

A STUDY ON AN EXTENSION OF EULER TOTIENT SEQUENCE SPACES VIA q -ANALOGUE

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This paper introduces the sequence spaces $\ell_p(\Phi_q)$ and $\ell_\infty(\Phi_q)$ for $q \in (0, 1)$, generated by an infinite matrix Φ_q derived from the q -Euler totient function φ_q . We establish key properties of φ_q and comprehensively analyze the new sequence spaces, detailing their Schauder bases, α -, β -, and γ -duals, matrix class characterizations, and geometric properties.

1. INTRODUCTION AND PRELIMINARIES

Consider ω as the collection of all sequences whose terms are real or complex numbers. Any linear subspace of ω is known as a sequence space. Familiar examples of these classical spaces include ℓ_p (absolutely p -summable sequences), ℓ_∞ (bounded sequences), c_0 (null sequences), and c (convergent sequences), where $0 < p < \infty$. Unless otherwise specified, we assume throughout this discussion that $p^{-1} + p'^{-1} = 1$ and $0 < p < \infty$. Additionally, we define the sequence spaces cs and bs as follows:

$$cs = \left\{ s = (s_m) \in \omega : \lim_{m \rightarrow \infty} \left| \sum_{d=1}^m s_d \right| \text{ exists} \right\},$$

$$bs = \left\{ s = (s_m) \in \omega : \sup_{m \in \mathbb{Z}^+} \left| \sum_{d=1}^m s_d \right| < \infty \right\}.$$

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2020 Mathematics Subject Classification. 40C05, 46A45, 46B45.

Keywords and Phrases. q -Euler totient function, Sequence spaces, Dual spaces, Matrix transformations, Geometric properties.

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A Banach space Ξ is known as a BK -space if the projection mapping f to its m^{th} component, f_m , is continuous for every m . It is well-established that the spaces $\Xi \in \{\ell_\infty, c, c_0\}$ and ℓ_p ($1 < p < \infty$) are BK -spaces when endowed with their standard norms: $\|s\|_\Xi = \sup_m |s_m|$ and $\|s\|_{\ell_p} = (\sum_m |s_m|^p)^{1/p}$, respectively. Moreover, the sequence space ℓ_p , for $0 < p \leq 1$, forms a complete p -normed space due to the p -norm $\|s\|_{\ell_p} = \sum_m |s_m|^p$. For convenience, here and in the rest of the discussion, we use the notations $\sum_m$ and $\sup_m$ in place of $\sum_{m=1}^\infty$ and $\sup_{m \in \mathbb{Z}^+}$, respectively, where $\mathbb{Z}^+$ represents the set of positive integers.

Consider Ξ and Π as two arbitrary sequence spaces, and let $G = (g_{md})$ represent an infinite matrix containing real or complex entries. For a given positive integer m , let G_m be the m^{th} row of G . For any sequence $s = (s_m) \in \omega$, we define a transformed sequence $t = (t_m)$ where each term is given by

$$t_m = (Gs)_m = \sum_d g_{md} s_d,$$

assuming the infinite sum converges. The resulting sequence t is called the G -transform of s . Furthermore, the matrix G induces a mapping from Ξ into Π if, for every sequence $s \in \Xi$, its G -transform $t = Gs \in \Pi$. The set of all such matrices that map from Ξ to Π is denoted by (Ξ, Π) . The set

$$\Xi_G = \{s = (s_d) \in \omega : Gs \in \Xi\}$$

forms a sequence space and is known as the domain of G in Ξ . Additionally, a matrix G is defined as a triangle if and only if its diagonal entries g_{dd} are non-zero and its entries above the main diagonal are zero (i.e., $g_{md} = 0$ for all $m > d$). A fundamental result states that if Ξ is a BK -space and G is a triangle, then Ξ_G is also a BK -space due to the norm $\|s\|_{\Xi_G} = \|Gs\|_\Xi$. To explore this result and various other significant discussions concerning BK -spaces defined by matrix domains, we recommend the books by Başar [4] and Mursaleen and Başar [21].

1.1 Euler Totient Function

One of the most intriguing arithmetic function that has numerous applications in the field of number theory is Euler totient function φ . Let $m \in \mathbb{Z}^+$. Then, $\varphi(m)$ counts the number of positive integers not exceeding m that are relatively prime to m . There are several interesting properties of φ that can be found in [2, 24]. We highlight few of them that are required for our purpose.

Let $m = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, where p_i represents distinct prime factors of m . Then, Euler totient function φ is also defined by

$$\varphi(m) = m \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right).$$

As special cases, if p denotes any prime, $\varphi(p) = p-1$ and $\varphi(p^r) = p^r - p^{r-1}$. Further, φ is a multiplicative function. That is, for any two $a, b \in \mathbb{Z}^+$, $\varphi(ab) = \varphi(a)\varphi(b)$, given that a and b are co-prime. Now we define the well-known Möbius function μ as follows:

$$\mu(m) = \begin{cases} 1 & , \quad m = 1, \\ (-1)^r & , \quad m = p_1 p_2 \cdots p_r, \\ 0 & , \quad m \text{ is square free,} \end{cases}$$

The Euler totient function φ and the Möbius function μ are closely related. More specifically, φ can be defined in terms of Möbius function μ as follows;

$$(1) \quad m = \sum_{d|m} \mu(d) \frac{m}{d}.$$

Here, and throughout the paper, the notations ' $d \mid m$ ' and ' $d \nmid m$ ' mean that d is a factor of m and d is not a factor m , respectively. By using the celebrated Möbius inversion formula [2] in (1), we get that

$$\sum_{d|m} \varphi(d) = m.$$

By using the fascinating properties of Euler totient function φ , İlkhān et al. [12, 13] introduced Euler totient sequence spaces $\Xi(\Phi)$, where $\Xi \in \{\ell_p, c_0, c, \ell_\infty\}$ and $\Phi = (\phi_{md})$ is a triangle given by

$$\phi_{md} = \begin{cases} \frac{\varphi(d)}{m} & , \quad d \mid m, \\ 0 & , \quad d \nmid m, \end{cases}$$

in the following manner:

$$\Xi(\Phi) = \left\{ s = (s_m) \in \omega : \Phi s = \left(\sum_{d|m} \frac{\varphi(d)}{m} s_d \right) \in \Xi \right\}.$$

They further studied the compactness of certain matrix operator on these spaces. It is shown that the sequence (s_m) with

$$s_m = \begin{cases} 1 & , \quad m = 2 \cdot 3 \cdot p_r, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

is Φ -convergent but does not converge in the usual sense, where p_r denotes successive prime numbers. A generalized form of the above sequence spaces came in the form of Jordan totient sequence spaces $\Xi(\Upsilon^n) = \Xi_{\Upsilon^n}$ introduced by İlkhān et al. [13, 14] for $\Xi \in \{\ell_p, c_0, c, \ell_\infty\}$. Here, $\Upsilon = (v_{md}^n)$ is Jordan totient matrix defined by

$$v_{md}^n = \begin{cases} \frac{J_n(d)}{m^n} & , \quad d \mid m, \\ 0 & , \quad d \nmid m, \end{cases}$$

and $J_n(m) = m^n \sum_{d|m} \frac{\mu(d)}{d^n}$ for $n \in \mathbb{Z}^+$. It is clear that when $n = 1$, Jordan totient sequence spaces are reduced to Euler totient sequence spaces. Besides, the readers may also like to explore the papers by Yaying et al. [29, 32, 33, 34] and Demiriz et al. [8] for study of new sequence spaces defined by using arithmetic functions.

Let $q \in (0, 1)$. Then, q -integer, denoted by $[m]_q$, is defined by

$$[m]_q = \begin{cases} \sum_{r=0}^{m-1} q^r & , \quad m \in \mathbb{Z}^+, \\ 0 & , \quad m = 0. \end{cases}$$

It is known that $[m]_q = m$ when $q \rightarrow 1^-$. Quite recently, many authors have shown their keen interests in developing q -analogue of well-known sequence spaces. The idea behind the development of q -sequence spaces is to construct a well defined q -analogue of popular triangular matrices, and develop the q -sequence spaces as the domain of the particular q -triangular matrix in the chosen classical sequence spaces. For instances, q -Cesàro sequence spaces are developed by Demiriz and Şahin [7] and Yaying et al. [30] and q -Catalan sequence spaces by Yaying et al. [28, 31].

A q -analogue φ_q of Euler's totient function φ is defined by Haukkanen [10] (also refer [11, p. 286]) in a poset setting as follows:

Definition 1. The q -Euler function $\varphi_{f_q}(x)$ is the number of 1-dimensional subspaces of $f_q(x)$ that are not contained in $f_q(y)$ for $y < x$. Then

$$(2) \quad \varphi_{f_q}(x) = \sum_{y \leq x} \mu(y, x) \frac{q^{\dim f_q(y)} - 1}{q - 1}.$$

It is suggested to refer [10, pp. 84-85] for the meaning of mathematical symbols used in the above definition. By substituting $f_q(n) = F_q(U_n)$, (2) returns the q -analogue of classical Euler's totient function (see [10, Example 5.1]) given by:

$$(3) \quad \varphi_q(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{q^d - 1}{q - 1}.$$

As expected, $\varphi_q = \varphi$ when $q \rightarrow 1^-$. Another q -analogue of φ is given by Campbell [6] (also see [3]) in the following manner:

$$(4) \quad \varphi_q(n) = \sum_{\substack{1 \leq r \leq n \\ \gcd(r, n) = 1}} q^r.$$

For our purpose, we follow the definition of q -Euler totient function as given by Haukkanen [10]. The Table 1 gives the value of $\varphi_q(m)$ for the first 16 positive integers: Indeed, by the application of well-known Möbius inversion formulae [24] in (3), we obtain that

$$(5) \quad \sum_{d|m} \varphi_q(d) = \frac{q^m - 1}{q - 1} = [m]_q.$$

Table 1: Value of $\varphi_q(m)$ for $m = 1, 2, \dots, 16$.

m	$\varphi_q(m)$
1	1
2	q
3	$q + q^2$
4	$q^2 + q^3$
5	$q + q^2 + q^3 + q^4$
6	$-q + q^3 + q^4 + q^5$
7	$q + q^2 + \dots + q^6$
8	$q^4 + q^5 + q^6 + q^7$
9	$q^3 + q^4 + \dots + q^8$
10	$-q + q^5 + q^6 + \dots + q^9$
11	$q + q^2 + \dots + q^{10}$
12	$-q^2 - q^3 + q^6 + q^7 + \dots + q^{11}$
13	$q + q^2 + \dots + q^{12}$
14	$-q + q^7 + q^8 + \dots + q^{13}$
15	$-q - q^2 + q^5 + q^6 + \dots + q^{14}$
16	$q^8 + q^9 + \dots + q^{15}$

The equalities (3) and (5) are the foundation concepts in defining the Φ_q -matrix transformation or its inverse and will be used throughout this paper.

The main purpose of this study is to construct a q -analogue Φ_q of Euler totient matrix Φ via q -Euler totient function φ_q , and discuss its domain in the sequence spaces ℓ_p and ℓ_∞ . Besides, prior to the above objectives, we examine some fundamental properties of the function φ_q . Discussion of some key concepts in the theory sequence spaces like Schauder basis, Alpha-, Beta- and Gamma-duals, and characterization of matrix classes related to the newly defined sequence spaces are our main motives. We also devote a section that investigates the geometric properties of the newly defined sequence spaces.

2. SOME FUNDAMENTAL PROPERTIES OF THE FUNCTION φ_q AND THE SEQUENCE SPACES $\ell_p(\Phi_q)$ AND $\ell_\infty(\Phi_q)$

Indeed, the q -analogue φ_q of the Euler totient function φ has been mentioned in some studies [10, 11] in the past. But many of their properties are still a research of interest for many researchers working in this field. Here, we discuss some fundamental properties of φ_q from prime factorization point of view.

Remark 2. The q -Euler totient function φ_q is not multiplicative, unlike the classical Euler totient function φ . More specifically, $\varphi_q(xy) \neq \varphi_q(x)\varphi_q(y)$ for $\gcd(x, y) = 1$. This result is clear from the facts that $\varphi_q(2) = q$ and $\varphi_q(3) = q + q^2$, but $\varphi_q(2)\varphi_q(3) = q^2 + q^3 \neq \varphi_q(6)$, in general (refer Table 1).

Theorem 3. *Let p be any prime number. Then,*

$$\varphi_q(p) = [p]_q - 1 = q[p - 1]_q.$$

Proof. The proof is straight forward from the definition of φ_q itself. We have that

$$\begin{aligned} \varphi_q(p) &= \sum_{d|p} \mu\left(\frac{p}{d}\right) [d]_q = \mu(p)[1]_q + \mu(1)[p]_q = [p]_q - 1 = q + q^2 + \dots + q^{p-1} \\ &= q[p - 1]_q. \end{aligned}$$

□

As an example, $\varphi_q(3) = q[2]_q = q(1 + q) = q + q^2$.

Theorem 4. *Let p be any prime number. Then,*

$$\varphi_q(p^r) = [p^r]_q - [p^{r-1}]_q.$$

Proof. It is known that p^r has $r + 1$ positive divisors viz. $1, p, p^2, \dots, p^r$. By using the definition of φ_q and Möbius function μ , we obtain that

$$\begin{aligned} \varphi_q(p^r) &= \sum_{d|p^r} \mu\left(\frac{p^r}{d}\right) [d]_q \\ &= \mu(p^r)[1]_q + \mu(p^{r-1})[p]_q + \dots + \mu(p)[p^{r-1}]_q + \mu(1)[p^r]_q \\ &= [p^r]_q - [p^{r-1}]_q. \end{aligned}$$

Here, we use the fact that $\mu(p^r) = 0$ for $r \geq 2$.

□

Clearly, $\varphi_q(4) = \varphi_q(2^2) = [4]_q - [2]_q = 1 + q + q^2 + q^3 - 1 - q = q^2 + q^3$.

Theorem 5. *Let $m = p_1^{r_1} p_2^{r_2}$, where p_1 and p_2 are two distinct prime numbers. Then,*

$$\varphi_q(m) = [m]_q - \sum_{1 \leq i \leq 2} \left[\frac{m}{p_i} \right]_q + \left[\frac{m}{p_1 p_2} \right]_q.$$

Proof. Let $m = p_1^{r_1} p_2^{r_2}$. Then, m has $(r_1 + 1)(r_2 + 1)$ number of positive divisors. These are 1 and

$$\begin{aligned} &p_1, p_1^2, p_1^3, \dots, p_1^{r_1}, \\ &p_2, p_2^2, p_2^3, \dots, p_2^{r_2}, \\ &p_1 p_2, p_1 p_2^2, p_1 p_2^3, \dots, p_1 p_2^{r_2}, \\ &p_1^2 p_2, p_1^2 p_2^2, p_1^2 p_2^3, \dots, p_1^2 p_2^{r_2}, \\ &\vdots \\ &p_1^{r_1-1} p_2, p_1^{r_1-1} p_2^2, p_1^{r_1-1} p_2^3, \dots, p_1^{r_1-1} p_2^{r_2}, \\ &p_1^{r_1} p_2, p_1^{r_1} p_2^2, p_1^{r_1} p_2^3, \dots, p_1^{r_1} p_2^{r_2}. \end{aligned}$$

Now,

$$\varphi_q(p_1^{r_1} p_2^{r_2}) = \sum_{d|p_1^{r_1} p_2^{r_2}} \mu\left(\frac{p_1^{r_1} p_2^{r_2}}{d}\right) [d]_q.$$

To avoid the complexity of the terms in the definition of $\varphi_q(p_1^{r_1} p_2^{r_2})$, we keep in account the fact that $\mu(p^r) = 0$ for $r \geq 2$. Due to this fact, many terms in the definition of $\varphi_q(p_1^{r_1} p_2^{r_2})$ vanishes. Thus, our calculation is reduced to

$$\begin{aligned} \varphi_q(m) &= \varphi_q(p_1^{r_1} p_2^{r_2}) \\ &= \mu(p_1 p_2) [p_1^{r_1-1} p_2^{r_2-1}]_q + \mu(p_1) [p_1^{r_1-1} p_2^{r_2}]_q + \mu(p_2) [p_1^{r_1} p_2^{r_2-1}]_q + \mu(1) [p_1^{r_1} p_2^{r_2}]_q \\ &= [p_1^{r_1-1} p_2^{r_2-1}]_q - [p_1^{r_1-1} p_2^{r_2}]_q - [p_1^{r_1} p_2^{r_2-1}]_q + [p_1^{r_1} p_2^{r_2}]_q \\ &= [m]_q - \sum_{1 \leq i \leq 2} \left[\frac{m}{p_i} \right]_q + \left[\frac{m}{p_1 p_2} \right]_q. \end{aligned}$$

This completes the proof. $\square$

In the similar fashion, if $m = p_1^{r_1} p_2^{r_2} p_3^{r_3}$, then

$$\varphi_q(m) = [m]_q - \sum_{1 \leq i \leq 3} \left[\frac{m}{p_i} \right]_q + \sum_{1 \leq i, j \leq 3} \left[\frac{m}{p_i p_j} \right]_q - \left[\frac{m}{p_1 p_2 p_3} \right]_q$$

for any three distinct prime factors p_1 , p_2 and p_3 . It is obvious that, this result can be further generalized to k^{th} prime factorization of m . We present the result without proof:

Theorem 6. Let $m = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, where p_i 's are distinct prime factors of m . Then,

$$\begin{aligned} \varphi_q(m) &= [m]_q - \sum_{1 \leq a \leq k} \left[\frac{m}{p_a} \right]_q + \sum_{1 \leq a, b \leq k} \left[\frac{m}{p_a p_b} \right]_q - \sum_{1 \leq a, b, c \leq k} \left[\frac{m}{p_a p_b p_c} \right]_q + \dots \\ &\quad + (-1)^k \left[\frac{m}{\prod_{r=1}^k p_r} \right]_q. \end{aligned}$$

Example 7. In particular, to verify the result presented above, we choose $m = 12 = 2^2 \times 3^1$. Then,

$$\begin{aligned} \varphi_q(12) &= \varphi_q(2^2 \times 3^1) \\ &= [12]_q - ([6]_q + [4]_q) + [2]_q \\ &= 1 + q + q^2 + \dots + q^{11} - (1 + q + q^2 + \dots + q^5) - (1 + q + q^2 + \dots + q^3) + 1 + q \\ &= -q^2 - q^3 + q^6 + q^7 + \dots + q^{11} \end{aligned}$$

which is true (cf. Table 1).

Now, we focus our attention in introducing q -analogue of Euler totient sequence spaces studied in [13]. Define the matrix $\Phi_q = (\phi_{md}^q)$ as given below:

$$\phi_{md}^q = \begin{cases} \frac{\varphi_q(d)}{[m]_q} & , \quad d \mid m, \\ 0 & , \quad d \nmid m. \end{cases}$$

Let us define a sequence $t = (t_r) \in \omega$ in terms of the sequence $s = (s_r)$ such that

$$(6) \quad t_m = (\Phi_q s)_m = \sum_{d \mid m} \frac{\varphi_q(d)}{[m]_q} s_d,$$

for any $m \in \mathbb{Z}^+$. The sequence t is known as the Φ_q -transformation of the sequence s .

Lemma 8. *The inverse Φ_q -transformation of a sequence $t = (t_d)$ is the sequence $s = (s_d)$ defined by*

$$(7) \quad s_m = \sum_{d \mid m} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} t_d.$$

Equivalently, let Ψ_q be the notation for the inverse of Φ_q -transformation such that

$$(\Psi_q t)_m = \sum_{d \mid m} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} t_d.$$

Then

$$\Phi_q \Psi_q = \Psi_q \Phi_q = I,$$

where I is the identity operator in ω .

Proof. It is clear that

$$\begin{aligned} (\Phi_q \Psi_q(t))_m &= (\Phi_q(\Psi_q t))_m \\ &= \left(\sum_{d \mid m} \frac{\varphi_q(d)}{[m]_q} (\Psi_q t)_d \right)_m \\ &= \left(\sum_{d \mid m} \frac{\varphi_q(d)}{[m]_q} \left(\sum_{k \mid d} \frac{\mu\left(\frac{d}{k}\right) [k]_q}{\varphi_q(d)} t_k \right) \right)_m \\ &= t_m = (I(t))_m, \end{aligned}$$

for all $m \in \mathbb{Z}^+$. Other equality can also be shown in a similar way. This completes the proof. $\square$

Now, let us introduce the sequence spaces $\ell_p(\Phi_q)$ and $\ell_\infty(\Phi_q)$ as follows:

$$\begin{aligned}\ell_p(\Phi_q) &= \left\{ s = (s_m) \in \omega : \sum_m \left(\sum_{d|m} \frac{\varphi_q(d)}{[m]_q} s_d \right)^p < \infty \right\}; \\ \ell_\infty(\Phi_q) &= \left\{ s = (s_m) \in \omega : \sup_m \sum_{d|m} \frac{\varphi_q(d)}{[m]_q} s_d < \infty \right\}.\end{aligned}$$

Equivalently, $\Xi(\Phi_q)$ contains those sequences whose Φ_q -transformation are in $\Xi \in \{\ell_p, \ell_\infty\}$. In the notation of domain, this can be expressed as:

$$\Xi(\Phi_q) = \Xi_{\Phi_q}$$

for $\Xi \in \{\ell_p, \ell_\infty\}$. This fact immediately leads us to the following result:

Theorem 9.

(a) For $1 < p < \infty$, the space $\ell_p(\Phi_q)$ is a BK-space with norm given by

$$\|s\|_{\ell_p(\Phi_q)} = \|\Phi_q s\|_{\ell_p} = \left(\sum_m \left| \sum_{d|m} \frac{\varphi_q(d)}{[m]_q} s_d \right|^p \right)^{1/p}.$$

(b) For $0 < p \leq 1$, the space $\ell_p(\Phi_q)$ is a complete p -normed space due to the norm

$$\|s\|_{\ell_p(\Phi_q)} = \|\Phi_q s\|_{\ell_p} = \sum_m \left| \sum_{d|m} \frac{\varphi_q(d)}{[m]_q} s_d \right|^p.$$

(c) The space $\ell_\infty(\Phi_q)$ is a BK-space due to the norm

$$\|s\|_{\ell_\infty(\Phi_q)} = \|\Phi_q s\|_{\ell_\infty} = \sup_m \left| \sum_{d|m} \frac{\varphi_q(d)}{[m]_q} s_d \right|.$$

Proof. It is known that ℓ_p and ℓ_∞ are BK-spaces (complete p -normed space in the case of ℓ_p , $0 < p \leq 1$). The result immediately follows from the fact that

$$\Xi(\Phi_q) = \Xi_{\Phi_q},$$

where $\Xi \in \{\ell_p, \ell_\infty\}$ and by using Theorem 4.3.2 of Wilansky [27]. □

Theorem 10. Let $\Xi \in \{\ell_p, \ell_\infty\}$. Then, $\Xi(\Phi_q) \cong \Xi$.

Proof. Let sequences $s = (s_m)$ and $t = (t_m)$ be related shown as in (6), and consider Ξ to be either ℓ_p or ℓ_∞ . Given that Φ_q is a triangle, it is therefore invertible. Consequently, the mapping $\mathcal{M}$, defined by

$$\begin{aligned} \mathcal{M} : \Xi(\Phi_q) &\longrightarrow \Xi \\ s &\longmapsto \mathcal{M}s = t = \Phi_q s \end{aligned}$$

is a linear bijection that preserves the norm (or p -norm). This implies that $\Xi(\Phi_q)$ and Ξ are linearly isomorphic spaces. Specifically, $\ell_p(\Phi_q) \cong \ell_p$ and $\ell_\infty(\Phi_q) \cong \ell_\infty$. $\square$

Theorem 11. *For $1 \leq p < \infty$, the space $\ell_p(\Phi_q)$ is not a Hilbert space for $p \neq 2$.*

Proof. We choose sequences $\tau = (\tau_m)$ and $v = (v_m)$ given by

$$\begin{aligned} \tau_m &= \begin{cases} \frac{\mu(m)}{\Phi_q(m)} & , \quad m \text{ is odd,} \\ \frac{1}{\Phi_q(m)} \left(\mu(m) + (1+q)\mu\left(\frac{m}{2}\right) \right) & , \quad m \text{ is even,} \end{cases} \\ v_m &= \begin{cases} \frac{\mu(m)}{\Phi_q(m)} & , \quad m \text{ is odd,} \\ \frac{1}{\Phi_q(m)} \left(\mu(m) - (1+q)\mu\left(\frac{m}{2}\right) \right) & , \quad m \text{ is even,} \end{cases} \end{aligned}$$

for all $m \in \mathbb{Z}^+$. Then, $\Phi_q \tau = (1, 1, 0, 0, \dots) \in \ell_p$ and $\Phi_q v = (1, -1, 0, 0, \dots) \in \ell_p$. Moreover

$$\|\tau + v\|_{\ell_p(\Phi_q)}^2 + \|\tau - v\|_{\ell_p(\Phi_q)}^2 = 8 \neq 2^{2+2/p} = 2 \left(\|\tau\|_{\ell_p(\Phi_q)}^2 + \|v\|_{\ell_p(\Phi_q)}^2 \right).$$

Thus, unless $p = 2$, parallelogram law is violated. This completes the proof. $\square$

Theorem 12. *The inclusion $\ell_p(\Phi_q) \subset \ell_\infty(\Phi_q)$ strictly holds.*

Proof. The inclusion part is trivial from the inequality

$$\|s\|_{\ell_\infty(\Phi_q)} \leq \|s\|_{\ell_p(\Phi_q)}$$

which holds for any sequence s in $\ell_p(\Phi_q)$. Therefore, we focus on demonstrating that this inclusion is strict.

Since the inclusion $\ell_p \subset \ell_\infty$ is strict, we take a sequence $y = (y_m) \in \ell_\infty \setminus \ell_p$.

We then define a sequence $x = (x_m)$ by the relation $x_m = \sum_{d|m} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} y_d$

for all $m \in \mathbb{Z}^+$. It follows that $\Phi_q x = y \in \ell_\infty \setminus \ell_p$. This result shows that $x \in \ell_\infty(\Phi_q) \setminus \ell_p(\Phi_q)$. $\square$

Theorem 13. *The inclusion $\ell_p(\Phi_q) \subset \ell_r(\Phi_q)$ strictly holds, where $1 \leq p < r < \infty$.*

Proof. We use an approach similar to the proof of Theorem 12 to establish the result. $\square$

We conclude this section by identifying the Schauder basis for the space $\ell_p(\Phi_q)$. In a normed space $(\Xi, \|\cdot\|)$, a sequence $s = (s_d)$ is defined as a Schauder basis of Ξ if, for every $x \in \Xi$, there exists a unique sequence of scalars (α_d) such that

$$\lim_{m \rightarrow \infty} \left\| x - \sum_{d=1}^m \alpha_d s_d \right\| = 0.$$

We apply Theorem 2.3 from Jarrah and Malkowsky [16], which asserts that the domain Ξ_T of a triangle T in a normed space Ξ possesses a basis if and only if Ξ has a basis. As a direct result of this theorem, we deduce that the inverse image of the basis of the space ℓ_p form the basis of the space $\ell_p(\Phi_q)$. This observation leads to the following Corollary:

Corollary 14. *Consider the sequence $a^{(d)} = (a_m^{(d)})$, $d \in \mathbb{Z}^+$, given by*

$$(8) \quad a_m^{(d)} = \begin{cases} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} & , \quad d \mid m, \\ 0 & , \quad d \nmid m. \end{cases}$$

Then

- (a) *the sequence $a^{(d)}$ is a Schauder basis for the space $\ell_p(\Phi_q)$, and every $s \in \ell_p(\Phi_q)$ is uniquely expressed as*

$$s = \sum_d t_d a^{(d)},$$

where $t_d = (\Phi_q s)_d$ for each $d \in \mathbb{Z}^+$.

- (b) *the space $\ell_\infty(\Phi_q)$ possesses no Schauder basis.*

Corollary 15. *The space $\ell_p(\Phi_q)$ is not separable.*

3. THE α -, β - AND γ -DUALS

Here, we determine the α -, β -, and γ -duals of the space $\ell_p(\Phi_q)$ separately for the cases where $0 < p \leq 1$, $1 < p < \infty$, and $p = \infty$. We begin by reviewing the definitions of these duals for any sequence space Ξ .

For sequence spaces $\Xi, \Pi \subset \omega$, we define the multiplier space $M(\Xi, \Pi)$ as the set

$$M(\Xi, \Pi) := \{a = (a_d) \in \omega : as = (a_d s_d) \in \Pi \text{ for all } s = (s_d) \in \Xi\}.$$

Specifically, when Π is taken to be ℓ_1 , cs , or bs , the resulting sets

$$\Xi^\alpha = M(\Xi, \ell_1), \quad \Xi^\beta = M(\Xi, cs), \quad \text{and} \quad \Xi^\gamma = M(\Xi, bs)$$

are known as the α -, β -, and γ -duals of the space Ξ , respectively.

The following lemmas will be utilized in our analysis. Throughout this discussion, we denote the collection of all finite subsets of the positive integers $\mathbb{Z}^+$ by $\mathcal{Z} = \{Z \subset \mathbb{Z}^+ : Z \text{ is finite}\}$.

Lemma 16. [9, 18, 26] *It holds that*

(i) $G = (g_{md}) \in (\ell_p, \ell_\infty)$ if and only if

$$(9) \quad \sup_m \sum_d |g_{md}|^{p'} < \infty, \quad (1 < p < \infty),$$

$$(10) \quad \sup_{m,d} |g_{md}|^p < \infty, \quad (0 < p \leq 1).$$

(ii) $G = (g_{md}) \in (\ell_p, c)$ if and only if

$$(11) \quad \exists g_d \in \mathbb{C} \ni \lim_{m \rightarrow \infty} g_{md} = g_d \text{ for each } d \in \mathbb{Z}^+,$$

and (9) holds for the case $1 < p < \infty$ and (10) holds for the case $0 < p \leq 1$.

(iii) $G = (g_{md}) \in (\ell_p, \ell_1)$ if and only if

$$(12) \quad \sup_{Z \in \mathcal{Z}} \sup_d \left| \sum_{m \in Z} g_{md} \right|^p < \infty, \quad (0 < p \leq 1),$$

$$(13) \quad \sup_{Z \in \mathcal{Z}} \sum_d \left| \sum_{m \in Z} g_{md} \right|^{p'} < \infty, \quad (1 < p < \infty).$$

(iv) $G = (g_{md}) \in (\ell_\infty, \ell_\infty)$ if and only if (9) holds with $p' = 1$.

(v) $G = (g_{md}) \in (\ell_\infty, c)$ iff (11) holds, and

$$(14) \quad \lim_{m \rightarrow \infty} \sum_d |g_{md}| = \sum_d \left| \lim_{m \rightarrow \infty} g_{md} \right|.$$

(vi) $G = (g_{md}) \in (\ell_\infty, \ell_1)$ if and only if (13) holds with $p' = 1$.

Theorem 17. Define the sets M_1 , M_2 and M_3 by

$$\begin{aligned} M_1 &:= \left\{ a = (a_d) \in \omega : \sup_{Z \in \mathbb{Z}} \sup_d \left| \sum_{\substack{m \in Z \\ d|m}} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} a_m \right|^p < \infty \right\}, \quad (0 < p \leq 1), \\ M_2 &:= \left\{ a = (a_d) \in \omega : \sup_{Z \in \mathbb{Z}} \sum_d \left| \sum_{\substack{m \in Z \\ d|m}} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} a_m \right|^{p'} < \infty \right\}, \quad (1 < p < \infty), \\ M_3 &:= \left\{ a = (a_d) \in \omega : \sup_{Z \in \mathbb{Z}} \sum_d \left| \sum_{\substack{m \in Z \\ d|m}} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} a_m \right| < \infty \right\}. \end{aligned}$$

Then,

$$(i) \quad \{\ell_p(\Phi_q)\}^\alpha = \begin{cases} M_1 & , \quad 0 < p \leq 1, \\ M_2 & , \quad 1 < p < \infty. \end{cases}$$

$$(ii) \quad \{\ell_\infty(\Phi_q)\}^\alpha = M_3.$$

Proof. Consider any sequence $a = (a_d) \in \omega$, and define the matrix $X = (x_{md})$ by

$$(15) \quad x_{md} = \begin{cases} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} a_m & , \quad d \mid m, \\ 0 & , \quad d \nmid m. \end{cases}$$

Then, for each $m \in \mathbb{Z}^+$, we arrive at the following expression:

$$a_m s_m = \sum_{d|m} \frac{\mu\left(\frac{m}{d}\right) [d]_q}{\varphi_q(m)} t_d a_m = (Xt)_m,$$

where the sequence $t = \Phi_q s$. It follows that $as = (a_d s_d) \in \ell_1$ whenever $s \in \ell_p(\Phi_q)$ if and only if $Xs \in \ell_1$ whenever $s \in \ell_p$. This implies that $a = (a_d) \in \{\ell_p(\Phi_q)\}^\alpha$ if and only if $X \in (\ell_p, \ell_1)$. Consequently, using Part (iii) of Lemma 16, we deduce that

$$\{\ell_p(\Phi_q)\}^\alpha = \begin{cases} M_1 & , \quad 0 < p \leq 1, \\ M_2 & , \quad 1 < p < \infty. \end{cases}$$

Part (ii) can be proven by utilizing a similar approach to the proof of Part (i), with the exception that Part (vi) of Lemma 16 is used in place of Part (iii) of Lemma 16. We omit the details to avoid redundancy. $\square$

Theorem 18. Define the sets M_4 , M_5 , M_6 and M_7 in the following manner:

$$\begin{aligned}
 M_4 &:= \left\{ a = (a_d) \in \omega : \lim_{m \rightarrow \infty} \sum_{\substack{e=d \\ d|e}}^m \frac{\mu\left(\frac{e}{d}\right) [d]_q}{\varphi_q(e)} a_e \text{ exists} \right\}, \\
 M_5 &:= \left\{ a = (a_d) \in \omega : \sup_{m,d} \left| \sum_{\substack{e=d \\ d|e}}^m \frac{\mu\left(\frac{e}{d}\right) [d]_q}{\varphi_q(e)} a_e \right|^p < \infty \right\}, \quad (0 < p \leq 1), \\
 M_6 &:= \left\{ a = (a_d) \in \omega : \sup_m \sum_d \left| \sum_{\substack{e=d \\ d|e}}^m \frac{\mu\left(\frac{e}{d}\right) [d]_q}{\varphi_q(e)} a_e \right|^{p'} < \infty \right\}, \quad (1 < p < \infty), \\
 M_7 &:= \left\{ a = (a_d) \in \omega : \lim_{m \rightarrow \infty} \sum_d \left| \sum_{\substack{e=d \\ d|e}}^m \frac{\mu\left(\frac{e}{d}\right) [d]_q}{\varphi_q(e)} a_e \right| = \sum_d \left| \lim_{m \rightarrow \infty} \sum_{\substack{e=d \\ d|e}}^m \frac{\mu\left(\frac{e}{d}\right) [d]_q}{\varphi_q(e)} a_e \right| \right\}.
 \end{aligned}$$

Then,

$$(i) \quad \{\ell_p(\Phi_q)\}^\beta = \begin{cases} M_4 \cap M_5 & , \quad 0 < p \leq 1, \\ M_4 \cap M_6 & , \quad 1 < p < \infty. \end{cases}$$

$$(ii) \quad \{\ell_\infty(\Phi_q)\}^\beta = M_3 \cap M_7.$$

Proof. Let $d = (d_k) \in \omega$ and define the matrix $Y = (y_{md})$ for all $m, d \in \mathbb{Z}^+$ by

$$(16) \quad y_{md} = \begin{cases} \sum_{\substack{e=d \\ d|e}}^m \frac{\mu\left(\frac{e}{d}\right) [d]_q}{\varphi_q(e)} a_e & , \quad 1 \leq d \leq m, \\ 0 & , \quad d > m. \end{cases}$$

Then, we arrive at the following equality:

$$\sum_{d=1}^m a_d s_d = \sum_{d=1}^m \sum_{e|d} \frac{\mu\left(\frac{d}{e}\right) [e]_q}{\varphi_q(d)} t_e a_d = \sum_{d=1}^m \sum_{\substack{e=d \\ d|e}}^m \frac{\mu\left(\frac{e}{d}\right) [d]_q}{\varphi_q(e)} a_e t_d = (Yt)_m.$$

We observe that $(\sum_{d=1}^m a_d s_d) \in c$ whenever $s \in \ell_p(\Phi_q)$ if and only if $Yt \in c$ whenever $t \in \ell_p$. Consequently, $a = (a_d) \in \{\ell_p(\Phi_q)\}^\beta$ if and only if $Y \in (\ell_p, c)$. Hence, from Part (ii) of Lemma 16, it follows that:

$$\{\ell_p(\Phi_q)\}^\alpha = \begin{cases} M_4 \cap M_5 & , \quad 0 < p \leq 1, \\ M_4 \cap M_6 & , \quad 1 < p < \infty. \end{cases}$$

The β -dual of $\ell_\infty(\Phi_q)$ is found in a similar manner, replacing Part (ii) of Lemma 16 with Part (v) in the above reasoning. We omit the repeated steps. $\square$

Theorem 19. *The following statements hold:*

- (i) $\{\ell_p(\Phi_q)\}^\gamma = \begin{cases} M_5 & , \quad 0 < p \leq 1, \\ M_6 & , \quad 1 < p < \infty. \end{cases}$
- (ii) $\{\ell_\infty(\Phi_q)\}^\gamma = M_6$ with $p' = 1$.

Proof. The proof uses a similar approach to that of Theorem 18, with the exception that Part (i) of Lemma 16 is used instead of Part (ii) when determining the γ -dual of $\ell_p(\Phi_q)$. To find the γ -dual of $\ell_\infty(\Phi_q)$, Part (iv) of Lemma 16 is used instead of Part (v) of Lemma 16. We omit further details to avoid redundancy. $\square$

4. MATRIX TRANSFORMATIONS

Here, we explore the characteristics of matrix classes $(\Xi(\Phi_q), \Pi)$ and $(\Xi, \Pi(\Phi_q))$ for any chosen sequence spaces Ξ and Π . Utilizing our primary result, we then characterize certain classes of infinite matrices between $\Xi(\Phi_q)$ and the spaces ℓ_1 , c_0 , c , and ℓ_∞ .

The result below forms the foundation of our analysis and follows directly from Theorem 4.1 in [17].

Theorem 20. *Let $\Xi \in \{\ell_p, \ell_\infty\}$. Then, $G = (g_{md}) \in (\Xi(\Phi_q), \Pi)$ if and only if $H^{(m)} = (h_{nd}^{(m)}) \in (\Xi, c)$ for all $m \in \mathbb{Z}^+$ and $H = (h_{md}) \in (\Xi, \Pi)$, where*

$$\begin{aligned} h_{nd}^{(m)} &= \begin{cases} \sum_{\substack{e=d \\ d|e}}^n \frac{\mu\left(\frac{e}{d}\right)[d]_q}{\varphi_q(e)} g_{me} & , \quad 1 \leq d \leq m \\ 0 & , \quad d > m, \end{cases} \\ h_{md} &= \sum_{\substack{e=d \\ d|e}}^\infty \frac{\mu\left(\frac{e}{d}\right)[d]_q}{\varphi_q(e)} g_{me} \end{aligned}$$

for all $m, n, d = 1, 2, 3, \dots$.

Proof. Let $G \in (\Xi(\Phi_q), \Pi)$ and $s \in \Xi(\Phi_q)$. By using the technique analogous to the proof of Theorem 18, we obtain the equality:

$$(17) \quad \sum_{d=1}^n g_{md} s_d = \sum_{d=1}^n \sum_{\substack{e|d}} \frac{\mu\left(\frac{d}{e}\right)[e]_q}{\varphi_q(d)} t_e g_{md} = \sum_{d=1}^n \sum_{\substack{e=d \\ d|e}}^n \frac{\mu\left(\frac{e}{d}\right)[d]_q}{\varphi_q(e)} g_{me} t_d = \sum_{d=1}^n h_{nd}^{(m)} t_d$$

for all $n, m \in \mathbb{Z}^+$. The existence of Gs implies $H^{(m)} \in (\Xi, c)$. Taking the limit as $n \rightarrow \infty$ in the equality (17) gives $Gs = Ht$. Since $Gs \in \Pi$, we have $Ht \in \Pi$, and thus $H \in (\Xi, \Pi)$.

Conversely, suppose that $H^{(m)} = (h_{nd}^{(m)}) \in (\Xi, c)$ for all $m \in \mathbb{Z}^+$, and $H = (h_{md}) \in (\Xi, \Pi)$. Let $s \in \Xi(\Phi_q)$. Then, for each $m \in \mathbb{Z}^+$, $(h_{md})_{d \in \mathbb{Z}^+} \in \Xi^\beta$ which

in turn implies the fact that $(g_{md})_{d \in \mathbb{Z}^+} \in \{\Xi(\Phi_q)\}^\beta$ for each $m \in \mathbb{Z}^+$. Again from (17), $Gs = Ht$ as $n \rightarrow \infty$. This implies that $G \in (\Xi(\Phi_q), \Pi)$. $\square$

Using Theorem 20 as our foundation, we proceed to characterize certain matrix classes (Ξ, Π) , where $\Xi \in \{\ell_1(\Phi_q), \ell_p(\Phi_q), \ell_\infty(\Phi_q)\}$ and $\Pi \in \{\ell_1, c_0, c, \ell_\infty\}$. We begin by listing the following conditions:

- (18) $\lim_{n \rightarrow \infty} h_{nd}^{(m)}$ exists for each $m, d \in \mathbb{Z}^+$;
- (19) $\sup_{n,d} |h_{nd}^{(m)}| < \infty$;
- (20) $\sup_n \sum_{d=1}^n |h_{nd}^{(m)}|^{p'} < \infty$ for each $m \in \mathbb{Z}^+$;
- (21) $\lim_{n \rightarrow \infty} \sum_{d=1}^n |h_{nd}^{(m)}| = \sum_{d=1}^n |h_{nd}^{(m)}|$ for each $m \in \mathbb{Z}^+$.
- (22) $\lim_{m \rightarrow \infty} \sum_d h_{md} = 0$.

Lemma 21. *Let $1 < p < \infty$. Then, the necessary and sufficient condition that $G = (g_{md}) \in (\Xi, \Pi)$, where $\Xi \in \{\ell_1(\Phi_q), \ell_p(\Phi_q), \ell_\infty(\Phi_q)\}$ and $\Pi \in \{\ell_1, c_0, c, \ell_\infty\}$ can be inferred from Table 2, where*

A. (18) and (19)	B. (18) and (20)
C. (18) and (21)	D. (13) with $p' = 1$ and h_{md} instead of g_{md}
E. (11) with $g_d = 0$ for all d and h_{md} instead of g_{md}	F. (11) with h_{md} instead of g_{md}
G. (9) with $p' = 1$ and h_{md} instead of g_{md}	H. (22)
I. (14) with h_{md} instead of g_{md}	J. (9) with h_{md} instead of g_{md}
K. (12) with $p = 1$ and h_{md} instead of g_{md}	L. (13) with h_{md} instead of g_{md}
M. (10) with $p = 1$ and h_{md} instead of g_{md}	

Table 2: Characterization of the matrix class (Ξ, Π) , where $\Xi \in \{\ell_1(\Phi_q), \ell_p(\Phi_q), \ell_\infty(\Phi_q)\}$ and $\Pi \in \{\ell_1, c_0, c, \ell_\infty\}$

From \ To	ℓ_1	c_0	c	ℓ_∞
$\ell_1(\Phi_q)$	A & K	A, E & M	A, F & M	A & M
$\ell_p(\Phi_q)$	B & L	B, E & J	B, F & J	B & J
$\ell_\infty(\Phi_q)$	C & D	C & H	C, F & I	C & G

Lemma 22. [5, Lemma 5.3] *Let $\Xi, \Pi \subset \omega$, G be an infinite matrix, and T be a triangle. Then, $G \in (\Xi, \Pi_T)$ iff $TG \in (\Xi, \Pi)$.*

Theorem 23. Let $\Xi \in \{\ell_1, c_0, c, \ell_\infty\}$, $Y \in \{\ell_p, \ell_\infty\}$ and $G = (g_{md})$ be an infinite matrix. Define the matrix $\Omega_q = (\omega_{md}^q)$ by

$$\omega_{md}^q = \sum_{e|m} \frac{\varphi_q(e)}{[m]_q} g_{ed}, \quad m, d = 1, 2, 3, \dots$$

Then, $G \in (\Xi, \Pi(\Phi_q))$ iff $\Omega_q \in (\Xi, \Pi)$.

Proof. This can be deduced straightly from Lemma 22. $\square$

The following conditions labeled as $\mathbf{A}'$ and $\mathbf{B}'$ are used in our next findings:

$$(23) \quad \mathbf{A}' := \sup_d \sum_m |\omega_{md}^q|^p < \infty;$$

$$(24) \quad \mathbf{B}' := \sup_{Z \in \mathcal{Z}} \sum_m \left| \sum_{d \in Z} \omega_{md}^q \right|^p < \infty;$$

Lemma 24. Let $1 < p < \infty$. Then, a matrix $G = (g_{md}) \in (\Xi, \Pi)$, where $\Xi \in \{\ell_1, c_0, c, \ell_\infty\}$ and $\Pi \in \{\ell_1(\Phi_q), \ell_p(\Phi_q), \ell_\infty(\Phi_q)\}$ if and only if the conditions listed in Table 3 are met. These conditions are derived from $\mathbf{K}$, $\mathbf{D}$, $\mathbf{G}$ and $\mathbf{M}$ in Lemma 21, by replacing each instance of ' h_{md} instead of g_{md} ' with 'with ω_{md}^q instead of g_{md} '.

Table 3: Characterization of the matrix class (Ξ, Π) , where $\Xi \in \{\ell_1, c_0, c, \ell_\infty\}$ and $\Pi \in \{\ell_1(\Phi_q), \ell_p(\Phi_q), \ell_\infty(\Phi_q)\}$

From \ To	$\ell_1(\Phi_q)$	$\ell_p(\Phi_q)$	$\ell_\infty(\Phi_q)$
ℓ_1	$\mathbf{K}$	$\mathbf{A}'$	$\mathbf{M}$
c_0	$\mathbf{D}$	$\mathbf{B}'$	$\mathbf{G}$
c	$\mathbf{D}$	$\mathbf{B}'$	$\mathbf{G}$
ℓ_∞	$\mathbf{D}$	$\mathbf{B}'$	$\mathbf{G}$

5. CERTAIN GEOMETRIC PROPERTIES OF $\ell_p(\Phi_q)$ AND $\ell_\infty(\Phi_q)$

Here, we investigate specific geometric features of the spaces $\ell_p(\Phi_q)$ ($1 \leq p < \infty$), and $\ell_\infty(\Phi_q)$ like AP (approximation) property, DP (Dunford-Pettis) property, HB (Hahn-Banach) extension property, and rotundity. The reader can refer to the papers [22] and [23] related to some geometric properties of the space generated from the Maddox's space $\ell(p)$, and references therein.

A linear operator $\mathcal{M} : \Xi \rightarrow \Pi$ between Banach spaces Ξ and Π is defined as compact if it maps any bounded subset $A \subset \Xi$ into a relatively compact subset $\mathcal{M}(A) \subset \Pi$ [20, Definition 3.4.1]. We use $\mathcal{C}(\Xi, \Pi)$ to denote the set of all such

compact operators and $\mathcal{B}(\Xi, \Pi)$ for the set of all bounded linear operators from Ξ to Π . It is a well-established fact that $\mathcal{C}(\Xi, \Pi)$ forms a closed subspace of $\mathcal{B}(\Xi, \Pi)$.

An operator between Banach spaces is considered to have finite rank if its range is a finite-dimensional vector space. Moreover, a compact linear operator between Banach spaces has a closed range iff it has a finite rank [20, Proposition 3.4.6].

To keep this presentation concise, we have omitted the definitions of AP property, weak compactness, DP property, HB extension theorem, and rotundity. For detailed accounts of these terms, we direct the reader to the papers by Megginson [20] and Philips [25]. Nevertheless, the following results are essential for our investigation:

Theorem 25. [20, Theorem 3.4.27] *For $1 \leq p < \infty$, the space ℓ_p possesses AP property.*

Lemma 26. [20, Exercise 3.50, p. 339] *Given Ξ and Π are two normed linear spaces, if $\mathcal{M} : \Xi \rightarrow \Pi$ is continuous and linear, then it maps weakly compact subsets of Ξ to weakly compact subsets of Π . That is to say that continuous linear operators preserve weak compactness.*

Theorem 27. [20, Art 1.9.6, p. 75](HB extension theorem) *A bounded linear functional $\mathfrak{L}_0$ defined on a subspace ψ of a Banach space Ξ can always be extended to a bounded linear functional $\mathfrak{L}$ on Ξ such that the norm is preserved, i.e., $\|\mathfrak{L}_0\| = \|\mathfrak{L}\|$.*

Theorem 28. [25] *A bounded linear operator $\mathfrak{L}_0$ mapping a subspace ψ of a Banach space Ξ into ℓ_∞ can always be extended to a bounded linear operator $\mathfrak{L}$ mapping Ξ into ℓ_∞ such that the operator norm is preserved, i.e., $\|\mathfrak{L}_0\| = \|\mathfrak{L}\|$.*

In this case, the space ℓ_∞ is said to possess HB extension property.

Proposition 29. [20, Proposition 5.1.9] *A normed space that is linearly isomorphic to a rotund space inherits the rotundity property*

Now, we proceed to the key findings of this section:

Theorem 30. *The AP property holds for the space $\ell_p(\Phi_q)$ whenever $1 \leq p < \infty$.*

Proof. Let $\mathcal{M} : \Xi \rightarrow \ell_p(\Phi_q)$ be a compact linear operator, where Ξ is an arbitrary Banach space and $1 \leq p < \infty$. Due to the compactness of $\mathcal{M}$, it follows that for any bounded sequence $s = (s_m)$ in Ξ , one can extract a subsequence $(\mathcal{M}s_{m_j})$ from $(\mathcal{M}s_m)$ that converges in the space $\ell_p(\Phi_q)$. In particular,

$$\|\mathcal{M}s_{m_i} - \mathcal{M}s_{m_j}\|_{\ell_p(\Phi_q)}^p = \|\mathcal{M}(s_{m_i} - s_{m_j})\|_{\ell_p(\Phi_q)}^p = \|(\Phi_q \mathcal{M})(s_{m_i} - s_{m_j})\|_{\ell_p}^p \rightarrow 0$$

as $i, j \rightarrow \infty$. This shows that the composition $\Phi_q \mathcal{M} : \Xi \rightarrow \ell_p$ is not only well defined but also compact. Now, turning attention to ℓ_p , recall that it possesses the

approximation property (A-P). Hence, there exists a sequence $(\mathcal{B}_m)$ of finite-rank, bounded linear maps from Ξ into ℓ_p such that

$$\|\Phi_q \mathcal{M} - \mathcal{B}_m\| \rightarrow 0$$

as $m \rightarrow \infty$.

Consequently, the sequence $(\Phi_q^{-1} \mathcal{B}_m)$, consisting of bounded linear operators mapping Ξ into $\ell_p(\Phi_q)$, is the required sequence of finite-rank operators. In addition,

$$\begin{aligned} \|\mathcal{M} - \Phi_q^{-1} \mathcal{B}_m\| &= \sup_{\|s\|=1} \|(\mathcal{M} - \Phi_q^{-1} \mathcal{B}_m) s\|_{\ell_p(\Phi_q)}^p \\ &= \sup_{\|s\|=1} \|\mathcal{M}s - (\Phi_q^{-1} \mathcal{B}_m) s\|_{\ell_p(\Phi_q)}^p \\ &= \sup_{\|s\|=1} \|\Phi_q \mathcal{M}s - \mathcal{B}_m s\|_{\ell_p}^p \\ &= \sup_{\|s\|=1} \|(\Phi_q \mathcal{M} - \mathcal{B}_m) s\|_{\ell_p}^p \\ &\rightarrow 0 \text{ as } m \rightarrow \infty. \end{aligned}$$

This ends the proof. $\square$

Theorem 31. *The DP property holds for the space $\ell_1(\Phi_q)$.*

Proof. Suppose that $\mathcal{M} : \ell_1(\Phi_q) \rightarrow \Xi$ is a weakly compact operator. We aim to prove that $\mathcal{M}$ is completely continuous by demonstrating that $\mathcal{M}\Phi_q^{-1}$ is a bounded linear operator from ℓ_1 to Ξ . Let B be a bounded subset of ℓ_1 . Then, $\Phi_q^{-1}B$ remains bounded in $\ell_1(\Phi_q)$. Since $\mathcal{M}$ is weakly compact, the set

$$\mathcal{M}(\Phi_q^{-1}B) = (\mathcal{M}\Phi_q^{-1})B$$

is weakly relatively compact in Ξ . Therefore, $\mathcal{M}\Phi_q^{-1}$ is a weakly compact operator mapping from ℓ_1 to Ξ . Since ℓ_1 has DP property, it follows that $\mathcal{M}\Phi_q^{-1}$ is completely continuous. Let C be a weakly compact subset of $\ell_1(\Phi_q)$. According to Lemma 26, $\Phi_q C$ is weakly compact in ℓ_1 . Since $\mathcal{M}\Phi_q^{-1}$ is completely continuous, we conclude that $\mathcal{M}\Phi_q^{-1}(\Phi_q C) = \mathcal{M}(C)$ is a compact subset of Ξ . Thus, $\mathcal{M}$ is completely continuous, as desired. $\square$

In the next finding, we establish that the space $\ell_\infty(\Phi_q)$ exhibits HB extension property.

Theorem 32. *Let Ξ be a Banach space and Λ a subspace of Ξ . For any bounded linear operator $\mathcal{M}_0 : \Lambda \rightarrow \ell_\infty(\Phi_q)$, there exists a bounded linear operator $\mathcal{M} : \Xi \rightarrow \ell_\infty(\Phi_q)$ which restricts to $\mathcal{M}_0$ on Λ and satisfies the norm equality $\|\mathcal{M}\| = \|\mathcal{M}_0\|$.*

Proof. Given a bounded linear operator $\mathcal{M}_0 \in \mathcal{B}(\Lambda, \ell_\infty(\Phi_q))$, the operator $\Phi_q \mathcal{M}_0 \in \mathcal{B}(\Lambda, \ell_\infty)$. The fact that ℓ_∞ has HB extension property, as given in Theorem 28,

allows us to extend $\Phi_q \mathcal{M}_0$ to an operator $\mathcal{L} \in \mathcal{B}(\Xi, \ell_\infty)$ while preserving its norm i.e. $\|\Phi_q \mathcal{M}_0\| = \|\mathcal{L}\|$. By setting $\mathcal{M} = \Phi_q^{-1} \mathcal{L}$, it is observed that $\mathcal{M} \in \mathcal{B}(\Xi, \ell_\infty(\Phi_q))$. Furthermore, for any $s \in \Lambda$, we have

$$\mathcal{M}s = (\Phi_q^{-1} \mathcal{L})s = \{\Phi_q\}^{-1}(\mathcal{L}s) = \Phi_q^{-1}((\Phi_q \mathcal{M}_0)s) = \mathcal{M}_0 s.$$

Additionally, we get

$$\|\mathcal{M}\| = \|\Phi_q^{-1} \mathcal{L}\| = \|\Phi_q^{-1}(\Phi_q \mathcal{M}_0)\| = \|\mathcal{M}_0\|,$$

as required. $\square$

Theorem 33. *For $1 < p < \infty$, the space $\ell_p(\Phi_q)$ is rotund.*

Proof. As ℓ_p is rotund for $1 < p < \infty$, this result follows from Proposition 29. $\square$

Theorem 34. *The spaces $\ell_1(\Phi_q)$ and $\ell_\infty(\Phi_q)$ are not rotund.*

Proof. Let us choose two sequences $s = (s_m), t = (t_m) \in \ell_1(\Phi_q)$ defined by

$$\begin{aligned} s_m &= \begin{cases} \frac{1}{\varphi_q(m)} & , \quad m = 1, \\ 0 & , \quad p^2 \mid m, \\ \frac{(-1)^r}{\varphi_q(m)} & , \quad m = p_1 p_2 \dots p_r. \end{cases} \\ t_m &= \begin{cases} 0 & , \quad m \text{ is odd}, \\ \frac{(1+q)\mu(\frac{m}{2})}{\varphi_q(m)} & , \quad m \text{ is even}. \end{cases} \end{aligned}$$

Then, $\Phi_q s = (1, 0, 0, 0, \dots)$, for,

$$\begin{aligned} (\Phi_q s)_m &= \frac{1}{[m]_q} \sum_{d \mid m} \varphi_q(d) s_d \\ &= \frac{1}{[m]_q} \left\{ \varphi_q(1) s_1 + \sum_{\substack{d \mid m \\ d > 1 \\ d \text{ is square free}}} \varphi_q(d) \frac{(-1)^{\omega(d)}}{\varphi_q(d)} \right\} \\ &= \frac{1}{[m]_q} \left\{ 1 + \sum_{\substack{d \mid m \\ d > 1 \\ d \text{ is square free}}} (-1)^{\omega(d)} \right\} \\ &= \frac{1}{[m]_q} \sum_{\substack{d \mid m \\ d \text{ is square free}}} (-1)^{\omega(d)} \end{aligned}$$

It is known that [2]

$$\sum_{\substack{d|m \\ d \text{ is square free}}} (-1)^{\omega(d)} = \begin{cases} 1 & , \quad m = 1, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Thus,

$$(\Phi_q s)_m = \begin{cases} 1 & , \quad m = 1, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In a similar way, we can show that $\Phi_q t = (0, 1, 0, 0, \dots)$. It follows that $\|s\|_{\ell_1(\Phi_q)} = 1$ and $\|t\|_{\ell_1(\Phi_q)} = 1$. That is $s, t \in S_{\ell_1(\Phi_q)}$.

Let $u = \frac{s+t}{2}$. Then, by using linearity of Φ_q , we deduce that

$$\Phi_q u = \frac{1}{2}(\Phi_q s + \Phi_q t) = \frac{1}{2}(1, 1, 0, 0, \dots).$$

Thus,

$$\|u\|_{\ell_1(\Phi_q)} = \|\Phi_q u\|_{\ell_1} = 1.$$

This shows that $\|u\|_{\ell_1(\Phi_q)} \geq 1$, implying that the space $\ell_1(\Phi_q)$ does not possess the rotund property. By analogy, it can be proven that $\ell_\infty(\Phi_q)$ also fails to be rotund. $\square$

6. CONCLUSION

In this study, we have introduced and examined the q -analogues of Euler totient sequence spaces $\ell_p(\Phi_q)$ and $\ell_\infty(\Phi_q)$, which are generalizations of the classical Euler totient spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ previously developed by İlkan and Kara [13]. The construction of these spaces is based on the q -Euler totient matrix Φ_q , defined via q -Euler totient function Φ_q , a natural q -analogue of the classical Euler totient function Φ .

The article presents some fundamental properties of Φ_q , including its non-multiplicative nature unlike its classical counterpart Φ , which plays a crucial role in establishing the algebraic and analytical foundations of the associated sequence spaces. Furthermore, we have investigated the existence of Schauder bases, as well as determined the α -, β -, and γ -duals of these spaces. The study also includes a section that devoted to characterization of matrix transformations and related results. In the final part of the work, geometric properties such as the Approximation Property (AP), the Dunford-Pettis Property (DP), rotundness, and the Hahn-Banach extension property were explored.

Overall, the present work not only extends the classical theory of sequence spaces into the field of q -calculus but also offers a perspective for further mathematical exploration in sequence spaces, summability theory, and number theory.

Let $q \in (0, 1)$. Define the sequence spaces $c(\Phi_q)$ and $c_0(\Phi_q)$ as follows:

$$\begin{aligned} c(\Phi_q) &= \{s \in \omega : \Phi_q s \in c\}, \\ c_0(\Phi_q) &= \{s \in \omega : \Phi_q s \in c_0\}. \end{aligned}$$

Equivalently, $c(\Phi_q) = c_{\Phi_q}$ and $c_0(\Phi_q) = (c_0)_{\Phi_q}$. A natural direction for future research would be to investigate these newly defined spaces in a manner analogous to the present work, with reference to the classical results given in [12].

Acknowledgements. The first author (T. Yaying) gratefully acknowledges the support received from the Science and Engineering Research Board, New Delhi, under grant number EEQ/2023/001069.

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(Received 31.08.2025.)

(Revised 27.03.2026.)

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