

A SOLUTION TO AN OPEN PROBLEM FOR GENERALIZED CUSA–HUYGENS INEQUALITY

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This paper completely resolves an open problem concerning the generalized Cusa–Huygens inequality $\left(\frac{\sin x}{x}\right)^a < \frac{2}{3} \left(1 + \frac{\cos^a x}{2}\right)$ for $x \in (0, \frac{\pi}{2})$. We obtain the following sharp results: the inequality holds for $a \geq a_0 = \frac{\ln(3/2)}{\ln(\pi/2)}$, and neither the inequality nor its reverse holds for $\frac{4}{5} < a < a_0$.

1. INTRODUCTION

The inequality

$$(1) \quad \frac{\sin x}{x} < \frac{\cos x}{3} + \frac{2}{3}, \quad x \in \left(0, \frac{\pi}{2}\right)$$

was first derived by Nicholas of Cusa, and the first rigorous proof was given by Huygens in [1]. This is the well-known **Cusa–Huygens inequality**, which reveals the relationship between $\frac{\sin x}{x}$ and the weighted arithmetic mean of 1 and $\cos x$.

The Cusa–Huygens inequality has been generalized by various methods [2, 3, 4, 5, 6]. Zhu studied the generalized form [7, 8]:

$$(2) \quad \left(\frac{\sin x}{x}\right)^a < \frac{2}{3} \left(1 + \frac{\cos^a x}{2}\right)$$

and concluded that: The inequality (2) holds for $a < 0$, and the reverse inequality holds for $0 < a \leq \frac{4}{5}$. Furthermore, an open problem was proposed:

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What is the largest range of a for which inequality (2) holds?

The generalized Cusa-Huygens inequality has important theoretical significance and has been extended in different forms in recent studies. Some representative results related to inequality (2) are as follows [9, 10, 11].

Theorem 1. *Let $0 < x < \frac{\pi}{2}$. Then*

1. *If $p \geq 1$, the double inequality*

$$(1 - \lambda) + \lambda(\cos x)^p < \left(\frac{\sin x}{x}\right)^p < (1 - \eta) + \eta(\cos x)^p$$

holds if and only if $\eta \leq \frac{1}{3}$ and $\lambda \geq 1 - (\frac{2}{\pi})^p$;

2. *If $0 \leq p \leq \frac{4}{5}$, the double inequality*

$$(1 - \eta) + \eta(\cos x)^p < \left(\frac{\sin x}{x}\right)^p < (1 - \lambda) + \lambda(\cos x)^p$$

holds if and only if $\eta \geq \frac{1}{3}$ and $\lambda \leq 1 - (\frac{2}{\pi})^p$;

3. *If $p < 0$, the inequality $(\frac{\sin x}{x})^p < (1 - \eta) + \eta(\cos x)^p$ holds if and only if $\eta \geq \frac{1}{3}$.*

Theorem 2. *Let $\alpha, \lambda \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, then the inequality*

$$\left(\frac{\sin x}{x}\right)^\alpha - \lambda \cos^\beta x > 1 - \lambda$$

holds for $x \in (0, \frac{\pi}{2})$ if (α, β) satisfies one of the following conditions:

1. $(\alpha, \beta) \in \mathbb{R}_1$ and

$$\lambda \geq \max \left\{ \frac{\alpha}{3\beta}, 1 - \left(\frac{2}{\pi}\right)^\alpha \right\},$$

where

$$\mathbb{R}_1 = \left\{ (\alpha, \beta) \mid \beta \geq \max \left\{ 1, \frac{5\alpha + 8}{15} \right\} \right\};$$

2. $(\alpha, \beta) \in \mathbb{R}_2$ and

$$\lambda \geq \max \left\{ \frac{a}{3\beta}, 1 - \left(\frac{2}{\pi}\right)^\alpha \right\},$$

where

$$\mathbb{R}_2 = \left\{ (\alpha, \beta) \mid \left(\alpha \geq \frac{\pi^2}{4} - 1 \text{ and } 0 < \beta \leq 1 \right) \right. \\ \left. \text{or } \left(\alpha \leq \frac{\pi^2}{4} - 1, 0 < \beta \leq \beta(\alpha) \right) \right\},$$

and $\beta(\alpha)$ is defined in (25) in [10];

3. $(\alpha, \beta) \in \mathbb{R}_3$ and

$$\lambda \geq \max \left\{ \frac{a}{3\beta}, 1 - \left(\frac{2}{\pi}\right)^\alpha \right\},$$

where

$$\mathbb{R}_3 = \left\{ (\alpha, \beta) \left| \begin{array}{l} 1 < \beta < \frac{5\alpha + 8}{15} \\ \text{or } \left(\frac{7}{5} < \alpha < \frac{\pi^2}{4} - 1 \text{ and } \beta = 1 \right) \end{array} \right. \right\};$$

4. $(\alpha, \beta) \in \mathbb{R}_4$ and

$$\lambda \geq \max \left\{ \left(\frac{2}{\pi} \right)^{(\alpha+1)} / \beta, \frac{a}{3\beta}, 1 - \left(\frac{2}{\pi} \right)^\alpha \right\},$$

where

$$\mathbb{R}_4 = \left\{ (\alpha, \beta) \left| \begin{array}{l} \max \left\{ \frac{5\alpha + 8}{15}, 0 \right\} < \beta < 1 \\ \text{or } \left(\frac{46}{35} < \alpha < \frac{7}{5} \text{ and } \beta = \frac{5\alpha + 8}{15} \right) \end{array} \right. \right\}.$$

Theorem 3. Let us consider the function defined for $x \in (0, \frac{\pi}{2})$ and $p, r \geq 0$

$$h(p, r, x) = \left(\frac{\sin x}{x} \right)^p - \left(\frac{p}{3r} \right) (\cos x)^r - 1 + \frac{p}{3r}$$

Then, for every integer $p \geq 3r - \frac{8}{5}$ one has $h(p, r, x) > 0$.

However, none of these results addresses the above open problem. It should be noted that the exponents of $\frac{\sin x}{x}$ and $\cos x$ in inequality (2) are equal and the above analysis does not cover the scenario where the exponent falls within the interval $(\frac{4}{5}, 1)$.

In this paper, we obtain the following sharp conclusions.

1. For $\frac{4}{5} < a < \frac{\ln(3/2)}{\ln(\pi/2)}$, neither the inequality (2) nor its reverse holds for $x \in (0, \frac{\pi}{2})$;
2. For $a \geq \frac{\ln(3/2)}{\ln(\pi/2)}$, the inequality (2) holds for $x \in (0, \frac{\pi}{2})$.

Combined with the results in [7, 8], they completely solve the open problem. Specifically, inequality (2) holds if and only if $a < 0$ or $a \geq a_0 = \frac{\ln(3/2)}{\ln(\pi/2)}$; the reverse inequality holds for $0 < a \leq \frac{4}{5}$; and for $\frac{4}{5} < a < a_0$ or $a = 0$, neither (2) nor its reverse holds on $(0, \frac{\pi}{2})$.

2. PRELIMINARIES

In this section, we introduce the core tool to deal with the mixed-trigonometric polynomials [12, 13] of the form $\sum_{i=1}^n \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x$, where $\alpha_i \in \mathbb{R} \setminus \{0\}$, $p_i, q_i, r_i \in \mathbb{N}_0$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), $n \in \mathbb{N}$, and the transcendental-polynomial [14, 15] of the form $f(x, \text{trans}_1(x), \dots, \text{trans}_m(x))$, where $f \in \mathbb{R}[x_1, \dots, x_m, x_{m+1}]$ and each $\text{tran}_i(x)$ is a transcendental function (A mixed-trigonometric polynomial is also a

transcendental-polynomial). And several lemmas are presented that will be employed in the subsequent proofs.

The main method adopted herein is the **Taylor substitution method** [13, 14, 15], whose basic idea is to construct a sequence of univariate polynomial interval nesting sequences to approximate the transcendental functions involved in the target inequality. This approach reduces the proof of the transcendental function inequality to the verification of a series of univariate polynomial inequalities, which can ultimately be accomplished using algebraic inequality proving tools.

Definition 1. For a transcendental-polynomial $F = f(x, \text{trans}_1(x), \dots, \text{trans}_n(x))$, on a certain interval I , if there are two algebraic functions sequences $\{T_{\min}(n, F)\}$ and $\{T_{\max}(n, F)\}$, and an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

1) $T_{\min}(n+1, F)(x) < T_{\min}(n, F)(x) < F(x)$, and for $n \rightarrow \infty$, $T_{\min}(n, F)(x) \rightarrow F(x)$ for all $x \in I$,

2) $T_{\max}(n+1, F)(x) > T_{\max}(n, F)(x) > F(x)$, and for $n \rightarrow \infty$, $T_{\max}(n, F)(x) \rightarrow F(x)$ for all $x \in I$.

We call $\{T_{\min}(n, F)(x)\}$ and $\{T_{\max}(n, F)(x)\}$ the lower limit polynomials sequence and upper limit polynomials sequence of $F(x)$ on interval I respectively, $T_{\min}(n, F)(x)$ is the lower limit polynomial of $F(x)$, and $T_{\max}(n, F)(x)$ is the upper limit of $F(x)$. n_0 is called the threshold.

Obviously, the lower limit polynomials and the upper limit polynomials of $F(x)$ satisfy:

$$T_{\min}(n_0, F)(x) < T_{\min}(n_0 + 1, F)(x) < T_{\min}(n_0 + 2, F)(x) < \dots < F(x) < \dots < T_{\max}(n_0 + 2, F)(x) < T_{\max}(n_0 + 1, F)(x) < T_{\max}(n_0, F)(x).$$

For convenience, we take the sum of the first n terms in Taylor expansion of function $f(x)$ at 0 as $T_f(n, x)$.

Definition 2. If the transcendental function $f(x)$ meets the following conditions on a certain intervals $[0, T]$,

1) For all $x \in [0, T]$, $f(x) \geq 0$ and for all $x \in (0, T]$, $f(x) > 0$;

2) The Taylor expansion of $f(x)$ at 0 is an alternating series, i.e. $T_f(n, x) = \sum_{i=1}^n (-1)^{i-1} f_i(x)$, where $f_i(x) = a_i x^{m_i}$, $0 < a_i \leq 1$, $m_{i-1} < m_i$, and converges to $f(x)$;

3) There exists a constant $n_0(f, T)$, such that for $n \geq n_0(f, T)$, $\text{taylor}(f, n) > 0$ and $f_n(x) > f_{n+1}(x) > 0$.

We say that $f(x)$ can be regularly expanded on the interval $[0, T]$, and the constant $n_0(f, T)$ is called its threshold.

For an elementary transcendental function f , such as $\sin x, \cos x$ etc., denote $f_{\min}(n, x) = T_f(2n, x)$, $f_{\max}(n, x) = T_f(2n - 1, x)$. Clearly, for $\forall n \geq n_0$, where n_0 is the corresponding threshold,

$$f_{\min}(n, x) < f(x) < f_{\max}(n, x).$$

Most of the common basic elementary transcendental functions can be regularly expanded on the corresponding intervals.

$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \cdots$ ($|x| \leq 1$), can be regularly expanded on $(0, 1]$, and the threshold $n_0 = 1$.

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k-1}x^k}{k} + \cdots$ ($0 < x < 1$), can be regularly expanded on $(0, 1)$, and the threshold $n_0 = 1$.

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{k-1}x^{2k-1}}{(2k-1)!} + \cdots$, can be regularly expanded on $(0, T]$, where $0 < T < \pi$ and the threshold $n_0 = \min\{n \in \mathbb{N}, \text{for } 0 < x \leq T, T_{\sin(x)}(n) > 0, T_{\sin(x)}(n+1) > 0\}$, e.g. for $T = \frac{\pi}{2}$, $n_0 = 1$; for $T = 3$, $n_0 = 3$; for $T = \frac{314}{100}$, $n_0 = 5$.

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots$, can be regularly expanded on $(0, T]$, where $0 < T < \frac{\pi}{2}$ and the threshold $n_0 = 1$. When $x = \frac{\pi}{2}$, as $T_{\cos(x)}(2n) < \cos(x) = 0$, so $\cos(x)$ can not be regularly expanded on $(0, \frac{\pi}{2}]$.

Denote the sums of the positive and negative terms of the expansion of polynomial f by f^+ and f^- respectively. Obviously, $f = f^+ + f^-$ and the following lemma clearly holds.

Lemma 4. For $f(x, y_1, \dots, y_n) \in \mathbb{R}[x, y_1, \dots, y_n]$, expressions T_{1i} and T_{2i} such that $T_{2i} > y_i > T_{1i} > 0$ for $i = 1, \dots, n$,

$$\begin{aligned} & f^+(x, T_{11}, T_{12}, \dots, T_{1n}) + f^-(x, T_{21}, T_{22}, \dots, T_{2n}) \\ & < f(x, y_1, \dots, y_n) < \\ & f^+(x, T_{21}, T_{22}, \dots, T_{2n}) + f^-(x, T_{11}, T_{12}, \dots, T_{1n}). \end{aligned}$$

Reference [15] presents the following theorem, which can be used to compute the upper (lower) limit polynomials of transcendental-polynomials, thereby reducing the proof of transcendental inequalities to the verification of univariate polynomials.

Theorem 5. For $f(x, y_1, \dots, y_t) \in \mathbb{R}[x, y_1, \dots, y_t]$, transcendental functions $\text{trans}_1(x), \dots, \text{trans}_t(x)$ can be regularly expanded on $(0, T)$, the thresholds are $n_1, \dots, n_t$ respectively. $n_0 = \max\{n_1, \dots, n_t\}$, $F(x) = f(x, \text{trans}_1(x), \dots, \text{trans}_t(x))$, when $n \geq n_0$,

$T_{\max}(n, F) = f^+(x, \text{taylor}(\text{trans}_1(x), 2n-1), \dots, \text{taylor}(\text{trans}_t(x), 2n-1)) + f^-(x, \text{taylor}(\text{trans}_1(x), 2n), \dots, \text{taylor}(\text{trans}_t(x), 2n))$ is an upper limit polynomial of $F(x)$,

$T_{\min}(n, F) = f^+(x, \text{taylor}(\text{trans}_1(x), 2n), \dots, \text{taylor}(\text{trans}_t(x), 2n)) + f^-(x, \text{taylor}(\text{trans}_1(x), 2n-1), \dots, \text{taylor}(\text{trans}_t(x), 2n-1))$ is a lower limit polynomial of $F(x)$.

We call the above scheme Successive Taylor-substitution, which is an effective scheme for dealing with transcendental functions. In the latter part of this section, we need to prove mixed-trigonometric polynomial inequalities and there are two options: one is to first convert $\sin(x)$ and $\cos(x)$ into rational expressions in $\tan(\frac{x}{2})$, then convert them into polynomials in $\arctan(y)$, and use the Taylor expansion of

$\arctan(y)$ to compute upper and lower limit polynomials; the other is to treat $\sin(x)$ and $\cos(x)$ directly as transcendental functions and compute their upper and lower limit polynomials. In practice, if the inequality contains $\cos(x)$ and the right endpoint of the interval is $\frac{\pi}{2}$, we choose option 1; if it does not contain $\cos(x)$, or the right endpoint of the interval is less than $\frac{\pi}{2}$, we use option 2. For the reasons, please refer to [14, 15].

To prove the univariate polynomial inequalities, which mostly involve operations with large integers and high-order polynomials, we choose to use the `xprove` command from the algebraic inequality proving package BOTTEMA[16]. While manual calculation is theoretically feasible and would be tedious. The calling syntax of `xprove` is: `xprove(ineq, [ineqs])`, where *ineq* denotes the algebraic inequality to be proved (with all its variables assumed to be non-negative), and *ineqs* represents a set of algebraic inequalities serving as assumed conditions (with all their variables also assumed to be non-negative). The command returns `true` if the non-strict inequality corresponding to *ineq* holds, and `false` otherwise.

Although the Taylor substitution method for proving inequalities can be fully automated by software, to enhance the readability of the latter proof process, we present key calculation steps and omit redundant details, all of which can be reproduced using Maple or manual calculation.

Since the transcendental number π is not convenient for direct comparison with rational numbers, we use rational approximation of π . One of the most famous formulas for calculating π is **Machin's Formula**:

$$\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right).$$

For $n \in \mathbb{N}$, the lower bound and the upper bound can be defined as

$$\pi_{\min}(n) = 16 \arctan_{\min}\left(n, \frac{1}{5}\right) - 4 \arctan_{\max}\left(n, \frac{1}{239}\right),$$

$$\pi_{\max}(n) = 16 \arctan_{\max}\left(n, \frac{1}{5}\right) - 4 \arctan_{\min}\left(n, \frac{1}{239}\right).$$

Then $\pi_{\min}(n) < \pi < \pi_{\max}(n)$ and $\lim_{n \rightarrow \infty} \pi_{\min}(n) = \pi$, $\lim_{n \rightarrow \infty} \pi_{\max}(n) = \pi$, thus we can perform a strong rational estimation of constants involving π .

Lemma 6.

$$\frac{89}{100} < \frac{\ln(3/2)}{\ln(\pi/2)} < \frac{9}{10}.$$

Proof. Denote $a_0 = \frac{\ln(3/2)}{\ln(\pi/2)}$. We have the Taylor polynomial bound for a_0 :

$$\frac{\ln_{\min}\left(n, \frac{1}{2}\right)}{\ln_{\max}\left(n, \frac{\pi}{2} - 1\right)} < a_0 < \frac{\ln_{\max}\left(n, \frac{1}{2}\right)}{\ln_{\min}\left(n, \frac{\pi}{2} - 1\right)}.$$

For $n = 5$ (We choose $n = 5$ because the 5th-order $\ln_{\min}$ and $\ln_{\max}$ can ensure that the approximation error of a_0 is sufficiently small to meet the requirements of

the subsequent inequality proof. The same in the later lemma proving process.), we get $a_1 < a_0 < a_2$, where

$$a_1 = \frac{\ln_{\min}(5, \frac{1}{2})}{\ln_{\max}(5, \frac{\pi}{2} - 1)}$$

$$= \frac{29053}{71680(\frac{9}{2}\pi + \frac{1}{4608}\pi^9 - \frac{7129}{2520} - \frac{9}{2048}\pi^8 - \frac{7}{32}\pi^6 + \frac{9}{224}\pi^7 + \frac{7}{2}\pi^3 - \frac{63}{32}\pi^4 + \frac{63}{80}\pi^5 - \frac{9}{2}\pi^2)}$$

$$a_2 = \frac{\ln_{\max}(5, \frac{1}{2})}{\ln_{\min}(5, \frac{\pi}{2} - 1)}$$

$$= 645120 \left(4\pi - \frac{761}{280} - \frac{1}{2048}\pi^8 - \frac{7}{96}\pi^6 + \frac{1}{112}\pi^7 + \frac{7}{3}\pi^3 - \frac{35}{32}\pi^4 + \frac{7}{20}\pi^5 - \frac{7}{2}\pi^2 \right).$$

For $n = 5$,

$$p_1 = \pi_{\min}(5) = \frac{1574353035058098242816726960753383747979148678087888676629836}{501132135402448902498734193931140174949627402324676513671875},$$

$$p_2 = \pi_{\max}(5) = \frac{3597144788622195741816151737039073386719688033186645699212537048}{1145006748252931350385207835661626317331906673927513885498046875},$$

and we have $p_1 < \pi < p_2$.

Substitute $\pi = p_2$ for the positive terms of π in the denominator of a_1 and $\pi = p_1$ for the negative terms of π , we get $a_0 > \frac{89}{100}$ (the calculation can be implemented by Maple while manual computation is labor-intensive). Substitute $\pi = p_1$ for the positive terms of π in the denominator of a_2 and $\pi = p_2$ for the negative terms of π , we get $a_0 < \frac{9}{10}$. This completes the proof. $\square$

In fact, from the proof of this lemma, we can derive a more precise bound

$$0.8943004098 < \frac{\ln(3/2)}{\ln(\pi/2)} < 0.8967667651.$$

Lemma 7.

$$3 \ln\left(\frac{\pi}{2}\right) > \frac{5}{4}.$$

Proof.

$$\begin{aligned} \ln\left(\frac{\pi}{2}\right) &= \ln\left(1 + \left(\frac{\pi}{2} - 1\right)\right) > \ln_{\min}\left(3, \frac{\pi}{2} - 1\right) \\ &= \frac{1}{2}\pi - 1 - \frac{1}{2}\left(\frac{1}{2}\pi - 1\right)^2 + \frac{1}{3}\left(\frac{1}{2}\pi - 1\right)^3 - \frac{1}{4}\left(\frac{1}{2}\pi - 1\right)^4 \\ &= 2\pi - \frac{25}{12} - \frac{3}{4}\pi^2 + \frac{1}{6}\pi^3 - \frac{1}{64}\pi^4 \\ &> 2\pi_{\min}(3) + \frac{1}{6}(\pi_{\min}(3))^3 - \frac{25}{12} - \frac{3}{4}(\pi_{\max}(3))^2 - \frac{1}{64}(\pi_{\max}(3))^4 \\ &= \frac{a_1}{a_2} \end{aligned}$$

where

$$\begin{aligned} a_1 &= 10928372565174158731562274589664654486722180173214739270580292077902 \\ &\quad 669567478152872279886122709782529993123063064895242033272590719897452543, \\ a_2 &= 2464985982782929416663755496945814551314187907489615699312732401944 \\ &\quad 9163760206871350613551764489615545341573224716528784483671188354492187500. \end{aligned}$$

It holds that

$$3 \ln \frac{\pi}{2} > 3 \frac{a_1}{a_2} > \frac{5}{4}.$$

This completes the proof. $\square$

Lemma 8. For $x \in (0, \frac{\pi}{2})$,

$$-\cos(x) \sin(x)^2 x^2 - \cos(x)^3 x^2 + \sin(x)^3 x + 2 \cos(x)^2 \sin(x) x - \cos(x) \sin(x)^2 > 0.$$

Proof. We transform the left-hand side of the inequality, denoted as $f(x)$, using tangent half-angle substitution into a rational function of $\tan(\frac{x}{2})$

$$\begin{aligned} f(x) &= -\frac{4(1 - \tan^2(\frac{x}{2})) \tan^2(\frac{x}{2}) x^2}{(1 + \tan^2(\frac{x}{2}))^3} - \frac{(1 - \tan^2(\frac{x}{2}))^3 x^2}{(1 + \tan^2(\frac{x}{2}))^3} + \frac{8 \tan^3(\frac{x}{2}) x}{(1 + \tan^2(\frac{x}{2}))^3} \\ &\quad + \frac{4(1 - \tan^2(\frac{x}{2}))^2 \tan(\frac{x}{2}) x}{(1 + \tan^2(\frac{x}{2}))^3} - \frac{4(1 - \tan^2(\frac{x}{2})) \tan^2(\frac{x}{2})}{(1 + \tan^2(\frac{x}{2}))^3} \\ &= \frac{\tan^6(\frac{x}{2}) x^2 + 4 \tan^5(\frac{x}{2}) x + \tan^4(\frac{x}{2}) x^2 + 4 \tan^4(\frac{x}{2})}{(1 + \tan^2(\frac{x}{2}))^3} \\ &\quad + \frac{-\tan^2(\frac{x}{2}) x^2 - 4 \tan^2(\frac{x}{2}) + 4 \tan(\frac{x}{2}) x - x^2}{(1 + \tan^2(\frac{x}{2}))^3}. \end{aligned}$$

To prove $f(x) > 0$, it suffices to show that the numerator of $f(x)$, denoted as g , is positive. Let $\tan(x/2) = y$, then $x = 2 \arctan y$ and $0 < y < \tan(\pi/4) = 1$, and we have

$$\begin{aligned} g &= 4y^6 \arctan^2 y + 4y^4 \arctan y + 8y^5 \arctan y + 4y^4 + 8y \arctan y - 4y^2 \\ &\quad - 4y^2 \arctan^2 y - 4 \arctan^2 y. \end{aligned}$$

The lower limit polynomial of $g(y)$

$$\begin{aligned} g_1(n, y) &= 4y^6 \arctan_{\min}^2(n, y) + 4y^4 \arctan_{\min}(n, y) + 8y^5 \arctan_{\min}(n, y) \\ &\quad + 4y^4 + 8y \arctan_{\min}(n, y) - 4y^2 - 4y^2 \arctan_{\max}^2(n, y) - 4 \arctan_{\max}^2(n, y). \end{aligned}$$

For $n = 2$, $g_1(2, y)$ is a specific polynomial

$$\begin{aligned} g_1(2, y) &= \left(2y - \frac{2}{3}y^3 + \frac{2}{5}y^5 - \frac{2}{7}y^7\right)^2 y^6 + \left(2y - \frac{2}{3}y^3 + \frac{2}{5}y^5 - \frac{2}{7}y^7\right)^2 y^4 + \\ &\quad 4 \left(2y - \frac{2}{3}y^3 + \frac{2}{5}y^5 - \frac{2}{7}y^7\right) y^5 + 4y^4 + 4 \left(2y - \frac{2}{3}y^3 + \frac{2}{5}y^5 - \frac{2}{7}y^7\right) y - \\ &\quad y^2 \left(2y - \frac{2}{3}y^3 + \frac{2}{5}y^5\right)^2 - \left(2y - \frac{2}{3}y^3 + \frac{2}{5}y^5\right)^2 - 4y^2 \end{aligned}$$

Calling the command `xprove( $g_1(2, y), [y < 1]$ )` returns `true`, which means $g > g_1(2, y) \geq 0$ holds for $y \in (0, 1)$. Thus the inequality $f(x) > 0$ holds for $x \in (0, \frac{\pi}{2})$. $\square$

Lemma 9. For $x \in (0, 1]$,

$$\frac{111}{100} \cos(x)^3 x^2 + \frac{111}{100} \cos(x) \sin(x)^2 x^2 + \frac{39}{50} \cos(x)^2 \sin(x) x - \frac{11}{100} \sin(x)^3 x - \frac{189}{100} \cos(x) \sin(x)^2 > 0.$$

Proof. Denote the left-hand side of the inequality as $g(x)$, and the lower limit polynomial of $g(x)$ is

$$g_1(n, x) = \frac{111}{100} \cos_{\min}(n, x) \cdot \sin_{\min}(n, x)^2 \cdot x^2 + \frac{111}{100} \cos_{\min}(n, x)^3 \cdot x^2 + \frac{39}{50} \cos_{\min}(n, x)^2 \cdot \sin_{\min}(n, x) \cdot x - \frac{11}{100} \sin_{\max}(n, x)^3 \cdot x - \frac{189}{100} \cos_{\max}(n, x) \cdot \sin_{\max}(n, x)^2.$$

where $n \in \mathbb{N}$. For $n = 2$, $g_1(2, x)$ is the specific polynomial

$$g_1(2, x) = -\frac{1}{609638400000} x^6 (37x^{16} - 2223x^{14} + 52254x^{12} - 264864x^{10} - 12171600x^8 + 296110080x^6 - 2911829760x^4 + 13404787200x^2 - 12192768000)$$

Furthermore, as $\cos_{\min}(2, x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 > 0$ on $(0, 1]$, we have $g(x) > g_1(2, x)$ for $x \in (0, 1]$. Calling the command `xprove( $g_1(2, x), [x < 1]$ )` returns `true`, which means $g_1(2, x) \geq 0$ and implies $g(x) > 0$ hold on $(0, 1]$. The lemma is proved. $\square$

Lemma 10. For $x \in (0, \frac{\pi}{2})$,

$$\begin{aligned} & \sin^4(x) \cos^2(x) x^4 - \cos^6(x) x^4 - 2 \sin^5(x) \cos(x) x^3 + 2 \sin(x) \cos^5(x) x^3 \\ & + \sin^6(x) x^2 - 2 \sin^3(x) \cos^3(x) x + \sin^4(x) \cos^2(x) > 0. \end{aligned}$$

Proof. We transform the left-hand side of the inequality, denoted as $g(x)$, using the tangent half-angle substitution, into a rational function of $\tan(\frac{x}{2})$, and denote $\tan(\frac{x}{2})$ by y , then $y \in (0, 1)$, to get

$$\begin{aligned} g(x) &= \frac{16y^4(-y^2+1)^2x^4}{(y^2+1)^6} - \frac{(-y^2+1)^6x^4}{(y^2+1)^6} - \frac{64y^5(-y^2+1)x^3}{(y^2+1)^6} \\ &+ \frac{4y(-y^2+1)^5x^3}{(y^2+1)^6} + \frac{64y^6x^2}{(y^2+1)^6} - \frac{16y^3(-y^2+1)^3x}{(y^2+1)^6} + \frac{16y^4(-y^2+1)^2}{(y^2+1)^6} \\ &= -\frac{1}{(y^2+1)^6} (x^4y^{12} - 6x^4y^{10} + 4x^3y^{11} - x^4y^8 - 20x^3y^9 + 12x^4y^6 - 24x^3y^7 \\ &\quad - 16xy^9 - x^4y^4 + 24x^3y^5 - 64x^2y^6 + 48xy^7 - 16y^8 - 6x^4y^2 \\ &\quad + 20x^3y^3 - 48xy^5 + 32y^6 + x^4 - 4x^3y + 16xy^3 - 16y^4) \end{aligned}$$

The denominator $(1+y^2)^6 > 0$, then it is only necessary to consider the numerator g_1 , transform $g_1 = g_{11} - g_{12}$, where

$$\begin{aligned} g_{11} &= 6x^4y^{10} + x^4y^8 + 20x^3y^9 + 24x^3y^7 + 16xy^9 + x^4y^4 + 64x^2y^6 + 16y^8 \\ &\quad + 6x^4y^2 + 48xy^5 + 4x^3y + 16y^4, \\ g_{12} &= x^4y^{12} + 4x^3y^{11} + 12x^4y^6 + 24x^3y^5 + 48xy^7 + 20x^3y^3 + 32y^6 + 4x^4 + 16xy^3. \end{aligned}$$

For $n \in \mathbb{N}$, the lower limit polynomial of $g_1(y)$

$$g_2(n, y) = g_{11}(x, y)|_{x=2 \arctan_{\min}(n, y)} - g_{12}(x, y)|_{x=2 \arctan_{\max}(n, y)}.$$

For $n = 3$, $g_2(3, y)$ is a specific polynomial

$$\begin{aligned} g_2(3, y) = & \frac{16}{144149438750625} y^{10} \left(59073603750 y^{44} - 278958684375 y^{42} \right. \\ & + 852660270000 y^{40} - 2174769976750 y^{38} + 5182130215500 y^{36} \\ & - 12981441675125 y^{34} + 26377801792650 y^{32} - 49740841671950 y^{30} \\ & + 87465024525450 y^{28} - 148772508532450 y^{26} + 243153402100620 y^{24} \\ & - 319836132538956 y^{22} + 415856839232406 y^{20} - 471302041755204 y^{18} \\ & + 461235264298578 y^{16} - 424233063221814 y^{14} - 164338326671904 y^{12} \\ & + 209611930691394 y^{10} - 1003616040686490 y^8 + 500680622208690 y^6 \\ & \left. - 1273557171244050 y^4 + 14578666916352000 y^2 + 683375117040000 \right) \end{aligned}$$

Calling the command `xprove( $g_2(3, y) > 0, [y < 1]$ )` returns `true`, which implies $g_1(y) > g_2(3, y) > 0$ for $y \in (0, 1)$. Thus we get $g(x) > 0$ holds on $(0, \frac{\pi}{2})$. $\square$

Lemma 11. For $x \in (1, \frac{\pi}{2})$,

$$\begin{aligned} & -\frac{11}{10} \sin^4(x) \cos^2(x) x^4 + \frac{11}{10} \cos^6(x) x^4 + \frac{1}{5} \sin^5(x) \cos(x) x^3 - \frac{11}{5} \sin(x) \cos^5(x) x^3 \\ & - \frac{1}{10} \sin^6(x) x^2 - 2 \sin^3(x) \cos^3(x) x^3 + 2 \sin^4(x) \cos^2(x) x^2 - \frac{19}{5} \sin^3(x) \cos^3(x) x \\ & + 3 \sin^2(x) \cos^4(x) x^2 + \frac{19}{10} \sin^4(x) \cos^2(x) < 0. \end{aligned}$$

Proof. Denote the left-hand side of the inequality as $f(x)$. By applying the tangent half-angle substitution, we obtain

$$\begin{aligned} f(x) = g(x, y) = & -\frac{88}{5} \frac{y^4(-y^2+1)^2 x^4}{(y^2+1)^6} + \frac{11}{10} \frac{(-y^2+1)^6 x^4}{(y^2+1)^6} + \frac{32}{5} \frac{y^5(-y^2+1) x^3}{(y^2+1)^6} \\ & - \frac{22}{5} \frac{y(-y^2+1)^5 x^3}{(y^2+1)^6} - \frac{32}{5} \frac{y^6 x^2}{(y^2+1)^6} - \frac{16 y^3(-y^2+1)^3 x^3}{(y^2+1)^6} + \frac{32 y^4(-y^2+1)^2 x^2}{(y^2+1)^6} \\ & - \frac{152}{5} \frac{y^3(-y^2+1)^3 x}{(y^2+1)^6} + \frac{12 y^2(-y^2+1)^4 x^2}{(y^2+1)^6} + \frac{152}{5} \frac{y^4(-y^2+1)^2}{(y^2+1)^6} \\ = & \frac{1}{10} \frac{1}{(y^2+1)^6} \left(11x^4 y^{12} - 66x^4 y^{10} + 44x^3 y^{11} - 11x^4 y^8 - 60x^3 y^9 + 120x^2 y^{10} \right. \\ & + 132x^4 y^6 - 104x^3 y^7 - 160x^2 y^8 + 304x y^9 - 11x^4 y^4 + 104x^3 y^5 + 16x^2 y^6 \\ & - 912x y^7 + 304y^8 - 66x^4 y^2 + 60x^3 y^3 - 160x^2 y^4 + 912x y^5 - 608y^6 \\ & \left. + 11x^4 - 44x^3 y + 120x^2 y^2 - 304x y^3 + 304y^4 \right) \end{aligned}$$

where $y = \tan(\frac{x}{2})$ and $\tan(\frac{1}{2}) < y < 1$. Let the numerator $g_1(y) = g_{11}(y) - g_{12}(y)$,

where

$$\begin{aligned} g_{11} &= 11x^4y^{12} + 44x^3y^{11} + 120x^2y^{10} + 132x^4y^6 + 304xy^9 \\ &\quad + 104x^3y^5 + 16x^2y^6 + 304y^8 + 60x^3y^3 + 912xy^5 + 11x^4 + 120x^2y^2 + 304y^4, \\ g_{12} &= 66x^4y^{10} + 11x^4y^8 + 60x^3y^9 + 104x^3y^7 + 160x^2y^8 \\ &\quad + 11x^4y^4 + 912xy^7 + 66x^4y^2 + 160x^2y^4 + 608y^6 + 44x^3y + 304xy^3. \end{aligned}$$

For $n \in \mathbb{N}$, the upper limit polynomial of $g_1(y)$

$$g_2(n, y) = g_{12}(x, y)|_{x=2 \arctan_{\max}(n, y)} - g_{11}(x, y)|_{x=2 \arctan_{\min}(n, y)}.$$

By the definitions of $\sin_{\min}$ and $\cos_{\max}$, we have $\tan(\frac{1}{2}) > \frac{\sin_{\min}(1, \frac{1}{2})}{\cos_{\max}(1, \frac{1}{2})} = \frac{184}{337}$. Calling the command `xprove`($g_2(20, y) < 0, [y > \frac{184}{337}, y < 1]$) returns **true**, which implies $g_1(y) < g_2(20, y) < 0$ holds for $y \in (\frac{184}{337}, 1)$, and thus holds for $y \in (\tan(\frac{1}{2}), 1)$. So we get $g(x) < 0$ for $x \in (1, \frac{\pi}{2})$. $\square$

Lemma 12. For $x \in (0, \frac{\pi}{2})$,

$$\frac{\sin^2(x)x}{\cos(x)(\sin(x) - x\cos(x))} > 3$$

Proof. The inequality is equivalent to

$$g(x) = \sin(x)^2 \cdot x - 3\cos(x) \cdot (\sin(x) - x\cos(x)) > 0.$$

Transform $g(x)$ using the tangent half-angle substitution and denote $\tan(\frac{x}{2})$ by y , and then $0 < y < 1$, into a rational function

$$\begin{aligned} g(x) &= \frac{4y^2x}{(y^2+1)^2} - \frac{3(-y^2+1)\left(\frac{2y}{y^2+1} - \frac{x(-y^2+1)}{y^2+1}\right)}{y^2+1} \\ &= \frac{3xy^4 - 2xy^2 + 6y^3 + 3x - 6y}{(y^2+1)^2} \end{aligned}$$

where the denominator $(1+y^2)^2 > 0$, and the numerator $g_1(x, y) = g_{11}(x, y) - g_{12}(x, y)$ with $g_{11}(x, y), g_{12}(x, y)$ being polynomials of x and y

$$g_{11} = 3xy^4 + 6y^3 + 3x, \quad g_{12} = 2xy^2 + 6y$$

For $n \in \mathbb{N}$, the lower limit polynomial of g_1 is

$$g_2(n, y) = g_{11}(x, y)|_{x=2 \arctan_{\min}(n, y)} - g_{12}(x, y)|_{x=2 \arctan_{\max}(n, y)}.$$

Calling the command `xprove`($g_2(1, y) > 0, [y < 1]$) returns **true**, which implies $g_1(y) > g_2(1, y) > 0$ holds for $y \in (0, 1)$. So we get $g(x) > 0$ for $x \in (0, \frac{\pi}{2})$. Then we conclude that the lemma holds. $\square$

Lemma 13. For $x \in (0, \frac{\pi}{2})$,

$$\begin{aligned} & -12 \cos(x)^5 x^3 - 4 \cos(x)^3 \sin(x)^2 x^3 + 8 \cos(x) \sin(x)^4 x^3 + 36 \cos(x)^4 \sin(x) x^2 \\ & + 38 \cos(x)^2 \sin(x)^3 x^2 + 2 \sin(x)^5 x^2 - 36 \cos(x)^3 \sin(x)^2 x - 34 \cos(x) \sin(x)^4 x \\ & + 12 \cos(x)^2 \sin(x)^3 > 0. \end{aligned}$$

Proof. Denote the left-hand side of the inequality as $g(x)$. We transform $g(x)$ using the tangent half-angle substitution and denote $\tan(\frac{x}{2})$ by y , and then $0 < y < 1$, into a rational function

$$\begin{aligned} g(x) &= \frac{128x^3y^4(1-y^2)}{5(y^2+1)^5} - \frac{16x^3y^2(1-y^2)^3}{5(y^2+1)^5} - \frac{12x^3(1-y^2)^5}{5(y^2+1)^5} \\ &+ \frac{64x^2y^5}{5(y^2+1)^5} + \frac{304x^2y^3(1-y^2)^2}{5(y^2+1)^5} + \frac{72x^2y(1-y^2)^4}{5(y^2+1)^5} \\ &- \frac{544xy^4(1-y^2)}{5(y^2+1)^5} - \frac{144xy^2(1-y^2)^3}{5(y^2+1)^5} + \frac{96y^3(1-y^2)^2}{5(y^2+1)^5} \\ &= \frac{4}{5(y^2+1)^5} \left(3x^3y^{10} - 11x^3y^8 + 18x^2y^9 - 14x^3y^6 + 4x^2y^7 + 36xy^8 \right. \\ &\quad \left. + 14x^3y^4 - 28x^2y^5 + 28xy^6 + 24y^7 + 11x^3y^2 + 4x^2y^3 - 28xy^4 \right. \\ &\quad \left. - 48y^5 - 3x^3 + 18x^2y - 36xy^2 + 24y^3 \right), \end{aligned}$$

where the denominator $(1+y^2)^5 > 0$, and the numerator $g_1(x, y) = g_{11}(x, y) - g_{12}(x, y)$ with $g_{11}(x, y), g_{12}(x, y)$ being polynomials of x and y

$$\begin{aligned} g_{11} &= 12x^3y^{10} + 72x^2y^9 + 16x^2y^7 + 144xy^8 + 56x^3y^4 + 112xy^6 \\ &\quad + 96y^7 + 44x^3y^2 + 16x^2y^3 + 72x^2y + 96y^3, \\ g_{12} &= 44x^3y^8 + 56x^3y^6 + 112x^2y^5 + 112xy^4 + 192y^5 + 12x^3 + 144xy^2. \end{aligned}$$

For $n \in \mathbb{N}$, the lower limit polynomial of g_1 ,

$$g_2(n, y) = g_{11}(x, y)|_{x=2 \arctan_{\min}(n, y)} - g_{12}(x, y)|_{x=2 \arctan_{\max}(n, y)}.$$

Calling the command `xprove( $g_2(6, y) > 0, [y < 1]$ )` returns `true`, which implies $g_1(y) > g_2(6, y) > 0$ holds for $y \in (0, 1)$. So we get $g(x) > 0$ for $x \in (0, \frac{\pi}{2})$. $\square$

Lemma 14.

$$\frac{\frac{\ln(1/\cos x)}{3 \ln(x/\sin x)} - 1}{\ln(\sin x/x) - \ln(\cos x)} < \frac{4}{5}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Proof. Let $u = \ln(\frac{x}{\sin x})$ and $v = -\ln \cos x$. Define the function

$$F(x) = 1 - \frac{4}{5}u + \frac{4}{5}v - \frac{v}{3u} = 1 - \frac{4}{5} \ln\left(\frac{x}{\sin x}\right) - \frac{4}{5} \ln(\cos x) + \frac{1}{3} \cdot \frac{\ln(\cos x)}{\ln(\frac{x}{\sin x})}.$$

Since $\ln(\frac{\sin x}{x}) - \ln \cos x = \ln(\frac{\tan x}{x}) > 0$ for $x \in (0, \frac{\pi}{2})$, the lemma is equivalent to $F(x) > 0$ for $x \in (\frac{\pi}{2})$.

As $\lim_{x \rightarrow 0^+} \ln\left(\frac{x}{\sin x}\right) = \lim_{x \rightarrow 0^+} \ln(\cos x) = 0$, $\lim_{x \rightarrow 0^+} \frac{\ln(\cos x)}{\ln\left(\frac{x}{\sin x}\right)} = -3$ we have $F(0^+) = 0$.

As $\lim_{x \rightarrow (\frac{\pi}{2})^-} \ln\left(\frac{x}{\sin x}\right)$ is a constant, $\frac{1}{3 \ln(\frac{\pi}{2})} - \frac{4}{5} < 0$ by Lemma 7, and $\lim_{x \rightarrow (\frac{\pi}{2})^-} \ln(\cos x) = -\infty$, we have $F((\frac{\pi}{2})^-) = +\infty$.

Next, we are to prove that $F(x)$ is increasing on $(0, \frac{\pi}{2})$. Take v as a function of u , the derivative of $F(x)$ with respect with x can be expressed as

$$F'(x) = \frac{dF}{du} \cdot \frac{du}{dx}.$$

Calculate the derivative of u :

$$\frac{du}{dx} = \frac{\frac{1}{\sin x} - \frac{x \cos x}{\sin^2 x}}{\frac{x}{\sin x}} = \frac{\sin x - x \cos x}{x \sin^3 x} > 0, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Thus we only need to show that

$$G(x) = \frac{dF}{du} = -\frac{4}{5} + \frac{4}{5} \frac{dv}{du} - \frac{\frac{dv}{du} \cdot u - v}{3u^2} > 0.$$

Denote $d = \frac{dv}{du}$, then

$$G(x) = \frac{4}{5}(d-1) - \frac{d-v/u}{3u}.$$

From Mitrinović-Adamović inequality $\left(\frac{\sin x}{x}\right)^3 > \cos x$ on $(0, \frac{\pi}{2})$, we have $3 \ln\left(\frac{\sin x}{x}\right) > \ln \cos x$, so $\frac{v}{u} = \frac{-\ln \cos x}{\ln(x/\sin x)} > 3$. Thus

$$G(x) > \frac{4}{5}(d-1) - \frac{d-3}{3u}.$$

Define the function

$$H(x) = 3 \cdot \frac{4}{5}u - \frac{d-3}{d-1} = \frac{12}{5}u - \frac{d-3}{d-1}.$$

As

$$d = \frac{dv}{du} = \frac{x}{\cos(x) \left(\frac{1}{\sin(x)} - \frac{x \cos(x)}{\sin^2(x)} \right)} = \frac{\sin^2(x) x}{\cos(x) (\sin(x) - x \cos(x))} > 3$$

by Lemma 12, then $d > 1$, we have $H(x) > 0$ if and only if $G(x) > 0$.

We now calculate the limit of $H(x)$ at $x = 0^+$.

$\lim_{x \rightarrow 0^+} u = \lim_{x \rightarrow 0^+} \ln(x/\sin x) = 0$. For the term $\frac{d-3}{d-1}$, we have:

$$\frac{d-3}{d-1} = \frac{\sin^2 x \cdot x + 3 \cos^2 x \cdot x - 3 \cos x \sin x}{\cos^2 x \cdot x + \sin^2 x \cdot x - \cos x \sin x}.$$

Using L'Hospital's Rule, we get:

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{d-3}{d-1} &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} (\sin^2 x \cdot x + 3 \cos^2 x \cdot x - 3 \cos x \sin x)}{\frac{d}{dx} (\cos^2 x \cdot x + \sin^2 x \cdot x - \cos x \sin x)} \\ &= \lim_{x \rightarrow 0^+} \frac{-4 \cos x \sin x \cdot x + 4 \sin^2 x}{2 \sin^2 x} = 0.\end{aligned}$$

Thus $\lim_{x \rightarrow 0^+} H(x) = 0$.

Next, we calculate the derivative of $H(x)$:

$$H'(x) = \frac{12}{5}u'(x) - \frac{d}{dx} \left(\frac{d-3}{d-1} \right) = \frac{H_1}{H_2},$$

where

$$\begin{aligned}H_1 &= -12 \cos(x)^5 x^3 - 4 \cos(x)^3 \sin(x)^2 x^3 + 8 \cos(x) \sin(x)^4 x^3 \\ &\quad + 36 \cos(x)^4 \sin(x) x^2 + 38 \cos(x)^2 \sin(x)^3 x^2 + 2 \sin(x)^5 x^2 \\ &\quad - 36 \cos(x)^3 \sin(x)^2 x - 34 \cos(x) \sin(x)^4 x + 12 \cos(x)^2 \sin(x)^3,\end{aligned}$$

and

$$H_2 = \sin x \cdot (\cos^2 x \cdot x + \sin^2 x \cdot x - \cos x \sin x)^2 \cdot x.$$

$H_2 > 0$ obviously, and from Lemma 13 we have $H_1 > 0$ on $(0, \frac{\pi}{2})$, so $H'(x) > 0$. Thus $H(x) > 0$ for all $x \in (0, \frac{\pi}{2})$, which implies $G(x) > 0$, $F'(x) > 0$, and finally we get $F(x) > 0$. The lemma is proved. $\square$

3. THE MAIN THEOREMS

Theorem 15. For $\frac{4}{5} < a < a_0 = \frac{\ln(3/2)}{\ln(\pi/2)}$, neither $F(a, x) = 2 + \cos^a x - 3 \left(\frac{\sin x}{x} \right)^a > 0$ nor its reverse inequality holds on $(0, \frac{\pi}{2})$.

Proof. First, we analyze the limit of $F(a, x)$ as $x \rightarrow 0^+$. Using the Taylor expansion

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5), \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

and for $u \rightarrow 0$

$$(1+u)^a = 1 + au + \frac{a(a-1)}{2}u^2 + O(u^3),$$

we have

$$\begin{aligned}\left(\frac{\sin x}{x} \right)^a &= \left(1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + o(x^4) \right) \right)^a = 1 - \frac{ax^2}{6} + \frac{a(5a-2)x^4}{360} + o(x^4), \\ (\cos x)^a &= \left(1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + o(x^4) \right) \right)^a = 1 - \frac{ax^2}{2} + \frac{a(3a-2)x^4}{24} + o(x^4).\end{aligned}$$

Substitute into $F(a, x)$,

$$\begin{aligned} F(a, x) &= 2 + 1 - \frac{ax^2}{2} + \frac{a(3a-2)x^4}{24} - 3 \left(1 - \frac{ax^2}{6} + \frac{a(5a-2)x^4}{360} \right) + o(x^4) \\ &= \frac{a(5a-4)x^4}{60} + o(x^4). \end{aligned}$$

Thus

$$\lim_{x \rightarrow 0^+} \frac{F(a, x)}{x^4} = \frac{a(5a-4)}{60}.$$

For $a > \frac{4}{5}$, the limit is positive, so there exists a constant $c \in (0, \frac{\pi}{2})$ such that $\frac{F(a, x)}{x^4} > 0$ holds on $(0, c)$, which implies $F(a, x) > 0$ on $(0, c)$.

Next,

$$\lim_{x \rightarrow (\frac{\pi}{2})^-} F(a, x) = 2 + 0 - 3 \left(\frac{2}{\pi} \right)^a = 2 - 3 \left(\frac{2}{\pi} \right)^a.$$

For $a < a_0 = \frac{\ln(3/2)}{\ln(\pi/2)}$, we have $(\frac{2}{\pi})^a > \frac{2}{3}$, so $2 - 3(\frac{2}{\pi})^a < 0$. Thus there exists a constant $d \in (c, \frac{\pi}{2})$ such that $f_a(x) < 0$ on $(d, \frac{\pi}{2})$.

Since $F(a, x) > 0$ on $(0, c)$ and $F(a, x) < 0$ on $(d, \frac{\pi}{2})$, neither $F(a, x) > 0$ nor its reverse holds on the entire interval $(0, \frac{\pi}{2})$. $\square$

Lemma 16. For $a_0 = \frac{\ln(3/2)}{\ln(\pi/2)}$, $f(x) = F(a_0, x) = 2 + \cos^{a_0} x - 3 \left(\frac{\sin x}{x} \right)^{a_0} > 0$ for $x \in (0, \frac{\pi}{2})$.

Proof. First, we have the limit properties:

$$\lim_{x \rightarrow 0^+} F(a_0, x) = 0, \quad \lim_{x \rightarrow (\frac{\pi}{2})^-} F(a_0, x) = 2 - 3 \left(\frac{2}{\pi} \right)^{a_0} = 0.$$

From the proof process of Theorem 15, there exists a constant $c > 0$ such that $F(a_0, x) > 0$ on $(0, c)$. So we only need to show that $f(x)$ is concave on $(0, \frac{\pi}{2})$, it suffices to prove $f'(x)$ has exactly one zero point on $(0, \frac{\pi}{2})$.

$$\begin{aligned} f'(x) &= a_0 \cos^{a_0-1} x (-\sin x) - 3a_0 \left(\frac{\sin x}{x} \right)^{a_0-1} \cdot \frac{x \cos x - \sin x}{x^2} \\ &= a_0 \left[3 \left(\frac{\sin x}{x} \right)^{a_0-1} \cdot \frac{\sin x - x \cos x}{x^2} - \cos^{a_0-1} x \sin x \right]. \end{aligned}$$

Let $f'(x) = 0$, we get

$$3 \left(\frac{\sin x}{x} \right)^{a_0-1} \cdot \frac{\sin x - x \cos x}{x^2} = \cos^{a_0-1} x \sin x.$$

Rearrange the terms to get

$$3 \left(\frac{\sin x}{x \cos x} \right)^{a_0-1} = \frac{\sin x}{(\sin x - x \cos x)/x^2}.$$

Take the natural logarithm on both sides,

$$\ln \left(3 \left(\frac{\sin x}{x \cos x} \right)^{a_0-1} \right) = \ln \left(\frac{\sin x}{(\sin x - x \cos x)/x^2} \right).$$

Define the function

$$g(a_0, x) = \ln \left(3 \left(\frac{\sin x}{x \cos x} \right)^{a_0-1} \right) - \ln \left(\frac{\sin x}{(\sin x - x \cos x)/x^2} \right).$$

It is easy to see that $g(a_0, x)$ and $f'(x)$ have the same zero points and the same sign on $(0, \frac{\pi}{2})$. We now prove that $g(a_0, x)$ has exactly one zero point on $(0, \frac{\pi}{2})$. We divide $(0, \frac{\pi}{2})$ into two subintervals: $(0, 1]$ and $(1, \frac{\pi}{2})$ for discussion.

Step 1: The limit of $g(a_0, x)$ at $x \rightarrow 0^+$.

Combine the two logarithmic terms:

$$g(a_0, x) = \ln \left(3 \left(\frac{\sin x}{x \cos x} \right)^{a_0-1} \cdot \frac{(\sin x - x \cos x)/x^2}{\sin x} \right).$$

Using the equivalent infinitesimals: $\sin x \sim x$, $(\sin x - x \cos x) \sim \frac{x^3}{3}$, $\cos x \sim 1$ ($x \rightarrow 0^+$), the expression inside the logarithm tends to $3 \cdot 1^{a_0-1} \cdot \frac{1/3}{1} = 1$, so

$$\lim_{x \rightarrow 0^+} g(a_0, x) = 0.$$

Step 2: The first derivative of $g(a_0, x)$.

$$\begin{aligned} g'(a_0, x) = & \frac{1}{\cos x \cdot x \cdot (-\cos x \cdot x + \sin x) \cdot \sin x} \left[-\cos^3 x \cdot a_0 x^2 \right. \\ & - \cos x \sin^2 x \cdot a_0 x^2 + 2 \cos^3 x \cdot x^2 + 2 \cos^2 x \sin x \cdot a_0 x + 2 \cos x \sin^2 x \cdot x^2 \\ & \left. + \sin^3 x \cdot a_0 x - \cos^2 x \sin x \cdot x - \cos x \sin^2 x \cdot a_0 - \sin^3 x \cdot x - \cos x \sin^2 x \right] \end{aligned}$$

The denominator is positive since $\frac{\sin x}{x} > \cos x$ and then $\sin x - x \cos x > 0$ for $x \in (0, \frac{\pi}{2})$. Denote the numerator as $g_1(a_0, x)$, and the coefficient of a in $g_1(a, x)$ is

$$-\cos(x)^3 x^2 - \cos(x) \sin(x)^2 x^2 + 2 \cos(x)^2 \sin(x) x + \sin(x)^3 x - \cos(x) \sin(x)^2$$

which is positive on $(0, \frac{\pi}{2})$ by Lemma 8. So $g_1(a, x)$ is increasing with respect to a . Thus, by Lemma 6 we have

$$g_1(a_0, x) > g_1\left(\frac{89}{100}, x\right).$$

$$\begin{aligned} g_1\left(\frac{89}{100}, x\right) = & \frac{111}{100} \cos(x)^3 x^2 + \frac{111}{100} \cos(x) \sin(x)^2 x^2 + \frac{39}{50} \cos(x)^2 \sin(x) x - \frac{11}{100} \sin(x)^3 x \\ & - \frac{189}{100} \cos(x) \sin(x)^2. \end{aligned}$$

From Lemma 9 we have $g_1\left(\frac{89}{100}, x\right) > 0$ on $(0, 1)$, so $g_1(a_0, x) > 0$ on $(0, 1)$. This implies $g'(a_0, x) > 0$ on $(0, 1)$, so $g(a_0, x) > g(a_0, 0^+) = 0$ on $(0, 1]$.

Step 3: Analysis on $(1, \frac{\pi}{2})$.

$$g(a_0, x) = (a_0 - 1) \ln\left(\frac{1}{\cos x}\right) + \ln\left(3\left(\frac{\sin x}{x}\right)^{a_0-1}\right) - \ln\left(\frac{\sin x}{(\sin x - x \cos x)/x^2}\right).$$

For $x \rightarrow (\frac{\pi}{2})^-$, the first term is $-\infty$ as $a_0 - 1 < 0$, and the other two terms are constants, we have $\lim_{x \rightarrow (\frac{\pi}{2})^-} g(a_0, x) = -\infty$.

Next, calculating the second derivative of $g(a_0, x)$ yields

$$\begin{aligned} g''(a_0, x) = & \frac{1}{\cos(x)^2 x^2 (-x \cos(x) + \sin(x))^2 \sin(x)^2} \left[\sin(x)^4 \cos(x)^2 a_0 x^4 - \right. \\ & \cos(x)^6 a_0 x^4 - 2 \sin(x)^5 \cos(x) a_0 x^3 - 2 \sin(x)^4 \cos(x)^2 x^4 \\ & + 2 \sin(x) \cos(x)^5 a_0 x^3 + 2 \cos(x)^6 x^4 + \sin(x)^6 a_0 x^2 + 2 \sin(x)^5 \cos(x) x^3 \\ & - 2 \sin(x)^3 \cos(x)^3 x^3 - 4 \sin(x) \cos(x)^5 x^3 - \sin(x)^6 x^2 + 2 \sin(x)^4 \cos(x)^2 x^2 \\ & - 2 \sin(x)^3 \cos(x)^3 a_0 x + 3 \sin(x)^2 \cos(x)^2 x^2 + \sin(x)^4 \cos(x)^2 a_0 \\ & \left. - 2 \sin(x)^3 \cos(x)^3 x + \sin(x)^4 \cos(x)^2 \right], \end{aligned}$$

whose denominator is obviously positive. Denote the numerator as $h(a_0, x)$.

The coefficient of a in $h(a, x)$ is

$$\begin{aligned} & \sin(x)^4 \cos(x)^2 x^4 - \cos(x)^6 x^4 - 2 \sin(x)^5 \cos(x) x^3 \\ & + 2 \sin(x) \cos(x)^5 x^3 + \sin(x)^6 x^2 - 2 \sin(x)^3 \cos(x)^3 x + \sin(x)^4 \cos(x)^2 \end{aligned}$$

which is positive function on $(0, \frac{\pi}{2})$ by Lemma 10, so $h(a, x)$ is increasing with respect to a for $0 < x < \frac{\pi}{2}$. Thus, on $(0, \frac{\pi}{2})$,

$$h\left(\frac{\ln(3/2)}{\ln(\pi/2)}, x\right) < h\left(\frac{9}{10}, x\right).$$

$$\begin{aligned} h\left(\frac{9}{10}, x\right) = & -\frac{11}{10} \sin(x)^4 \cos(x)^2 x^4 + \frac{11}{10} \cos(x)^6 x^4 + \frac{1}{5} \sin(x)^5 \cos(x) x^3 \\ & - \frac{11}{5} \sin(x) \cos(x)^5 x^3 - \frac{1}{10} \sin(x)^6 x^2 - 2 \sin(x)^3 \cos(x)^3 x^3 \\ & + 2 \sin(x)^4 \cos(x)^2 x^2 - \frac{19}{5} \sin(x)^3 \cos(x)^3 x + 3 \sin(x)^2 \cos(x)^4 x^2 \\ & + \frac{19}{10} \sin(x)^4 \cos(x)^2 \end{aligned}$$

From Lemma 11 we have $h\left(\frac{9}{10}, x\right) < 0$ on $(1, \frac{\pi}{2})$, so $h(a_0, x) < 0$ on $(1, \frac{\pi}{2})$, which implies $g''(a_0, x) < 0$ on $(1, \frac{\pi}{2})$, i.e., $g(a_0, x)$ is **concave** on $(1, \frac{\pi}{2})$.

Combining the above results: $g(a_0, x) > 0$ on $(0, 1]$, $g(a_0, x)$ is concave on $(1, \frac{\pi}{2})$, and $\lim_{x \rightarrow (\frac{\pi}{2})^-} g(a_0, x) = -\infty$, we can conclude that $g(a_0, x)$ has exactly one zero point $x_0 \in (1, \frac{\pi}{2})$ on $(0, \frac{\pi}{2})$.

This implies that $f'(x) > 0$ on $(0, x_0)$ and $f'(x) < 0$ on $(x_0, \frac{\pi}{2})$. So $F(a_0, x)$ is increasing on $(0, x_0)$ and decreasing on $(x_0, \frac{\pi}{2})$. Since $F(a_0, 0^+) = F(a_0, (\frac{\pi}{2})^-) = 0$, we have $F(a_0, x) > 0$ on $(0, \frac{\pi}{2})$. $\square$

Lemma 17. $F(a, x) = 2 + \cos^a x - 3 \left(\frac{\sin x}{x} \right)^a$ is increasing with respect to a on $\left(\frac{\ln(3/2)}{\ln(\pi/2)}, +\infty \right)$ for $x \in (0, \frac{\pi}{2})$.

Proof. Calculate the partial derivative of $F(a, x)$ with respect to a :

$$\frac{\partial F(a, x)}{\partial a} = \cos^a x \ln \cos x - 3 \left(\frac{\sin x}{x} \right)^a \ln \left(\frac{\sin x}{x} \right).$$

Let $\frac{\partial F(a, x)}{\partial a} = 0$, rearrange to get:

$$\cos^a x \ln \left(\frac{1}{\cos x} \right) = 3 \left(\frac{\sin x}{x} \right)^a \ln \left(\frac{x}{\sin x} \right).$$

Take the natural logarithm on both sides and solve for a :

$$a = \frac{\ln \left(\frac{\ln(1/\cos x)}{3 \ln(x/\sin x)} \right)}{\ln(\sin x/x) - \ln \cos x}.$$

Denote the right-hand side as $g(x)$. Since $\ln(1+x) < x$ for $x > 0$, we have

$$g(x) < g_1(x) = \frac{\frac{\ln(1/\cos x)}{3 \ln(x/\sin x)} - 1}{\ln(\sin x/x) - \ln \cos x}.$$

From Lemma 14, we have $g_1(x) < \frac{4}{5}$ on $(0, \frac{\pi}{2})$. As $a_0 = \frac{\ln(3/2)}{\ln(\pi/2)} > \frac{4}{5}$, for $a > a_0$, $a > g_1(x) > g(x)$ on $(0, \frac{\pi}{2})$, which implies

$$a > \frac{\ln \left(\frac{\ln(1/\cos x)}{3 \ln(x/\sin x)} \right)}{\ln(\sin x/x) - \ln \cos x}$$

and then

$$\cos^a x \ln \left(\frac{1}{\cos x} \right) < 3 \left(\frac{\sin x}{x} \right)^a \ln \left(\frac{x}{\sin x} \right),$$

so,

$$\cos^a x \ln(\cos x) > 3 \left(\frac{\sin x}{x} \right)^a \ln \left(\frac{\sin x}{x} \right).$$

Then we get that $\frac{\partial F(a, x)}{\partial a} > 0$ and $F(a, x)$ is increasing with respect to a on $(a_0, +\infty)$ for $x \in (0, \frac{\pi}{2})$. $\square$

From Lemma 16 and Lemma 17, we get the following theorem.

Theorem 18. For $a \geq \frac{\ln(3/2)}{\ln(\pi/2)}$, $F(a, x) = 2 + \cos^a x - 3 \left(\frac{\sin x}{x} \right)^a > 0$ holds on $(0, \frac{\pi}{2})$.

4. CONCLUSION

In this paper, we completely solve the open problem on the generalized Cusa-Huygens inequality $\left(\frac{\sin x}{x}\right)^a < \frac{2}{3} \left(1 + \frac{\cos^a x}{2}\right)$ for $x \in (0, \frac{\pi}{2})$. The main conclusions are:

1. The inequality holds for $a \geq \frac{\ln(3/2)}{\ln(\pi/2)}$;
2. For $\frac{4}{5} < a < \frac{\ln(3/2)}{\ln(\pi/2)}$, neither the inequality nor its reverse holds on the interval $(0, \frac{\pi}{2})$.

The proof method adopted in this paper provides an effective way for the proof of similar transcendental inequalities with parameters as exponents, and can be extended to the study of other trigonometric inequalities.

REFERENCES

1. Huygens, C., Oeuvres completes, publiees par la Societe hollandaise des science. Haga, 1888–1940(20 volumes)
2. Bagul, Y.J., Banjac, B., Chesneau, C., Kostić, M., Malešević, B.: New refinements of Cusa-Huygens inequality, Results Math 76(2) (2021), Paper No. 107, 16 pp.
3. Bagul, Y.J., Chesneau, C., Refined forms of Oppenheim and Cusa-Huygens type inequalities. Acta Comment. Univ. Tartu. Math. 24(2) (2020), 183–194.
4. Bagul, Y.J., Chesneau, C., Kostić, M.: On the Cusa-Huygens inequality. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 115(1) (2021), Paper No. 29, 12 pages.
5. Banjac, B., Malešević, B., Micović, M., Mihailović, B., Savatović, M.: The best possible constants approach for Wilker-Cusa-Huygens inequalities via stratification. Appl. Anal. Discrete Math. 18(1) (2024), 244–288.
6. Chen, C.-P., Mortici, C., The relationship between Huygens' and Wilker's inequalities and further remarks. Appl. Anal. Discrete Math. 17(1) (2023), 92–100.
7. Kuang, J.C. (Eds), Applied Inequalities, Shandong Science and Technology Press, 4th ed.(in Chinese), Shandong, 2010.
8. Zhu, L., Proceedings of the Fourth National Conference on Inequalities(in Chinese), 2009 (pp. 32-41).
9. Zhu, L., A source of inequalities for circular functions, Computers and Mathematics with Applications, 58 (2009), 1998–2004
10. Zhao, T., ON A UNIFIED GENERALIZATION OF CUSA-HUYGENS AND MITRINOVIĆ-ADAMOVIĆ INEQUALITIES, Appl. Anal. Discrete Math. 19 (2025), 370–393. <https://doi.org/10.2298/AADM240914017Z>
11. Chouikha, A.R., New Look of Trigonometric and Hyperbolic Inequalities, Int. J. Open Problems Compt. Math., Vol. 18, No. 4, December 2025.
12. B. Malešević, M. Makragić: A Method for Proving Some Inequalities on Mixed Trigonometric Polynomial Functions, Journal of Mathematical Inequalities, Volume 10, Number 3 (2016), 849–876 (DOI: 10.7153/jmi-10-69).

13. Chen, S., Automated Proof of Mixed Trigonometric-polynomial Inequalities, Journal of Symbolic Computation 101C (2020) pp. 318-329.
14. Chen, S., Liu, Z., Automated Proof of class of transcendental function inequalities, Journal of Systems Science and Mathematical Sciences(in Chinese), 2019.
15. Chen, S., Ge, X., Reachability analysis of linear systems. Acta Informatica 61, 231–260 (2024). <https://doi.org/10.1007/s00236-024-00458-8>
16. Yang L., Xia B., *Inequality Machine Proof and Automatic Discovery*(in Chinese), Science Press, 2008.

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