

THE SPECTRAL RADIUS FOR THE HYPERGRAPHS WITHOUT BERGE-GRAPHS

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For the hypergraphs among the set of the connected linear 3-uniform hypergraphs on n vertices without the Berge- C_l , we present two upper bounds for their spectral radius and α -spectral radius, where $l \geq 5$, $n \geq 3$ and $0 \leq \alpha < 1$. In addition, for the hypergraphs among the set of the connected linear k -uniform hypergraphs on n vertices without the Berge- $\{B_s, K_{2,t}\}$, we derive two upper bounds for their spectral radius and α -spectral radius, where $n, k \geq 3$, $s \geq 2$, $1 \leq t \leq \frac{1}{2}(6k^2 - 15k + 10)(s - 1) + 1$, and $0 \leq \alpha < 1$.

1. INTRODUCTION

Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real and complex numbers, respectively. A k -ordered and n -dimensional real tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_k})$ over $\mathbb{R}$ is a multi-dimensional array with n^k entries, where $a_{i_1 i_2 \dots i_k} \in \mathbb{R}$ with $i_1, i_2, \dots, i_k \in [n] = \{1, 2, \dots, n\}$. The concept of tensor eigenvalues and the spectra of tensors are independently introduced by Qi [23] and Lim [19] as follows. If there exist a number $\lambda \in \mathbb{C}$ and an eigenvector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$ satisfying

$$(1) \quad \sum_{i_2, \dots, i_k=1}^n a_{i i_2 \dots i_k} x_{i_2} \cdots x_{i_k} = \lambda x_i^{k-1}, \text{ for any } 1 \leq i \leq n,$$

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then λ is called an eigenvalue of $\mathcal{A}$ and $\mathbf{x}$ an eigenvector of $\mathcal{A}$ corresponding to λ . The spectral radius of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$, is the largest modulus of all the eigenvalues of $\mathcal{A}$.

A hypergraph H is a pair $(V(H), E(H))$, where $V(H)$ and $E(H)$ stand for the sets of the vertices and the edges of H , respectively. For an arbitrary edge $e \in E(H)$, if $|e| = k$, then H is a k -uniform hypergraph, where $k \geq 2$. For simplicity, a k -uniform hypergraph is hereinafter called a k -graph, where $k \geq 2$. Obviously, a 2-graph is a graph. A k -graph H is linear if any two edges of $E(H)$ intersect at most one vertex. For $u, v \in V(H)$, if there exists an edge $e \in E(H)$ such that $\{u, v\} \subseteq e$, then u and v are adjacent, and u is incident with e . The number of the edges of H incident with v , denoted by $d_H(v)$, is the degree of v . Without confusion, $d_H(v)$ is simplified as d_v .

Let $H = (V(H), E(H))$ be a k -graph on n vertices, where $V(H) = \{v_1, \dots, v_n\}$ and $k, n \geq 2$. The adjacency tensor of H is $\mathcal{A}(H) = (a_{i_1 i_2 \dots i_k})$, where $a_{i_1 i_2 \dots i_k} = \frac{1}{(k-1)!}$ if $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \in E(H)$ and $a_{i_1 i_2 \dots i_k} = 0$ otherwise [4]. The degree diagonal tensor of H is $\mathcal{D}(H) = (d_{i_1 i_2 \dots i_k})$, where $d_{i_1 i_2 \dots i_k} = d_{v_i}$ for any $v_i \in V(H)$ if $i_1 = i_2 = \dots = i_k = i$ with $i \in [n]$ and $d_{i_1 i_2 \dots i_k} = 0$ otherwise. Let $0 \leq \alpha < 1$. Lin et al. [18] introduced the convex linear combination of $\mathcal{D}(H)$ and $\mathcal{A}(H)$ for a hypergraph H as follows: $\mathcal{A}_\alpha(H) = \alpha \mathcal{D}(H) + (1 - \alpha) \mathcal{A}(H)$. The α -spectral radius of H , denoted by $\rho_\alpha(H)$, is the spectral radius of $\mathcal{A}_\alpha(H)$. Obviously, when $\alpha = 0$, $\mathcal{A}_\alpha(H)$ is the adjacency tensor and $\rho_0(H)$ is equal to the spectral radius of H (denoted by $\rho(H)$). When $\alpha = \frac{1}{2}$, $2\mathcal{A}_\alpha(H)$ is the signless Laplacian tensor of H and $2\rho_{\frac{1}{2}}(H)$ is the signless Laplacian spectral radius of H .

Let $\mathcal{F}$ be a given family of hypergraphs. If a hypergraph H does not contain any member of $\mathcal{F}$ as a subhypergraph, then H is $\mathcal{F}$ -free. If $\mathcal{F} = \{F\}$, then H is F -free. The Turán type extremal problem is one of the most fundamental problems in the extremal combinatorics, which is to determine the Turán number of $\mathcal{F}$, namely the maximum number of edges in $\mathcal{F}$ -free hypergraphs on n vertices. For the results about the Turán type extremal problems for graphs and hypergraphs, one can refer to surveys [8, 20] and Refs. [5, 9, 12, 13, 14, 16], respectively.

Let $G = (V(G), E(G))$ be a given graph and $H = (V(H), E(H))$ be a hypergraph. H is called a Berge- G if there exists a bijection $\varphi : E(G) \rightarrow E(H)$ such that $e \subseteq \varphi(e)$ for any edge $e \in E(G)$ [12]. Namely, H is obtained from G by replacing each edge of G with a hyperedge that contains it. For a given graph G , a hypergraph H is Berge- G -free if H does not contain a subhypergraph isomorphic to a Berge- G . Recently, the study on the Turán numbers of hypergraphs without a Berge- G attracted the attention of researchers [11, 26].

The spectral version of the Turán type problem for graphs is to determine the maximum spectral radius among all the G -free graphs on n vertices, which was first proposed by Nikiforov [21]. For details, one can refer to Refs. [1, 3, 25, 27, 30, 31].

For the spectral analogue of the Turán type problem for hypergraphs, Hou et al. [15] proposed the following question: “What is the maximum spectral radius of hypergraphs of order n , not containing a given hypergraph F ?” This problem attracts researchers’ interest and some results are obtained. Keevash et al. [17]

determined the maximum p -spectral radius of any 3-graphs on n vertices not containing the Fano plane when n is sufficiently large. Hou et al. [15] derived the upper bounds of the maximum spectral radii for the connected linear k -graphs on n vertices without the Berge- C_4 or with the girth of at least five. For a given graph G , the r -expansion G^+ of G is the r -uniform hypergraph obtained from G by enlarging each edge of G with $r - 2$ new vertices which are disjoint from the vertex set of G such that distinct edges are enlarged by disjoint subsets. Gao et al. [10] obtained the maximum spectral radius of the connected linear r -graphs on n vertices without K_{r+1}^+ , where K_{r+1} is a complete graph on $r + 1$ vertices. She et al. [24] presented the sharp (or asymptotic) bounds of the maximum number of edges and the maximum spectral radius of all F^+ -free linear r -uniform hypergraphs on n vertices, where F is a graph with chromatic number k .

Let P_k be a path of length k and we write P_k as $v_1 v_2 \cdots v_{k+1}$, where $k \geq 1$. A cycle of length k , denoted by C_k , is obtained from P_k by identifying v_1 with v_{k+1} together. Let sC_3 be the union of C_3 and the number of C_3 is s , where $s \geq 2$. An s -book, denoted by B_s , is a connected graph obtained from sC_3 in such a way that all C_3 share a common edge. Let $K_{s,t}$ be a complete bipartite graph with two parts of size s and t , respectively, where $s, t \geq 1$.

The research on the spectral analogue of the Turán-type problem for hypergraphs is an interesting field. Let $\mathcal{H}_{n,3,l}$ be the set of the connected linear 3-graphs on n vertices without the Berge- C_l , where $n \geq 3$ and $l \geq 5$. Let $\mathcal{H}_{n,k,s,t}$ be the set of the connected linear k -graphs on n vertices without the Berge- $\{B_s, K_{2,t}\}$, where $n, k \geq 3$, $s \geq 2$ and $t \geq 1$. In this article, for the hypergraphs among $\mathcal{H}_{n,3,l}$ and $\mathcal{H}_{n,k,s,t}$, we aim to characterize the upper bounds of their maximum spectral radius and α -spectral radius.

The rest of this article is organized as follows. In Section 2, some known necessary lemmas are quoted and some new tools for us to study the α -spectral radius of hypergraphs are developed. In Section 3, for $H \in \mathcal{H}_{n,3,l}$ with $n \geq 3$ and $l \geq 5$, we obtain the upper bounds for $\rho(H)$ and $\rho_\alpha(H)$, which depend on n , l and α . In Section 4, for $H \in \mathcal{H}_{n,k,s,t}$ with $n, k \geq 3$, $s \geq 2$ and $1 \leq t \leq \frac{1}{2}(6k^2 - 15k + 10)(s - 1) + 1$, the upper bounds for $\rho(H)$ and $\rho_\alpha(H)$ are also derived, which are related with n , k , s , and α .

2. PRELIMINARIES

In this section, we will introduce several necessary lemmas for subsequent proofs.

In 2008, Chang et al. [2] gave the definition of the irreducible tensor.

Definition 1. [2] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_k})$ be a nonnegative tensor of order k and dimension n . For any nonempty proper index subset $I \subset [n]$, if there is at least one entry $a_{i_1 i_2 \dots i_k} > 0$, where $i_1 \in I$ and $i_j \in [n] \setminus I$ for $j = 2, 3, \dots, k$, then $\mathcal{A}$ is called a nonnegative irreducible tensor.

The nonnegative weakly irreducible tensor was defined by Friedland et al. [7] and was represented by Yang et al. [28] as follows.

Definition 2. [7, 28] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_k})$ be a nonnegative tensor of order k and dimension n . For any nonempty proper index subset $I \subset [n]$, if there is at least one entry $a_{i_1 i_2 \dots i_k} > 0$, where $i_1 \in I$ and at least one $i_j \in [n] \setminus I$ for $j = 2, 3, \dots, k$, then $\mathcal{A}$ is called a nonnegative weakly irreducible tensor.

Let $\mathbb{R}_+^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \geq 0, \forall i \in [n]\}$ and $\mathbb{R}_{++}^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_i > 0, \forall i \in [n]\}$.

Lemma 1. [2, 7, 29] (The Perron–Frobenius theorem for nonnegative tensors) Let $\mathcal{A}$ be a nonnegative tensor of order k and dimension n , where $k, n \geq 2$. Then we have the following statements.

(i). $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector $\mathbf{x} \in \mathbb{R}_+^n$ corresponding to it.

(ii). If $\mathcal{A}$ is irreducible, then $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ with the nonnegative eigenvector $\mathbf{x} \in \mathbb{R}_+^n$, up to a positive scaling coefficient.

(iii). If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ with the positive eigenvector $\mathbf{x} \in \mathbb{R}_{++}^n$, up to a positive scaling coefficient.

Lemma 2. [22] A k -graph H is connected if and only if $\mathcal{A}(H)$ is weakly irreducible.

If $\mathcal{A}(H)$ is weakly irreducible, then according to the definition of $\mathcal{A}_\alpha(H)$, $\mathcal{A}_\alpha(H)$ is also weakly irreducible. If H is a connected k -graph, then by Lemmas 1 and 2, there exists the unique vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ corresponding to $\rho_\alpha(H)$ and each $x_i > 0$ with $1 \leq i \leq n$. This vector $\mathbf{x}$ is referred to as the α -Perron vector of H . It is convenient for us to normalize $\mathbf{x}$ such that the maximum entry of $\mathbf{x}$ is 1. If there are many such vertices, then among them we can arbitrarily choose one and fix it. For any $v \in V(H)$, from (1), we have

$$(2) \quad \rho_\alpha(H)x_v^{k-1} = \alpha d_v x_v^{k-1} + (1 - \alpha) \sum_{\{v, v_{i_2}, \dots, v_{i_k}\} \in E(H)} x_{v_{i_2}} x_{v_{i_3}} \cdots x_{v_{i_k}}.$$

Lemma 3. [6] (Erdős–Gallai Theorem) Let G be a graph on n vertices and $e(G)$ be the number of edges of G . If G is P_k -free, then $e(G) \leq \frac{k-1}{2}n$ with the equality if and only if G is composed of multiple K_k which are disjoint from each other.

Let H be a k -graph and let $v \in V(H)$, where $k \geq 2$. For a subset $X \subseteq V(H)$, let $E_t(X) = \{e \mid e \in E(H) \text{ and } |e \cap X| = t\}$, where $t \geq 1$. Namely, each edge in $E_t(X)$ intersects X in t vertices. We define $|E_t(X)| = e_t(X)$. Furthermore, let $E_t^v(X) = \{e \mid e \in E_t(X) \text{ and } v \in e\}$ and $|E_t^v(X)| = e_t^v(X)$. Namely, $E_t^v(X)$ is a subset of $E_t(X)$ satisfying that each edge in $E_t^v(X)$ contains v . We define $N_v = \{w \mid w \in V(H), \{v, w\} \text{ is contained in an edge of } H\}$ and call N_v the neighbourhood of v . In H , let $V(H) = \{v_1, v_2, \dots, v_n\}$. A Berge walk of length h in H is an alternating sequence of vertices and edges $v_1 e_1 v_2 e_2 v_3 \cdots v_h e_h v_{h+1}$, where $v_i \neq v_{i+1}$ and $\{v_i, v_{i+1}\} \subseteq e_i$ for $1 \leq i \leq h$. We use $w_h(v)$ to denote the number of the Berge walks of length h in H which start at a vertex $v \in V(H)$. By

the definition of $w_h(v)$, we have

$$(3) \quad w_1(v) = (k-1)d_v,$$

$$(4) \quad w_{h+1}(v) = \sum_{\{v, v_{i_2}, \dots, v_{i_k}\} \in E(H)} \left[w_h(v_{i_2}) + w_h(v_{i_3}) + \dots + w_h(v_{i_k}) \right].$$

Lemma 4. [10] *Let H be a connected linear k -graph, where $k \geq 3$. For any $v \in V(H)$, we have*

$$w_2(v) = (k-1) \sum_{i=1}^k i e_i(N_v).$$

To obtain the upper bound of the α -spectral radius for the hypergraphs, we introduce some new tools, which are shown in Theorem 1, Corollaries 1 and 2.

Theorem 1. *Let $H = (V(H), E(H))$ be a connected k -graph on n vertices and $\mathbf{x} = (x_1, \dots, x_n)^T$ be its α -Perron vector with the maximum entry in $\mathbf{x}$ being 1, where $k, n \geq 3$. For any $v \in V(H)$, we have*

$$\rho_\alpha^2(H) x_v^{k-1} \leq \frac{\alpha}{(k-1)^2} w_1^2(v) + \frac{1-\alpha}{(k-1)^2} w_2(v).$$

Proof. Let v be an arbitrary vertex in $V(H)$. Since $x_i \leq 1$ for any $1 \leq i \leq n$, by (2), we get

$$\begin{aligned} \rho_\alpha(H) x_v^{k-1} &= \alpha d_v x_v^{k-1} + (1-\alpha) \sum_{\{v, i_2, \dots, i_k\} \in E(H)} x_{i_2} \cdots x_{i_k} \\ &\leq \alpha d_v + (1-\alpha) \sum_{\{v, i_2, \dots, i_k\} \in E(H)} 1 \\ (5) \quad &= d_v = \frac{1}{k-1} w_1(v), \end{aligned}$$

where (5) follows from (3). We obtain

$$\begin{aligned} \rho_\alpha^2(H) x_v^{k-1} &= \rho_\alpha(H) \cdot \rho_\alpha(H) x_v^{k-1} \\ &= \rho_\alpha(H) \left[\alpha d_v x_v^{k-1} + (1-\alpha) \sum_{\{v, i_2, \dots, i_k\} \in E(H)} x_{i_2} \cdots x_{i_k} \right] \\ (6) \quad &\leq \alpha d_v \rho_\alpha(H) x_v^{k-1} + \frac{1-\alpha}{k-1} \sum_{\{v, i_2, \dots, i_k\} \in E(H)} \rho_\alpha(H) (x_{i_2}^{k-1} + \dots + x_{i_k}^{k-1}) \end{aligned}$$

$$(7) \quad \leq \alpha d_v^2 + \frac{1-\alpha}{(k-1)^2} \sum_{\{v, i_2, \dots, i_k\} \in E(H)} (w_1(i_2) + \dots + w_1(i_k))$$

$$(8) \quad = \frac{\alpha}{(k-1)^2} w_1^2(v) + \frac{1-\alpha}{(k-1)^2} w_2(v),$$

where (6) follows from the Arithmetic Mean–Geometry Mean inequality, (7) is derived from (5), and (8) is deduced from (3) and (4). $\square$

By Theorem 1, we directly get Corollary 1.

Corollary 1. *Let $H = (V(H), E(H))$ be a connected k -graph on n vertices and $\mathbf{x} = (x_1, \dots, x_n)^T$ be its α -Perron vector, where $k, n \geq 3$. If $u \in V(H)$ is the vertex having the maximum entry in $\mathbf{x}$, then*

$$\rho_\alpha^2(H) \leq \frac{\alpha}{(k-1)^2} w_1^2(u) + \frac{1-\alpha}{(k-1)^2} w_2(u).$$

Corollary 2. *Let $H = (V(H), E(H))$ be a connected linear k -graph on n vertices and $\mathbf{x} = (x_1, \dots, x_n)^T$ be its α -Perron vector, where $k, n \geq 3$. If $u \in V(H)$ is the vertex having the maximum entry in $\mathbf{x}$, then*

$$(9) \quad \rho_\alpha^2(H) \leq \alpha d_u^2 + \frac{1-\alpha}{k-1} \sum_{i=1}^k i e_i(N_u) = \alpha d_u^2 + \frac{1-\alpha}{k-1} \sum_{v \in N_u} d_v.$$

Proof. By combining Corollary 1, Lemma 4, (3), and $\sum_{i=1}^k i e_i(N_u) = \sum_{v \in N_u} d_v$, we obtain Corollary 2. $\square$

It is noted that (9) with $\alpha = 0$ was obtained by Hou et al. [15] and Gao et al. [10]. It can be seen from Corollary 2 that we can use a unified method to study the upper bounds of the maximum spectral radius and α -spectral radius for a connected linear k -graph on n vertices.

3. THE UPPER BOUND OF THE MAXIMUM SPECTRAL RADIUS OF THE CONNECTED LINEAR 3-GRAPHS WITHOUT THE $\text{BERGE-}C_l$

In this section, we will characterize the upper bounds of the maximum spectral radius and α -spectral radius for the hypergraphs among the set of the connected linear 3-graphs on n vertices without the $\text{Berge-}C_l$, where $n \geq 3$ and $l \geq 5$.

If H is a linear hypergraph, for any two adjacent vertices x and y of H , there exists the unique edge containing x and y . We denote this edge by l_{xy} .

Theorem 2. *Let $H \in \mathcal{H}_{n,3,l}$, where $n \geq 3$ and $l \geq 5$. Let $\mathbf{x} = (x_1, \dots, x_n)^T$ be the α -Perron vector of H . If $u \in V(H)$ is the vertex having the maximum entry in $\mathbf{x}$, then*

$$\rho_\alpha^2(H) \leq \alpha d_u^2 + (1-\alpha) \left[\frac{(2l-7)(n-1)}{4} + (2l-3)d_u \right].$$

Proof. To obtain Theorem 2, we firstly prove Claims 3.1 and 3.2.

Let $B_u = V(H) \setminus (N_u \cup \{u\})$. Since $|N_u| = 2d_u$, we get $|B_u| = n - 2d_u - 1$.

Claim 3.1. $e_1(N_u) \leq \frac{2l-7}{2}(n-2d_u-1)$.

Proof. We will prove Claim 3.1 by constructing an auxiliary graph. Let $G_1 = (V(G_1), E(G_1))$, where $V(G_1) = B_u$ and $E(G_1) = \{v_1v_2 \mid v_1, v_2 \in B_u, l_{v_1v_2} \in E_1(N_u)\}$. By the definition of G_1 , we have $|V(G_1)| = |B_u| = n-2d_u-1$ and $|E(G_1)| = e_1(N_u)$.

We prove $|E(G_1)| \leq \frac{2l-7}{2}(n-2d_u-1)$ by contradiction. Otherwise, we suppose $|E(G_1)| > \frac{2l-7}{2}(n-2d_u-1) = \frac{2l-7}{2}|V(G_1)|$. Obviously, the average degree of G_1 is greater than $2l-7$. When $l \geq 5$, we have $2l-7 \geq 3$. In G_1 , if we delete the vertex having degree less than 3, then the average degree of the resulting graph does not decrease. Therefore, in G_1 , if we delete all the vertices having degrees less than 3, then we get a resulting graph (denoted by G_2) such that the average degree of G_2 is greater than $2l-7$ and the minimum degree of G_2 is greater than or equal to 3. For G_2 , since its average degree is greater than $2l-7$, we get $|E(G_2)| > \frac{2l-7}{2}|V(G_2)|$. By Lemma 3, we obtain that G_2 contains a path P_{2l-6} as a subgraph, where $l \geq 5$. Let $P_{2l-6} = v_1e_1v_2 \cdots v_{2l-6}e_{2l-6}v_{2l-5}$.

Since the minimum degree of G_2 is greater than or equal to 3, there exists a vertex (denoted by w) in B_u such that $wv_{l-2} \in E(G_2)$, where $w \neq v_{l-3}, v_{l-1}$. According to the definition of G_1 , in H , there exist five vertices $w_1, \dots, w_5 \in N_u$ and five edges $f_1, \dots, f_5$, where $f_1 = \{v_1, v_2, w_1\}$, $f_2 = \{v_{l-3}, v_{l-2}, w_2\}$, $f_3 = \{v_{l-2}, v_{l-1}, w_3\}$, $f_4 = \{v_{2l-6}, v_{2l-5}, w_4\}$, and $f_5 = \{v_{l-2}, w, w_5\}$. It is noted that $w_1, w_2, \dots, w_5$ may be the same. Thus, we obtain $l_{uw_1} = l_{uw_3}$. Otherwise, if $l_{uw_1} \neq l_{uw_3}$, then there exists a Berge- C_l in H which is formed by $l_{uw_1}, l_{w_1v_2}, l_{v_2v_3}, \dots, l_{v_{l-2}w_3}, l_{uw_3}$. This contradicts the fact that H is Berge- C_l -free.

By the same analysis as those for $l_{uw_1} = l_{uw_3}$, we also get $l_{uw_1} = l_{uw_5}$. Therefore, we have $l_{uw_1} = l_{uw_3} = l_{uw_5}$. Furthermore, we claim $l_{uw_2} = l_{uw_4}$. Otherwise, if $l_{uw_2} \neq l_{uw_4}$, then there exists a Berge- C_l in H which is formed by $l_{uw_2}, l_{w_2v_{l-2}}, l_{v_{l-2}v_{l-1}}, \dots, l_{v_{2l-6}w_4}, l_{uw_4}$. This contradicts the fact that H is Berge- C_l -free. Similarly, we get $l_{uw_4} = l_{uw_5}$. Therefore, we obtain $l_{uw_2} = l_{uw_4} = l_{uw_5}$. Thus, $l_{uw_2} = l_{uw_3} = l_{uw_5}$. This means that an edge of H contains u, w_2, w_3 , and w_5 . Since $v_{l-2} \in f_2 \cap f_3 \cap f_5$, $w_2 \in f_2$, $w_3 \in f_3$, $w_5 \in f_5$, and H is a linear hypergraph, we deduce that w_2, w_3 , and w_5 are different from each other. Therefore, an edge of H has at least four distinct vertices. This contradicts the fact that H is a 3-graph. Thus, we get $e_1(N_u) = |E(G_1)| \leq \frac{2l-7}{2}(n-2d_u-1)$. Namely, Claim 3.1 holds. $\square$

Claim 3.2. $2e_2(N_u) + 3e_3(N_u) \leq (6l-13)d_u$.

Proof. Since H is a 3-graph, if an edge of H is incident with u , then this edge and N_u intersect at two vertices. In H , the number of such edges which, containing u , intersect with N_u at two vertices is d_u . Namely, $|E_2^u(N_u)| = d_u$.

Next, we will prove the following Fact 1 by constructing an auxiliary graph.

In H , let E' be the set of such edges which, not containing u , intersect with N_u at two vertices or three vertices.

Fact 1: $|E'| \leq (2l-5)d_u$.

Let $G_3 = (V(G_3), E(G_3))$, where $V(G_3) = N_u$ and the edges of $E(G_3)$ are

formed by the edges of E' according to the following one-to-one mapping relationship. Let e be an arbitrary edge of E' . If $|e \cap N_u| = 2$, namely, there exist two vertices $x, y \in N_u$ such that $e \cap N_u = \{x, y\}$, then let $xy \in E(G_3)$. If $|e \cap N_u| = 3$, namely, there exist three vertices $x, y, z \in N_u$ such that $e \cap N_u = \{x, y, z\}$, then we take any two vertices from $\{x, y, z\}$ to form an edge of G_3 . Obviously, according to the mapping relationship, each edge in E' corresponds to the unique edge of $E(G_3)$. Therefore, we get $|E(G_3)| = |E'|$.

Next, we prove that G_3 does not contain P_{2l-4} . Otherwise, if G_3 contains $P_{2l-4} = v_1e_1v_2 \cdots v_{2l-4}e_{2l-4}v_{2l-3}$, then by the construction of G_3 , for any $i = 1, 2, \dots, 2l-3$, we get $v_i \in N_u$. Since $v_{l-1} \in N_u$, in H , there exists an edge $f_1 \in E(H)$ such that $\{u, v_{l-1}\} \subseteq f_1$. As H is a 3-graph, $v_1 \in f_1$ and $v_{2l-3} \in f_1$ can not hold simultaneously. Without loss of generality, let $v_1 \notin f_1$. Since $v_1 \in N_u$, in H , there exists an edge f_2 such that $\{u, v_1\} \subseteq f_2$ and $f_2 \neq f_1$. Therefore, in H , $f_1, f_2, l_{v_1v_2}, \dots, l_{v_{l-2}v_{l-1}}$ form a Berge- C_l . This contradicts the fact that H is Berge- C_l -free. Therefore, G_3 is P_{2l-4} -free. By Lemma 3, we get $|E(G_3)| \leq \frac{2l-4-1}{2}|V(G_3)| = \frac{2l-5}{2}|N_u|$. Since H is a connected linear 3-graph, $|N_u| = 2d_u$. Thus, $|E'| = |E(G_3)| \leq (2l-5)d_u$. Namely, Fact 1 holds.

Bearing $|E_2^u(N_u)| = d_u$ and Fact 1 in mind, we obtain

$$\begin{aligned}
 2e_2(N_u) + 3e_3(N_u) &= 2|E_2(N_u)| + 3|E_3(N_u)| \\
 &= 2|E_2^u(N_u)| + 2|E_2(N_u) \setminus E_2^u(N_u)| + 3|E_3(N_u)| \\
 &\leq 2|E_2^u(N_u)| + 3(|E_2(N_u) \setminus E_2^u(N_u)| + |E_3(N_u)|) \\
 (10) \quad &= 2|E_2^u(N_u)| + 3|E'| \leq (6l-13)d_u.
 \end{aligned}$$

□

Since H is a connected linear 3-graph, by Corollary 2, Claims 3.1 and 3.2, and (10), we have

$$\begin{aligned}
 \rho_\alpha^2(H) &\leq \alpha d_u^2 + \frac{1-\alpha}{2} \sum_{i=1}^3 i e_i(N_u) \\
 &\leq \alpha d_u^2 + \frac{1-\alpha}{2} \left[\frac{2l-7}{2}(n-2d_u-1) + (6l-13)d_u \right] \\
 (11) \quad &= \alpha d_u^2 + (1-\alpha) \left[\frac{(2l-7)(n-1)}{4} + (2l-3)d_u \right].
 \end{aligned}$$

Thus, we get Theorem 2. □

We can see from (11) that the upper bound of $\rho_\alpha(H)$ has a close relationship with d_u when n, l and α are fixed. Therefore, by Theorem 2, we directly obtain Theorem 3.

Theorem 3. *Let H be a connected linear 3-graph on $n \geq 3$ vertices with the maximum degree Δ . If H is Berge- C_l -free, where $l \geq 5$, then*

$$\rho_\alpha^2(H) \leq \alpha \Delta^2 + (1-\alpha) \left[\frac{(2l-7)(n-1)}{4} + (2l-3)\Delta \right].$$

If H is a connected linear 3-graph on $n \geq 3$ vertices, then its maximum degree Δ is less than or equal to $\frac{n-1}{2}$. Thus, by Theorem 3, for $H \in \mathcal{H}_{n,3,l}$ with $n \geq 3$ and $l \geq 5$, we obtain the upper bounds of $\rho(H)$ and $\rho_\alpha(H)$, which are shown in Theorems 4 and 5, respectively.

Theorem 4. *Let H be a connected linear 3-graph on n vertices with $n \geq 3$. If H is Berge- C_l -free, where $l \geq 5$, then*

$$\rho(H) \leq \sqrt{\frac{(6l-13)(n-1)}{4}}.$$

Theorem 5. *Let H be a connected linear 3-graph on n vertices with $n \geq 3$. If H is Berge- C_l -free, where $l \geq 5$, then*

$$\rho_\alpha^2(H) \leq \alpha \frac{(n-1)^2}{4} + (1-\alpha) \frac{(6l-13)(n-1)}{4}.$$

4. THE UPPER BOUND OF THE MAXIMUM SPECTRAL RADIUS FOR THE k -UNIFORM LINEAR HYPERGRAPHS WITHOUT THE $\text{BERGE-}\{B_s, K_{2,t}\}$

In this section, we will derive the upper bounds of the maximum spectral radius and α -spectral radius for the hypergraphs among the set of the connected linear k -graphs on n vertices without the Berge- $\{B_s, K_{2,t}\}$, where $n, k \geq 3$, $s \geq 2$ and $t \geq 1$.

Theorem 6. *Let $H \in \mathcal{H}_{n,k,s,t}$, where $k, n \geq 3$, $s \geq 2$, and $t \geq 1$. Let $\mathbf{x} = (x_1, \dots, x_n)^T$ be the α -Perron vector of H . If $u \in V(H)$ is the vertex having the maximum entry in $\mathbf{x}$, then*

$$(12) \quad \rho_\alpha^2(H) \leq \alpha d_u^2 + (1-\alpha) \left[\frac{(t-1)(n-1)}{(k-1)^2} + \frac{(6k^2 - 15k + 10)(s-1) - 2(t-1)}{2(k-1)} d_u \right].$$

Proof. To obtain Theorem 6, we firstly prove Claims 4.1–4.4.

For two vertices x and y in H , let N_{xy} be the set of the common neighbourhood of x and y . Namely, each vertex in N_{xy} is adjacent to x and y simultaneously. Furthermore, we define $N'_{xy} = N_{xy} \setminus l_{xy}$.

Claim 4.1. For any two adjacent vertices $x, y \in V(H)$, we have $|N_{xy}| \leq (k-1)(2s-1) - s$.

Proof. We prove Claim 4.1 by contradiction. Suppose that there exist two adjacent vertices $x, y \in V(H)$ such that $|N_{xy}| \geq (k-1)(2s-1) - s + 1 = (2k-3)(s-1) + k - 1$. Then $|N'_{xy}| \geq (2k-3)(s-1) + 1$. Let $V_0 = N'_{xy}$ and $E_0 = \{l_{xy}\}$. Next, we will prove that there exists a Berge- B_s in H according to the following recursive method.

Step (1): Since $V_0 = N'_{xy} \neq \emptyset$, we can choose one vertex from V_0 and denote this vertex by w_1 . Let $E_1 = E_0 \cup \{l_{xw_1}, l_{yw_1}\}$ and $V_1 = V_0 \setminus (l_{xw_1} \cup l_{yw_1})$. Since

$x, y \notin V_0$ and $|(l_{xw_1} \cup l_{yw_1}) \cap V_0| \leq 2k - 3$, we get $|V_1| \geq |V_0| - (2k - 3) = |N'_{xy}| - (2k - 3) \geq (2k - 3)(s - 2) + 1$. When $s \geq 2$ and $k \geq 3$, we have $|V_1| \geq 1$.

Step (2): Since $|V_1| \geq 1$, we can choose one vertex from V_1 and let this vertex be w_2 . Let $E_2 = E_1 \cup \{l_{xw_2}, l_{yw_2}\}$ and $V_2 = V_1 \setminus \{l_{xw_2}, l_{yw_2}\}$. Similarly, we have $|V_2| \geq |V_1| - (2k - 3) \geq (2k - 3)(s - 3) + 1$.

Step (r): Let $3 \leq r \leq s - 1$. Since $|(l_{xw_{r-1}} \cup l_{yw_{r-1}}) \cap V_{r-2}| \leq 2k - 3$, we obtain $|V_{r-1}| = |V_{r-2} \setminus \{l_{xw_{r-1}}, l_{yw_{r-1}}\}| \geq |V_{r-2}| - (2k - 3) \geq (2k - 3)(s - r + 1) + 1 - (2k - 3) = (2k - 3)(s - r) + 1 \geq 1$. Thus, from V_{r-1} , we can choose one vertex and denote it by w_r . Let $E_r = E_{r-1} \cup \{l_{xw_r}, l_{yw_r}\}$ and $V_r = V_{r-1} \setminus \{l_{xw_r}, l_{yw_r}\}$. Similarly, we get $|V_r| \geq |V_{r-1}| - (2k - 3) \geq (2k - 3)(s - r - 1) + 1$.

When $r = s - 1$, we have $|V_{s-1}| \geq 1$. Thus, from V_{s-1} , we can choose a vertex and denote it by w_s . Since w_s is adjacent to x and y , two edges (denoted by l_{xw_s} and l_{yw_s}) exist in H . Let $E_s = E_{s-1} \cup \{l_{xw_s}, l_{yw_s}\}$. Therefore, in H , E_s contains $2s + 1$ edges and these $2s + 1$ edges form a Berge- B_s in H . This contradicts the fact that H is Berge- B_s -free. Thus, for any two adjacent vertices $x, y \in V(H)$, we obtain $|N_{xy}| \leq (k - 1)(2s - 1) - s$. $\square$

Claim 4.2. For any vertex $y \in V(H)$ and any vertex $x \in N_y$, we get $\sum_{i=2}^k e_i^x(N_y) \leq 2(k - 1)(s - 1)$, where $k \geq 3$ and $s \geq 2$.

Proof. For any vertex $y \in V(H)$ and any vertex $x \in N_y$, we will firstly prove

$\sum_{i=2}^k e_i^x(N_y) \leq (2k - 3)(s - 1) + 1$ by contradiction. Otherwise, we suppose that there

exists a vertex $x \in N_y$ such that $\sum_{i=2}^k e_i^x(N_y) \geq (2k - 3)(s - 1) + 2$. This means

$|\bigcup_{i=2}^k E_i^x(N_y)| \geq (2k - 3)(s - 1) + 2$. Since $x \in N_y$, we get $l_{xy} \in E(H)$. Obviously, $l_{xy} \in E_{k-1}^x(N_y)$. Thus, we get $|\bigcup_{i=2}^k E_i^x(N_y) \setminus l_{xy}| \geq (2k - 3)(s - 1) + 1$. Namely, except for l_{xy} , H has at least $(2k - 3)(s - 1) + 1$ edges, each of which contains x and at least two vertices in N_y .

Let $E^* = \{f_i \mid f_i \in E(H), |f_i \cap N_y| \geq 2, x \in f_i, f_i \neq l_{xy}, 1 \leq i \leq (2k - 3)(s - 1) + 1\}$. For any edge $f_i \in E^*$, since $|f_i \cap N_y| \geq 2$ and $x \in f_i$, among f_i , we can choose a vertex (denoted by z_i) which belongs to N_{xy} , where $z_i \neq x, y$ and $1 \leq i \leq (2k - 3)(s - 1) + 1$. Since H is a linear hypergraph, $z_1, z_2, \dots, z_{(2k-3)(s-1)+1}$ are different from each other. Since $z_i \in N_{xy}$ and $l_{xy} \setminus \{x, y\} \subseteq N_{xy}$, we get $|N_{xy}| \geq (2k - 3)(s - 1) + 1 + k - 2 = (k - 1)(2s - 1) - s + 1$. This contradicts Claim 4.1.

Therefore, for any vertex $y \in V(H)$ and any vertex $x \in N_y$, we have $\sum_{i=2}^k e_i^x(N_y) \leq (2k - 3)(s - 1) + 1 \leq 2(k - 1)(s - 1)$, where $s \geq 2$. $\square$

Let $B_u = V(H) \setminus (N_u \cup \{u\})$. For $v \in N_u$, let $S_v = \{w \mid w \in B_u, \exists l \in E_1^v(N_u), w \in l\}$. By the definition of S_v , we get $S_v \subseteq B_u$. We denote, by $l_i = \{u, i_2, \dots, i_k\}$, the i -th edge which is incident with u . Furthermore, let $V_i = \bigcup_{j=2}^k S_{i_j}$, where $i = 1, 2, \dots, d_u$.

Claim 4.3. $|V_i| \geq \sum_{j=2}^k |S_{i_j}| - \binom{k-1}{2}(2k-3)(s-1)$, where $1 \leq i \leq d_u$.

Proof. For any i with $1 \leq i \leq d_u$ and $2 \leq p < q \leq k$, it follows from Claim 4.1 that $|S_{i_p} \cap S_{i_q}| \leq |N_{i_p i_q} \setminus (N_u \cup \{u\})| \leq |N_{i_p i_q} \setminus (l_i \setminus \{i_p, i_q\})| \leq (k-1)(2s-1) - s - (k-2) = (2k-3)(s-1)$. Thus, we get

$$\begin{aligned} |V_i| &= \left| \bigcup_{j=2}^k S_{i_j} \right| \geq \sum_{j=2}^k |S_{i_j}| - \sum_{2 \leq p < q \leq k} |S_{i_p} \cap S_{i_q}| \\ &\geq \sum_{j=2}^k |S_{i_j}| - \binom{k-1}{2}(2k-3)(s-1). \end{aligned}$$

□

Claim 4.4. We have $\sum_{i=1}^{d_u} |V_i| \leq (t-1)[n - (k-1)d_u - 1]$, where $t \geq 1$.

Proof. For any $v \in N_u$, since $S_v \subseteq B_u$, we have $V_i = \bigcup_{j=2}^k S_{i_j} \subseteq B_u$, where $1 \leq i \leq d_u$. Thus, we get $\bigcup_{i=1}^{d_u} V_i \subseteq B_u$. Next, we consider $\sum_{i=1}^{d_u} |V_i|$. For any vertex in B_u , we will prove that it contributes at most $t-1$ times in $\sum_{i=1}^{d_u} |V_i|$. Otherwise, we suppose that there exists a vertex $w \in B_u$ such that it contributes at least t times in $\sum_{i=1}^{d_u} |V_i|$. This means that at least t sets of $V_1, V_2, \dots, V_{d_u}$ contain w simultaneously.

Without loss of generality, we suppose $w \in V_i$ with $i = 1, 2, \dots, t$. According to the definition of V_i , there exists an edge $f_i \in E_1(N_u)$ such that $f_i \cap (l_i \setminus u) \neq \emptyset$ and $w \in f_i$, where $1 \leq i \leq t$. Let $1 \leq i < j \leq t$. We will prove $f_i \neq f_j$ for any $1 \leq i < j \leq t$. Otherwise, if there exist $1 \leq i < j \leq t$ such that $f_i = f_j$, then $f_i \cap (l_i \setminus u) \neq \emptyset$ and $f_i \cap (l_j \setminus u) \neq \emptyset$. Since H is a linear hypergraph and $l_i \cap l_j = \{u\}$, we have $(l_i \setminus u) \cap (l_j \setminus u) = \emptyset$. Furthermore, since $l_i \setminus u, l_j \setminus u \subseteq N_u$, we obtain $|f_i \cap N_u| \geq 2$. This contradicts $f_i \in E_1(N_u)$, where $1 \leq i \leq t$. Therefore, $f_1, f_2, \dots, f_t$ are different from each other. Thus, $f_1, f_2, \dots, f_t, l_1, l_2, \dots, l_t$ form a Berge- $K_{2,t}$ in H . This contradicts that H is Berge- $K_{2,t}$ -free. Finally, we get that each vertex in B_u contributes at most $t-1$ in $\sum_{i=1}^{d_u} |V_i|$. Furthermore, since

$|B_u| = n - (k-1)d_u - 1$, we have $\sum_{i=1}^{d_u} |V_i| \leq (t-1)(n - (k-1)d_u - 1)$. Thus, we get Claim 4.4. □

From Claim 4.2, for any $v \in N_u$, we have $d_v = \sum_{i=1}^k e_i^v(N_u) \leq e_1^v(N_u) + 2(k-1)(s-1)$. Therefore, for any $v \in N_u$, $e_1^v(N_u) \geq d_v - 2(k-1)(s-1)$. Since H is a connected linear k -graph, we have $|S_v| = (k-1)e_1^v(N_u)$. Thus,

$|S_v| \geq (k-1)d_v - 2(k-1)^2(s-1)$. Bearing $|N_u| = (k-1)d_u$ in mind, we get

$$\begin{aligned}
 \sum_{v \in N_u} |S_v| &\geq \sum_{v \in N_u} [(k-1)d_v - 2(k-1)^2(s-1)] \\
 &= \sum_{v \in N_u} (k-1)d_v - 2(k-1)^2(s-1)|N_u| \\
 (13) \quad &= \sum_{v \in N_u} (k-1)d_v - 2(k-1)^3(s-1)d_u.
 \end{aligned}$$

By (13) and the fact $\sum_{v \in N_u} |S_v| = \sum_{i=1}^{d_u} \sum_{j=2}^k |S_{i_j}|$, we have

$$(14) \quad \sum_{v \in N_u} (k-1)d_v \leq \sum_{i=1}^{d_u} \sum_{j=2}^k |S_{i_j}| + 2(k-1)^3(s-1)d_u.$$

Furthermore, from Claims 4.3 and 4.4, (14) can be transformed into

$$\begin{aligned}
 \sum_{v \in N_u} (k-1)d_v &\leq \sum_{i=1}^{d_u} \left[|V_i| + \binom{k-1}{2} (2k-3)(s-1) \right] + 2(k-1)^3(s-1)d_u \\
 &= \sum_{i=1}^{d_u} |V_i| + \binom{k-1}{2} (2k-3)(s-1)d_u + 2(k-1)^3(s-1)d_u \\
 &\leq (t-1)(n-1) + \left[\frac{1}{2}(k-1)(k-2)(2k-3)(s-1) \right. \\
 &\quad \left. + 2(k-1)^3(s-1) - (k-1)(t-1) \right] d_u.
 \end{aligned}$$

Therefore, we have

$$\sum_{v \in N_u} d_v \leq \frac{(t-1)(n-1)}{k-1} + \left[\frac{1}{2}(6k^2 - 15k + 10)(s-1) - (t-1) \right] d_u.$$

By Corollary 2, we get

$$\begin{aligned}
 \rho_\alpha^2(H) &\leq \alpha d_u^2 + \frac{1-\alpha}{k-1} \sum_{v \in N_u} d_v \\
 &\leq \alpha d_u^2 + (1-\alpha) \left[\frac{(t-1)(n-1)}{(k-1)^2} + \frac{(6k^2 - 15k + 10)(s-1) - 2(t-1)}{2(k-1)} d_u \right].
 \end{aligned}$$

Therefore, we obtain Theorem 6. $\square$

It is noted that when $(6k^2 - 15k + 10)(s-1) - 2(t-1) \geq 0$, namely, $t \leq \frac{1}{2}(6k^2 - 15k + 10)(s-1) + 1$, the value of the right-hand side of (12) is related with d_u . Therefore, we obtain Theorem 7 as follows.

Theorem 7. Let H be a connected linear k -graph on n vertices with the maximum degree Δ , where $n, k \geq 3$. If H is Berge- $\{B_s, K_{2,t}\}$ -free, where $s \geq 2$ and $1 \leq t \leq \frac{1}{2}(6k^2 - 15k + 10)(s - 1) + 1$, then

$$\rho_\alpha^2(H) \leq \alpha\Delta^2 + (1 - \alpha) \left[\frac{(t-1)(n-1)}{(k-1)^2} + \frac{(6k^2 - 15k + 10)(s-1) - 2(t-1)}{2(k-1)} \Delta \right].$$

If H is a connected linear k -graph on n vertices, then its maximum degree is less than or equal to $\frac{n-1}{k-1}$, where $n, k \geq 3$. Therefore, by Theorem 7, for $H \in \mathcal{H}_{n,k,s,t}$ with $n, k \geq 3$, $s \geq 2$ and $1 \leq t \leq \frac{1}{2}(6k^2 - 15k + 10)(s - 1) + 1$, we get the upper bounds of $\rho(H)$ and $\rho_\alpha(H)$, which are shown in Theorems 8 and 9, respectively.

Theorem 8. Let H be a connected linear k -graph on n vertices, where $n, k \geq 3$. If H is Berge- $\{B_s, K_{2,t}\}$ -free, where $s \geq 2$ and $1 \leq t \leq \frac{1}{2}(6k^2 - 15k + 10)(s - 1) + 1$, then

$$\rho(H) \leq \sqrt{\frac{(6k^2 - 15k + 10)(s-1)(n-1)}{2(k-1)^2}}.$$

Theorem 9. Let H be a connected linear k -graph on n vertices, where $n, k \geq 3$. If H is Berge- $\{B_s, K_{2,t}\}$ -free, where $s \geq 2$ and $1 \leq t \leq \frac{1}{2}(6k^2 - 15k + 10)(s - 1) + 1$, then

$$\rho_\alpha^2(H) \leq \alpha \frac{(n-1)^2}{(k-1)^2} + (1 - \alpha) \frac{(6k^2 - 15k + 10)(s-1)(n-1)}{2(k-1)^2}.$$

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