + 2p)} x^2;$$

(2) *If $p \in [\sqrt{2} + 1, +\infty)$, then*

$$(27) \quad \mathcal{F}_p(x) < -\frac{(p - 1 - \sqrt{2})(p - 1 + \sqrt{2})}{2p(1 + p)(1 + 2p)} x^2.$$

Proof. Define the remainder term

$$\mathcal{R}_p(x) := \mathcal{F}_p(x) - \frac{(1 + \sqrt{2} - p)(p - 1 + \sqrt{2})}{2p(1 + p)(1 + 2p)} x^2 = \sum_{n=3}^{\infty} (u_n + v_n) x^n,$$

where u_n and v_n are the coefficients appearing in the power series expansion of $\mathcal{F}_p$ (cf. Eq. (25) in Lemma 5).

We proceed by cases.

Case 1 $p \in (1, 2.142]$.

For $p \in (1, 2]$, Lemma 5(i) ensures that $u_n + v_n \geq 0$ for all $n \geq 1$, hence $\mathcal{R}_p(x)$ is absolutely monotonic on $(0, 1)$, and the desired inequality follows immediately.

For $p \in (2, 2.142]$, Lemma 5(i) implies that the coefficient sequence $\{u_n + v_n\}_{n \geq 3}$ is either nonnegative (absolute monotonicity) or of PN type. A direct computation gives $u_3 + v_3 > 0$, so in the PN-type case, Proposition 2 applies. It remains to verify the endpoint condition. Here the specific upper bound 2.142 is chosen as a numerically convenient threshold such that

$$\mathcal{R}_p(1^-) = \vartheta(p) - \frac{(1 + \sqrt{2} - p)(p - 1 + \sqrt{2})}{2p(1 + p)(1 + 2p)} \geq \vartheta(2.142) - \frac{1}{60} > 0,$$

where the last inequality follows from Lemma 1 and a direct numerical evaluation. The value $1/60$ arises as a uniform upper bound for the fractional terms being subtracted in the above expression over the interval $(2, 2.142]$, ensuring the positivity of the endpoint limit. Thus, by Proposition 2, $\mathcal{R}_p(x) > 0$ for all $x \in (0, 1)$, which yields (26).

Case 2 $p \in [\sqrt{2} + 1, +\infty)$.

By Lemma 5(iii), $-\mathcal{F}_p(x)$ is absolutely monotonic on $(0, 1)$, so $u_n + v_n \leq 0$ for all $n \geq 3$. Consequently, $-\mathcal{R}_p$ is absolutely monotonic, hence $\mathcal{R}_p(x) < 0$ for all $x \in (0, 1)$, which proves (27). $\square$

Lemma 6. For $p \in (1, +\infty)$, define $\overline{\mathcal{F}}_p : (0, 1) \mapsto \mathbb{R}$ by

$$\overline{\mathcal{F}}_p(x) = p(1 - x)^{1/p} F\left(1, \frac{1}{p}; 1 + \frac{1}{p}; x\right) + F\left(1, \frac{1}{p}; 1 + \frac{1}{p}; x\right) - (p + 1).$$

Then $\overline{\mathcal{F}}_p(x)$ is absolutely monotonic on $(0, 1)$ if and only if $p \in (1, \sqrt{2} + 1]$. In particular, there is no $p \in (1, +\infty)$ for which $\overline{\mathcal{F}}_p(x) < 0$ holds for all $x \in (0, 1)$.

Proof. We begin by expanding $\overline{\mathcal{F}}_p(x)$ as a power series. Let $\bar{u}_n$ denote the coefficients of $(1 - x)^{1/p} F(1, \frac{1}{p}; 1 + \frac{1}{p}; x)$ (cf. Lemma 3), and define

$$\bar{v}_n := \frac{1}{p(1 + pn)}.$$

Then

$$(28) \quad \overline{\mathcal{F}}_p(x) = p + p \sum_{n=1}^{\infty} \bar{u}_n x^n + 1 + \sum_{n=1}^{\infty} \frac{x^n}{1 + pn} - (p + 1) = p \sum_{n=1}^{\infty} (\bar{u}_n + \bar{v}_n) x^n.$$

To determine the sign of the coefficients $\bar{u}_n + \bar{v}_n$, we introduce

$$\overline{D}_n := \bar{u}_{n+1} - \frac{\bar{v}_{n+1}}{\bar{v}_n} \bar{u}_n,$$

which governs the monotonicity of $\{\bar{u}_n/\bar{v}_n\}_{n \geq 1}$, since

$$\frac{\bar{u}_{n+1}}{\bar{v}_{n+1}} - \frac{\bar{u}_n}{\bar{v}_n} = \frac{\bar{D}_n}{\bar{v}_{n+1}}$$

and $\bar{v}_{n+1} > 0$.

Using the recurrence for $\bar{u}_n$ (Lemma 3, Eq. (17)), we obtain

$$(29) \quad \bar{D}_n = \bar{\alpha}_n^* \bar{u}_n - \bar{\beta}_n \bar{u}_{n-1},$$

where

$$\bar{\alpha}_n^* := \bar{\alpha}_n - \frac{\bar{v}_{n+1}}{\bar{v}_n} = \frac{(p-1)(1+4p) + p(n-2)(pn+2p-1)}{p(n+1)(pn+p+1)} > 0 \quad (n \geq 2).$$

Moreover, for $n \geq 1$, $\bar{D}_n$ satisfies the recurrence

$$(30) \quad \bar{D}_{n+1} = \bar{\alpha}_{n+1}^* \bar{D}_n + \left[\bar{\alpha}_{n+1}^* (\bar{\alpha}_n - \bar{\alpha}_n^*) - \bar{\beta}_{n+1} \right] \bar{u}_n,$$

and a direct computation yields

$$\bar{\alpha}_{n+1}^* (\bar{\alpha}_n - \bar{\alpha}_n^*) - \bar{\beta}_{n+1} = -\frac{(\sqrt{2}+1-p)(\sqrt{2}+p-1)+pn}{(n+2)p(np+p+1)(np+2p+1)} < 0$$

for all $n \geq 1$ and $p > 1$.

We now prove the equivalence.

Necessity.

If $\bar{\mathcal{F}}_p(x)$ is absolutely monotonic on $(0, 1)$, then all its power-series coefficients must be nonnegative. In particular, the coefficient of x^2 must be nonnegative, which is equivalent to

$$\lim_{x \rightarrow 0^+} \frac{\bar{\mathcal{F}}_p(x)}{x^2} = \frac{(\sqrt{2}+1-p)(p-1+\sqrt{2})}{2p(1+p)(1+2p)} \geq 0.$$

This inequality simplifies to $p \leq \sqrt{2}+1$.

Sufficiency.

Assume $p \in (1, \sqrt{2}+1]$. Then

$$\bar{D}_1 = \frac{(1+\sqrt{2}-p)(p-1+\sqrt{2})}{2p^2(1+p)(1+2p)} \geq 0.$$

We claim that $\bar{D}_n \geq 0$ for all $n \geq 1$. Indeed, suppose $\bar{D}_n \geq 0$ for some $n \geq 1$. Since $\bar{\alpha}_{n+1}^* > 0$, $\bar{u}_n < 0$ (by Lemma 3), and

$$\bar{\alpha}_{n+1}^* (\bar{\alpha}_n - \bar{\alpha}_n^*) - \bar{\beta}_{n+1} < 0,$$

the second term in (30) is nonnegative. Hence $\bar{D}_{n+1} \geq 0$, and the claim follows by induction. Consequently, the sequence $\{\bar{u}_n/\bar{v}_n\}_{n \geq 1}$ is increasing, which gives

$$\bar{u}_n + \bar{v}_n = \bar{v}_n(\bar{u}_n/\bar{v}_n + 1) \geq \bar{v}_n(\bar{u}_1/\bar{v}_1 + 1) = 0$$

for all $n \geq 1$. Thus, by (28), all coefficients of $\overline{\mathcal{F}}_p$ are nonnegative, so $\overline{\mathcal{F}}_p$ is absolutely monotonic on $(0, 1)$.

In particular, for each $p \in (1, +\infty)$, it follows from (13) that $\overline{\mathcal{F}}_p(x) \rightarrow +\infty$ as $x \rightarrow 1^-$. Thus $\overline{\mathcal{F}}_p$ is eventually positive, ruling out the possibility that $\overline{\mathcal{F}}_p(x) < 0$ for all $x \in (0, 1)$. $\square$

3. THE MAIN THEOREMS

We now present the principal results of this paper. The proofs rely on the equivalence established in Lemma 5 and Lemma 6, which reduce each inequality to the sign analysis of a corresponding auxiliary function on the interval $(0, 1)$.

Theorem 1. *For $p \in (1, +\infty)$, the inequality*

$$(31) \quad \frac{\sin_p x}{x} < \frac{\cos_p x + p}{p + 1}$$

holds for all $x \in (0, \pi_p/2]$ if and only if $p \in (1, p_*]$, while the reverse inequality holds for all $x \in (0, \pi_p/2]$ if and only if $p \in [\sqrt{2} + 1, +\infty)$. Here $p_* = 2.16398 \dots$ denotes the unique root of $\vartheta(p) = \frac{\pi}{\sin(\pi/p)} - p - 1$ as defined in Lemma 1.

Proof. Set $t = \sin_p^p x \in (0, 1)$. By the definitions of generalized trigonometric functions and hypergeometric functions (Eqs. (2) and (14)), we rewrite (31) as:

$$(32) \quad \begin{aligned} & \left(\cos_p x + p \right) \frac{x}{\sin_p x} - (p + 1) \\ &= (1 - t)^{1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; t\right) + p F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; t\right) - (p + 1) = \mathcal{F}_p(t), \end{aligned}$$

where $\mathcal{F}_p$ is defined in Lemma 5. Consequently, inequality (31) and its reverse are equivalent to $\mathcal{F}_p(t) > 0$ and $\mathcal{F}_p(t) < 0$, respectively, for all $t \in (0, 1)$. The asserted statements then follow directly from Lemma 5. $\square$

By substituting $t = \sin_p^p x$ into the refined estimates of Corollary 1, we obtain the following strengthened forms of (31) and its reverse.

Proposition 3. *For all $x \in (0, \pi_p/2)$,*

(i) If $p \in (1, 2.142]$, then

$$\frac{\sin_p x}{x} \left[1 + \frac{1 + 2p - p^2}{2p(1 + p)^2(1 + 2p)} \sin_p^{2p} x \right] < \frac{\cos_p x + p}{p + 1};$$

(ii) If $p \in [\sqrt{2} + 1, +\infty)$, then

$$\frac{\sin_p x}{x} \left[1 - \frac{p^2 - 2p - 1}{2p(1 + p)^2(1 + 2p)} \sin_p^{2p} x \right] > \frac{\cos_p x + p}{p + 1}.$$

The next theorem addresses the long-standing Conjecture 1 from [15] for a substantial range of the parameter p .

Theorem 2. For $p \in [\sqrt{2} + 1, +\infty)$, the inequality

$$(33) \quad \frac{\sinh_p x}{x} < \frac{p+1}{p + \cos_p x}$$

holds for all $x \in (0, \pi_p/2)$.

Proof. For $p \in [\sqrt{2} + 1, +\infty)$, Theorem 1 yields the reverse of (31):

$$\frac{\sin_p x}{x} > \frac{p + \cos_p x}{p+1}, \quad x \in (0, \pi_p/2).$$

Combining this with the known mixed-type inequality (10) and rearranging gives

$$\frac{x}{\sinh_p x} > \frac{p + \cos_p x}{p+1},$$

which is equivalent to (33). $\square$

Remark 2. Theorem 2 confirms Conjecture 1 for $p \in [\sqrt{2} + 1, +\infty)$. The only remaining range to fully settle the conjecture is the interval $p \in (2, \sqrt{2} + 1)$.

We next turn to the classical form of the Cusa–Huygens inequality, where the constant 2 replaces the parameter p in the numerator of the right-hand side.

Theorem 3. For $p \in (1, +\infty)$, the inequality

$$(34) \quad \frac{\sin_p x}{x} < \frac{\cos_p x + 2}{3}$$

holds for all $x \in (0, \pi_p/2]$ if and only if $p \in (1, 2]$. Moreover, if $p \in [p^*, +\infty)$, where $p^* = 2.16193 \dots$ is the unique root of

$$1 + 6p + 13p^2 + 12p^3 + 4p^4 - 6p^5 = 0,$$

then the reverse of (34) holds for all $x \in (0, \pi_p/2]$. In particular, for $p \in (1, 2]$, the refined inequality

$$(35) \quad \frac{\sin_p x}{x} < \frac{\cos_p x + 2}{3} - \frac{(2-p) \sin_p^{p+1} x}{3p(p+1)x}$$

holds for all $x \in (0, \pi_p/2]$.

Proof. As in the proof of Theorem 1, inequality (34) is equivalent to

$$(36) \quad \begin{aligned} \mathcal{G}_p(t) &= (1-t)^{1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; t\right) + 2F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; t\right) - 3 \\ &= \sum_{n=1}^{\infty} v_n \left(\frac{u_n}{v_n} + \frac{2}{p}\right) t^n > 0 \end{aligned}$$

for all $t \in (0, 1)$, where $t = \sin_p^p x$ and u_n, v_n are as defined in Lemmas 3 and 4.

For $p \in (1, 2]$, the validity of (34) is already established in [15, Theorem 3.22]; we only need to verify the necessary condition

$$\lim_{t \rightarrow 0^+} \frac{\mathcal{G}_p(t)}{t} = \frac{2-p}{p(p+1)} \geq 0,$$

which gives $p \leq 2$.

For $p \in [p^*, +\infty)$, direct computation yields

$$\frac{u_3}{v_3} + \frac{2}{p} = \frac{2(1+6p+13p^2+12p^3+4p^4-6p^5)}{p(1+p)^2(1+2p)^2} \leq 0$$

and

$$D_3 = -\frac{p(3p-3-2\sqrt{3})(3p-3+2\sqrt{3})}{6(1+p)(1+2p)(1+3p)(1+4p)} < 0,$$

since $p^* > 2.16 > 1 + \frac{2\sqrt{3}}{3}$. Together with Lemma 4, this implies $D_n \leq 0$ for $n \geq 3$, so the sequence $\{u_n/v_n\}$ is decreasing for $n \geq 3$. Hence

$$\frac{u_n}{v_n} + \frac{2}{p} \leq \frac{u_3}{v_3} + \frac{2}{p} \leq 0 \quad n \geq 3.$$

Combining this with the direct checks $\frac{u_1}{v_1} + \frac{2}{p} < 0$ and $\frac{u_2}{v_2} + \frac{2}{p} < 0$, we conclude that $\mathcal{G}_p(t) < 0$ for all $t \in (0, 1)$, which proves the reverse of (34).

In particular, for $p \in (1, 2]$, Lemma 4(iii) gives that $\{u_n/v_n\}_{n \geq 1}$ is increasing, and thus

$$\frac{u_n}{v_n} + \frac{2}{p} \geq \frac{u_2}{v_2} + \frac{2}{p} = \frac{2[1+p(1+p)(2-p)]}{p(1+p)^2} > 0, \quad n \geq 2.$$

Therefore the remainder term $\mathcal{G}_p(t) - \left(u_1 + \frac{2}{p}v_1\right)t$ is positive. Substituting $t = \sin_p^p x$ into this remainder analysis yields the refined bound (35). $\square$

Remark 3. Theorem 1 already implies that for $p \in [\sqrt{2} + 1, +\infty)$,

$$\frac{\sin_p x}{x} > \frac{\cos_p x + p}{p+1} = \frac{\cos_p x + 2}{3} + \frac{(p-2)(1-\cos_p x)}{3(p+1)} > \frac{\cos_p x + 2}{3},$$

so the reverse of (34) holds in this parameter range. Theorem 3 sharpens this result by identifying the smaller threshold p^* for the reverse inequality.

We now present a complete extension of the reverse inequality (9), originally proved only for $p \in [2, +\infty)$, to the full range $p \in (1, +\infty)$.

Theorem 4. For $p \in (1, +\infty)$, the inequality

$$(37) \quad \frac{\sin_p x}{x} > \frac{p-1+\cos_p x}{p}$$

holds for all $x \in (0, \pi_p/2)$.

Proof. The case $p \in [2, +\infty)$ was already established in (9). It remains to prove the result for $p \in (1, 2)$. Set $t = \sin_p^p x \in (0, 1)$. Inequality (37) is equivalent to

$$\begin{aligned} & \left((p-1) + \cos_p x \right) \frac{x}{\sin_p x} - p \\ (38) \quad &= (1-t)^{1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; t\right) + (p-1) F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; t\right) - p \\ &= \sum_{n=1}^{\infty} v_n \left(\frac{u_n}{v_n} + \frac{p-1}{p} \right) t^n := \mathcal{H}_p(t), \end{aligned}$$

For $p \in (1, 2)$, Lemma 4(iii) shows that $\{u_n/v_n\}_{n \geq 1}$ is increasing. Note that

$$\frac{u_1}{v_1} + \frac{p-1}{p} = -\frac{1}{p} < 0.$$

If $\frac{u_n}{v_n} + \frac{p-1}{p} \leq 0$ for all $n \geq 1$, then $\mathcal{H}_p(t) < 0$. Otherwise, let $m > 1$ be the first index such that $\frac{u_m}{v_m} + \frac{p-1}{p} > 0$; then $\mathcal{H}_p$ is an NP-type series. Moreover, with $q = (\frac{1}{p} - \frac{1}{2})\pi \in (0, \pi/2)$, we have

$$\mathcal{H}_p(1^-) = \frac{(p-1)\pi}{p \sin(\pi/p)} - p = -\frac{2\pi h(q)}{(\pi + 2q) \cos q} < 0,$$

where $h(q) = \cos q - \frac{\pi}{4} + \frac{q^2}{\pi} > 0$ (this follows from $h(\pi/2) = 0$ and $h'(q) = \frac{2q}{\pi} - \sin q < 0$, by Jordan's inequality). Applying Proposition 2 for NP-type series yields $\mathcal{H}_p(t) < 0$ for all $t \in (0, 1)$. Hence (37) holds in all cases. $\square$

Finally, we characterize the validity of the hyperbolic analogue of the Cusa–Huygens inequality.

Theorem 5. *For $p \in (1, +\infty)$, the inequality*

$$(39) \quad \frac{\sinh_p x}{x} < \frac{\cosh_p x + p}{1 + p}$$

holds for all $x \in (0, \infty)$ if and only if $p \in (1, \sqrt{2} + 1]$. In particular, there is no $p \in (1, +\infty)$ for which the inverse of (39) holds for all $x \in (0, +\infty)$.

Proof. Let $t = \tanh_p^p x \in (0, 1)$. Using the hyperbolic identity (6) and the definition of $\overline{\mathcal{F}}_p(t)$ from Lemma 6, we rewrite (39) as

$$\begin{aligned} & \left(\cosh_p x + p \right) \frac{x}{\sinh_p x} - (p+1) \\ &= \frac{\operatorname{arctanh}_p \sqrt[p]{t}}{\sqrt[p]{t}} + p \frac{(1-t)^{1/p} \operatorname{arctanh}_p \sqrt[p]{t}}{\sqrt[p]{t}} - (p+1) = \overline{\mathcal{F}}_p(t). \end{aligned}$$

Thus (39) is equivalent to $\overline{\mathcal{F}}_p(t) > 0$ for all $t \in (0, 1)$. By Lemma 6, $\overline{\mathcal{F}}_p(t)$ is absolutely monotonic (hence positive) on $(0, 1)$ if and only if $p \in (1, \sqrt{2} + 1]$.

Moreover, Lemma 6 shows that there is no $p \in (1, +\infty)$ for which $\overline{\mathcal{F}}_p$ is negative throughout $(0, 1)$. Hence, the reverse of (39) cannot hold globally for any p . $\square$

Proposition 4. For $p \in (1, \sqrt{2} + 1]$ and all $x \in (0, +\infty)$, the refined inequality

$$(40) \quad \frac{\sinh_p x}{x} \left[1 + \frac{1 + 2p - p^2}{2p(1+p)(1+2p)} \tanh_p^{2p} x \right] < \frac{\cosh_p x + p}{p + 1}.$$

holds.

Proof. For $p \in (1, \sqrt{2} + 1]$, Lemma 6 ensures that $\overline{\mathcal{F}}_p(t)$ is absolutely monotonic on $(0, 1)$, so its remainder term is positive. In particular,

$$\overline{\mathcal{F}}_p(t) > p(\bar{u}_2 + \bar{v}_2)t^2 = \frac{1 + 2p - p^2}{2p(1+p)(1+2p)} t^2.$$

Substituting $t = \tanh_p x$ gives the desired estimate (40). $\square$

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Tiehong Zhao

School of Mathematics,
Hangzhou Normal University,
Hangzhou 311121, Zhejiang, China
E-mail: tiehong.zhao@hznu.edu.cn
<https://orcid.org/0000-0002-6394-1049>

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