

EXTENDING THE ANDERSON-LIVINGSTON ZERO-DIVISOR GRAPH VIA UNFAITHFUL MODULES

Fida Moh'd  and *Mamoon Ahmed* * 

Given a commutative ring R with unity and a left R -module M , the graph $\Gamma(R, M)$ is defined on the set of nonzero zero-divisors of R modulo $\text{Ann}_R(M)$. A vertex r lies in $\Gamma(R, M)$ if $r \in R - \text{Ann}_R(M) : rr' = 0 \pmod{\text{Ann}_R(M)}$ for some $r' \in R - \text{Ann}_R(M)$. Two distinct vertices r and s are adjacent if $rsM = 0$. This graph generalizes the Anderson–Livingston zero-divisor graph $\Gamma(R)$. We study its structure and establish formulas that link the invariants of $\Gamma(R, M)$ and $\Gamma(R)$.

1. INTRODUCTION

Algebra and graph theory intersect in powerful and insightful ways. This intersection yields the development of algebraic graph theory, where algebraic tools are applied to analyze graphs, or when the graph tools are used to explore algebraic structures. Through this interplay, new insights and methods emerge, allowing for a deeper understanding of both disciplines and their applications.

One important intersection between graphs and algebraic structures is the zero-divisor graphs of commutative rings. This concept was first introduced by I. Beck in 1988 [6]. Beck's interest was in the computation of the chromatic number of the graph by using the algebraic properties of the commutative ring. In 1999, the concept of zero-divisor graphs was further refined by D. F. Anderson and P. S. Livingston [5], where the set of vertices was shrunk to the set of all nonzero

*Corresponding author. Mamoon Ahmed

2020 Mathematics Subject Classification. 05C07; 05C25; 05C38.

Keywords and Phrases. Zero-divisor graphs, Commutative rings, Semiprime ideals, Cliques.

 Open Access ©2026 This work is licensed under the Creative Commons Attribution 4.0 International License.

zero-divisors. Later on, the researchers expanded and generalized this concept to other algebraic structures such as modules and graded rings, as one can observe for example in [1, 2, 3, 5, 8, 10, 12].

Influenced by the intensive efforts in studying the generalized zero-divisor graphs, we develop, in this paper, an interesting generalization of Anderson and P. S. Livingston zero-divisor graph for commutative rings. The very significance of our graph is being determined by a given unfaithful module, in contrast to the generalizations found in the literature, where the direction of generalization goes from rings to modules. Given a commutative ring R , and an unfaithful left R -module M , the graph $\Gamma(R, M)$ is a simple graph whose vertices (form the set $Z(R, M)$) are the nonzero zero-divisors of R modulo $\text{Ann}_R(M)$, where $\text{Ann}_R(M)$ is the annihilator of M in R , and two distinct vertices r, s are adjacent if $rsM = 0$. The graph $\Gamma(R, M)$ extends Anderson-Livingston zero-divisor graph $\Gamma(R)$. As indicated throughout the paper, there are a strong tie between $\Gamma(R, M)$ and $\Gamma(R)$. We shall see the connections between the two graphs through the interaction between the algebra-theoretic properties of the ring R and the graph-theoretic properties of $\Gamma(R, M)$. Besides, other types of connections appear in the formulas that provide estimations or calculations for the clique number, the independence number, the chromatic number, the girth and other graphic invariants. For instance, we show in Theorem 17 that if R is a commutative ring, M an unfaithful left R -module, and $r \in Z(R, M)$, then

$$\deg_{\Gamma(R, M)}(r) = \begin{cases} |\text{Ann}_R(M)| \cdot \deg_{[\Gamma]}([r]), & \text{if } x \notin (\text{Ann}_R(M))^{1/2}, \\ |\text{Ann}_R(M)| \cdot \deg_{[\Gamma]}([r]) \\ \quad + |\text{Ann}_R(M)| - 1, & \text{if } x \in (\text{Ann}_R(M))^{1/2}. \end{cases}$$

where $[\Gamma]$ denotes the Anderson-Livingston zero-divisor graph of $R/\text{Ann}_R(M)$. Another example is displayed in Theorem 28 which states that if R is a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal, and if $\Gamma(R, M)$ is incomplete, then

$$\alpha(\Gamma(R/\text{Ann}_R(M))) \leq \alpha(\Gamma(R, M)) \leq |\text{Ann}_R(M)| \cdot \alpha(\Gamma(R/\text{Ann}_R(M))),$$

where $\alpha(G)$ represents the independence number of the graph G . Also, in Theorem 34, if $\text{Ann}_R(M)$ is semiprime, then $\Gamma(R, M)$ and $[\Gamma] = \Gamma(R/\text{Ann}_R(M))$ share the same chromatic number. The previous formulas and other formulas are not only useful to compute the graphic numbers of $\Gamma(R, M)$ in terms of their counterparts in $\Gamma(R/\text{Ann}_R(M))$, but they also provide solid manners to compute the graphic numbers of $\Gamma(R)$ in terms of their counterparts of $\Gamma(R, M)$. More precisely, we could conclude outcomes about $\Gamma(R)$ by investigating and analyzing the environment of its extension $\Gamma(R, M)$. An important note about the results in this article is that they will be very helpful in solving coloring problems and related optimization problems such as in [9], which were used to solve optimization problems in [11].

In Section 2, we present the fundamental preliminaries. In Section 3, we introduce and study $\Gamma(R, M)$, examining aspects such as connectivity, completeness,

cycles, and providing estimates and calculations of the diameter, girth, independence number, clique number, chromatic number, domination number, and vertex degree, sometimes under restrictions on the ring or the module. In Section 4, we present applications of the results obtained in Section 3.

Many other properties of this new graph remain unexplored and are left for future work by interested researchers.

2. PRELIMINARIES

This section is concerned with the basic concepts of commutative rings, modules and graphs. All notions presented here are found in [4, 7]. Throughout this paper, R is a nonzero commutative ring with unit $1 \neq 0$, and M is a left R -module. The units set of R is denoted by $U(R)$.

Definition 1. Let I be a proper ideal of a R .

1. I is maximal if I is not contained in another proper ideal of R .
2. I is semiprime if whenever $r^2 \in I$, then $r \in I$.
3. I is prime if whenever $rr' \in I$, then $r \in I$ or $r' \in I$.
4. The square root of I is the set $I^{\frac{1}{2}} = \{r \in R : r^2 \in I\} \subseteq \text{Rad}(I)$, where $\text{Rad}(I)$ is the radical of I . In general, $I^{\frac{1}{2}}$ is not necessarily an ideal of R .

Theorem 2. [4] Let I be a proper ideal of a commutative ring R .

1. If I is maximal, then I is prime.
2. I is semiprime if and only if $I = \text{Rad}(I)$.
3. $I^{1/2}$ is an ideal of R if $\text{char}\left(\frac{R}{I}\right) = 2$.

Definition 3. The annihilator of a left R -module M in R is defined to be the set $\text{Ann}_R(M) = \{r \in R : rM = 0\}$.

The set $\text{Ann}_R(M)$ is an ideal of R . Also, $\text{Ann}_R(M)$ is a proper ideal of R if and only if $M \neq 0$. In addition, $\text{Ann}_R(M) = 0$ if and only if M is faithful.

Next, we display some background of undirected graphs. The letter G denotes an undirected graph. The set of vertices of G is denoted by $V(G)$, and the cardinality of $V(G)$ is called the order of G . If $V(G) = \emptyset$, then G is called an empty graph. We express the adjacency of two vertices g and h symbolically by $g \leftrightarrow h$.

Definition 4. Let v be a vertex in G . The neighborhood $N(v)$ of v is the set of all vertices adjacent to v .

Because G is a simple undirected graph, $v \notin N(v)$. If $N(v) = \emptyset$, then v is said to be an isolated vertex. If $V(G) \neq \emptyset$ and every vertex in G is isolated, we call G a null graph.

Definition 5. *The degree of a vertex v of G is the number of edges incident to v . The degree of v is denoted by $\deg_G(v)$ (or $\deg(v)$ if there is no confusion with the underlined graph).*

In case of a simple graph, $\deg(v) = |N(v)|$, where $||$ represents the cardinality of the set inside $||$. So, v is isolated if and only if $\deg(v) = 0$.

Definition 6. *Given two vertices t and h , the distance $d(t, h)$ between t and h is the length of a shortest path between them. The diameter $\text{diam}(G)$ of G is the maximum of the set $\{d(t, h) : g, h \in V(G)\}$.*

Definition 7. *A graph G is connected if there is a path joining any two distinct vertices of G .*

Definition 8. *By the girth of G , we mean the length of a shortest cycle in G . The girth of G is denoted by $g(G)$. If G has no cycles, then $g(G) = \infty$.*

Definition 9. *A graph is called complete if it is a simple graph and every two distinct vertices are adjacent. The complete graph on n vertices is denoted by K_n .*

Definition 10. *A complete subgraph of G is called a clique of G . The order of a clique with the largest number of vertices is called the clique number of G , denoted by $\omega(G)$.*

Definition 11. *A dominating set D of G is a nonempty subset of $V(G)$ such that every vertex of G not in D is adjacent to at least one vertex in D . The minimum of the set $\{|D| : D \text{ is a dominating set of } G\}$ is called the domination number of G , and is denoted by $\gamma(G)$.*

Definition 12. *A subset $A \subseteq G$ is said to be independent if no two vertices in A are adjacent. An independent set is maximal if it can be included in another independent set. The cardinality of a maximal independent set with the largest number of vertices is called the independence number of G , and is denoted by $\alpha(G)$.*

Throughout this paper, if W is a set of vertices of a graph G , then $G - W$ denotes the subgraph of G obtained by removing the vertices of W along with all edges incident to the removed vertices. If $W = \{v\}$, the notation $G - W$ is replaced with $G - v$.

3. THE RING ZERO-DIVISOR GRAPH INDUCED BY AN UNFAITHFUL MODULE

In this section, we define the graph $\Gamma(R, M)$ associated with a commutative ring R , and allocated by a left R -module M . Afterward, we investigate the graph

$\Gamma(R, M)$ through several aspects like connectivity, completeness, cycles. Further, we estimate the values of diameter, girth, independence number, clique number, chromatic number, domination number, and vertex degree and present a way to calculate them perhaps under some constraints on R , $\text{Ann}_R(M)$, or M . We define the set $Z(R, M)$ to be the set of all nonzero zero-divisors of R modulo $\text{Ann}_R(M)$, that is,

$$\left\{ r \in R - \text{Ann}_R(M) : \begin{array}{l} rr' = 0 \pmod{\text{Ann}_R(M)} \\ \text{for some } r' \in R - \text{Ann}_R(M), r' \neq r \end{array} \right\}.$$

Definition 13. Let R be a commutative ring, and M a left R -module. The zero-divisor graph of R induced by M , denoted by $\Gamma(R, M)$, is the simple graph whose vertices are the elements of $Z(R, M)$, and two distinct vertices r and r' are adjacent if $rsM = 0$.

Notice that $Z(R, M) \cap [U(R) \cup \text{Ann}_R(M)] = \emptyset$. Also, if M is a faithful module, then $\Gamma(R, M) = \Gamma(R)$. Since $\Gamma(R)$ has been extensively studied in the literature, **we shall focus in studying $\Gamma(R, M)$ on the case where M is unfaithful.**

In the next work, $[r] = r + \text{Ann}_R(M)$ the coset of $\text{Ann}_R(M)$ represented by $r \in R$. The upcoming lemma plays an important role in this paper.

Lemma 14. Let R be a commutative ring, and M an unfaithful left R -module. Then

1. $\Gamma(R, M)$ is an empty graph (i.e., $Z(R, M) = \emptyset$) if and only if $M = 0$ or $\text{Ann}_R(M)$ is a prime ideal of R .
2. Let $r \notin \text{Ann}_R(M)$. The coset $[r]$ is a clique if and only if $r \in (\text{Ann}_R(M))^{1/2}$. Notice that if $[r]$ is not a clique in $\Gamma(R, M)$, then it is an independent set.
3. For two distinct vertices r and r' in $\Gamma(R, M)$, $r \leftrightarrow r'$ if and only if $xr \leftrightarrow yr'$ for every $x, y \in R$ provided that $rx \neq yr' \pmod{\text{Ann}_R(M)}$.
4. For every two distinct vertices r and r' in $\Gamma(R, M)$ and $a, a' \in \text{Ann}_R(M)$, we have $r \leftrightarrow r'$ if and only if $r + a \leftrightarrow r' + a'$. In particular, if $r \in Z(R, M)$, then $r + \text{Ann}_R(M) \subseteq Z(R, M)$.
5. For two distinct vertices r and r' in $\Gamma(R, M)$, the following statements are equivalent:

$$(a) \ r \leftrightarrow r' \text{ in } \Gamma(R, M).$$

$$(b) \ rr' \in \text{Ann}_R(M).$$

$$(c) \ [r] \leftrightarrow [r'] \text{ in } \Gamma\left(\frac{R}{\text{Ann}_R(M)}\right) \text{ provided that } r \neq r' \pmod{\text{Ann}_R(M)}.$$

$$6. \ Z(R, M) = \bigcup_{[r] \in Z(R/\text{Ann}_R(M))^*} [r].$$

- Proof.* 1. Assume $\Gamma(R, M)$ is empty. We know that $M = 0$ iff $R = \text{Ann}_R(M)$. Therefore, $Z(R, 0) = \emptyset$. Assume $M \neq 0$. Then $\text{Ann}_R(M)$ is a proper ideal of R . Since M is unfaithful, $\text{Ann}_R(M) \neq 0$. Let $r, r' \notin \text{Ann}_R(M)$ be distinct elements. Then $rr' \neq 0 \pmod{\text{Ann}_R(M)}$. That is, $rr' \notin \text{Ann}_R(M)$. Besides, if $r^2 \in \text{Ann}_R(M)$, then r and $r + a$, where $0 \neq a \in \text{Ann}_R(M)$, are distinct elements outside $\text{Ann}_R(M)$ such that $r(r + a)M = (r^2 + ra)M = 0$. Thus, $r \in Z(R, M)$ which is a contradiction. Therefore, we obtain that $r^2 \notin \text{Ann}_R(M)$. We conclude that $\text{Ann}_R(M)$ is a prime ideal of R . For the converse, suppose $M = 0$ or $\text{Ann}_R(M)$ is a prime ideal. If $M = 0$, then $\text{Ann}_R(M) = R$ and hence $Z(R, M) = \emptyset$. Assume $M \neq 0$ is a prime ideal of R . Since M is unfaithful, we have $|R - \text{Ann}_R(M)| \geq 2$. Let $r, r' \notin \text{Ann}_R(M)$ be distinct elements. Since $\text{Ann}_R(M)$ is prime, $rr' \notin \text{Ann}_R(M)$. This implies $Z(R, M) = \emptyset$.
2. Assume the coset $[r]$ is a clique. Since M is unfaithful, there exists $0 \neq a \in \text{Ann}_R(M)$. We have r and $r + a$ are distinct elements outside $\text{Ann}_R(M)$ which belong to the same coset $[r]$. By the assumption, $r \leftrightarrow r + a$. Hence, $r^2M = r^2M + 0M = (r^2 + ra)M = r(r + a)M = 0$, which implies $r^2 \in \text{Ann}_R(M)$ and so $r \in (\text{Ann}_R(M))^{1/2}$. For the converse, assume $r \in (\text{Ann}_R(M))^{1/2} - \text{Ann}_R(M)$. The unfaithfulness of M guarantees that $\text{Ann}_R(M) \neq 0$. Let $a, a' \in \text{Ann}_R(M)$ be distinct elements. Then $r + a$ and $r + a'$ are distinct elements outside $\text{Ann}_R(M)$ that lie in the same coset $[r]$. Moreover, $(r + a)(r + a')M \subseteq r^2M + raM + ra'M + aa'M = 0$. Thus, $r + a \leftrightarrow r + a'$. We conclude that the coset $[r]$ forms a clique in $\Gamma(R, M)$. The rest is easy.
3. The proof follows from $\text{Ann}_R(M)$ being an ideal.
4. The proof follows from $\text{Ann}_R(M)$ being an ideal.
5. The equivalence of (a) and (b) is obvious. We prove the equivalence of (a) and (c). Assume $r \leftrightarrow r'$ in $\Gamma(R, M)$ and $r \neq r' \pmod{\text{Ann}_R(M)}$. Then $rr' \in \text{Ann}_R(M)$. Thus $[r][r'] = \text{Ann}_R(M)$ with $[r] \neq [r']$. So, $[r] \leftrightarrow [r']$ in $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$. Conversely, Assume $[r] \leftrightarrow [r']$ in $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$ where $[r] \neq [r']$. Then $[r][r'] = \text{Ann}_R(M)$ which implies $rr' \in \text{Ann}_R(M)$ or equivalently $r \leftrightarrow r'$ in $\Gamma(R, M)$.
6. The proof is straight forward from (5).

□

Proposition 15. *Let R be a commutative ring, and M an unfaithful left R -module. Then $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$ is embedded in $\Gamma(R, M)$ as a subgraph. Moreover, there are at least*

$$|\text{Ann}_R(M)|^{\left|Z\left(\frac{R}{\text{Ann}_R(M)}\right)\right|^*} \text{ different copies of } \Gamma\left(\frac{R}{\text{Ann}_R(M)}\right) \text{ inside } \Gamma(R, M).$$

Proof. From each vertex (coset) $[r]$ in $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$, Fix $a \in \text{Ann}_R(M)$ and a representative r for $[r]$. Define the function $f_a : Z\left(\frac{R}{\text{Ann}_R(M)}\right)^* \rightarrow Z(R, M)$ by $f_a([r]) = r + a$. It is not difficult to see that f is a one-to-one map from the graph $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$ to $\Gamma(R, M)$ that preserves adjacency between vertices by Lemma 14. So, $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$ is embedded in $\Gamma(R, M)$. Since there are $|\text{Ann}_R(M)|$ ways to select an embedding function $a \in \text{Ann}_R(M)$, by the product rule we have at least $|\text{Ann}_R(M)| \left| Z\left(\frac{R}{\text{Ann}_R(M)}\right)^* \right|$ different embeddings f_a (or copies) of $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$ inside $\Gamma(R, M)$. $\square$

In the next work, and in order to concise the notations, we shall sometimes denote $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$ by $[\Gamma]$ or $\Gamma(R/\text{Ann}_R(M))$.

Corollary 16. *Let R be a commutative ring, and M an unfaithful left R -module. Then $\Gamma(R, M)$ is empty if and only if $\Gamma\left(\frac{R}{\text{Ann}_R(M)}\right)$ is empty.*

Proof. The proof follows directly from Lemma 14 and Proposition 15. $\square$

Theorem 17. *Let R be a commutative ring, M an unfaithful left R -module, and $r \in Z(R, M)$. Then*

$$\deg_{\Gamma(R, M)}(r) = \begin{cases} |\text{Ann}_R(M)| \cdot \deg_{[\Gamma]}([r]), & \text{if } r \notin (\text{Ann}_R(M))^{1/2}, \\ |\text{Ann}_R(M)| \cdot \deg_{[\Gamma]}([r]) + |\text{Ann}_R(M)| - 1, & \text{if } r \in (\text{Ann}_R(M))^{1/2}. \end{cases}$$

Proof. Let $r \notin (\text{Ann}_R(M))^{1/2}$. As stated in (2) of Lemma 14, r is not adjacent to any of the other elements in the coset $[r]$. Thus, from (5) of Lemma 14, $r \leftrightarrow r'$, where $[r] \neq [r']$, gives that r is adjacent to every element in the coset $[r']$. Consequently, $\deg_{\Gamma(R, M)}(r) = |\text{Ann}_R(M)| \cdot \deg_{[\Gamma]}([r])$. Next, for $r \in (\text{Ann}_R(M))^{1/2}$, a parallel argument to that one above leads to the equality $\deg_{\Gamma(R, M)}(r) = |\text{Ann}_R(M)| \cdot \deg_{[\Gamma]}([r]) + |\text{Ann}_R(M)| - 1$. $\square$

The previous results can be illustrated by the following example.

Example 18. 1. $\Gamma(\mathbb{Z}, \mathbb{Z}/11\mathbb{Z}) = \emptyset$ since $\text{Ann}_{\mathbb{Z}}(\mathbb{Z}/11\mathbb{Z}) = 11\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$.

2. Let R be a commutative ring. Then $\Gamma(R, R) = \Gamma(R)$.

3. In $\Gamma(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$, we have $\text{Ann}_{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) = 6\mathbb{Z}$ and $Z(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = (2 + 6\mathbb{Z}) \cup (3 + 6\mathbb{Z}) \cup (4 + 6\mathbb{Z})$. The graph $\Gamma(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$ is described as follows:

- $2 + 6n \leftrightarrow 3 + 6m$ for all $m, n \in \mathbb{Z}$.
- $4 + 6k \leftrightarrow 3 + 6m$ for all $m, k \in \mathbb{Z}$.
- $2 + 6n \not\leftrightarrow 4 + 6k$ for all $k, n \in \mathbb{Z}$.
- Each of the sets $2 + 6\mathbb{Z}$, $3 + 6\mathbb{Z}$, and $4 + 6\mathbb{Z}$ is an independent set.

4. In $\Gamma(\mathbb{Z}_5[x], \mathbb{Z}_5[x]/\langle x^2 \rangle)$,

$$\text{Ann}_{\mathbb{Z}_5[x]}(\mathbb{Z}_5[x]/\langle x^2 \rangle) = \langle x^2 \rangle$$

and

$$Z(\mathbb{Z}_5[x], \mathbb{Z}_5[x]/\langle x^2 \rangle) = \langle x \rangle,$$

which is a maximal ideal. Consequently, $\Gamma(\mathbb{Z}_5[x], \mathbb{Z}_5[x]/\langle x^2 \rangle)$ is a complete graph.

5. In $\Gamma(\mathbb{Z}_2[x]/\langle x^4 \rangle, \langle x + \langle x^4 \rangle \rangle)$, where $M = \langle x + \langle x^4 \rangle \rangle$ is the principal ideal of $\mathbb{Z}_2[x]/\langle x^4 \rangle$ generated by the coset $x + \langle x^4 \rangle$, we have $\text{Ann}_{\mathbb{Z}_2[x]/\langle x^4 \rangle}(\langle x + \langle x^4 \rangle \rangle) = \langle x^3 + \langle x^4 \rangle \rangle$. This graph is not complete as displayed in Figure 1 below:

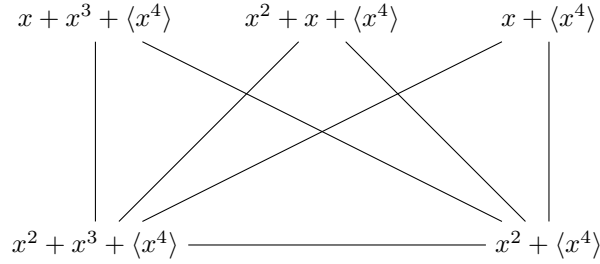


Figure 1

3.1 Connectedness and Completeness

In this subsection, we prove that $\Gamma(R, M)$ is connected with diameter at most 3. In addition, we study the conditions that ensure the completeness of $\Gamma(R, M)$. Also, we link the connectedness and the completeness of $\Gamma(R, M)$ to those of $\Gamma(R/\text{Ann}_R(M))$.

Theorem 19. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. Then $\Gamma(R, M)$ is connected with diameter at most 3.*

Proof. Since $\text{Ann}_R(M)$ is not prime, then $Z(R, M) \neq \emptyset$, and since M is unfaithful, $\text{Ann}_R(M) \neq 0$. From (4) of Lemma 14, we obtain that $|\text{Ann}_R(M)| \geq 2$ and $|Z(R, M)| \geq 2$. Let $r, r' \in Z(R, M)$. If $r \leftrightarrow r'$, then $d(r, r') = 1$. Suppose $r \not\leftrightarrow r'$. Then there exist $\alpha, \alpha' \notin \text{Ann}_R(M)$ such that $r \leftrightarrow \alpha$ and $r' \leftrightarrow \alpha'$. We consider the following cases:

1. Assume $\alpha = \alpha'$. Then we have the path $r \leftrightarrow \alpha \leftrightarrow r'$ and hence $d(r, r') = 2$.
2. Assume $\alpha \neq \alpha'$ and $r \leftrightarrow \alpha'$ or $r' \leftrightarrow \alpha$. We have the elements r, r', α , and α' are pairwise distinct. So, we obtain the path $r \leftrightarrow \alpha' \leftrightarrow r'$ or $r \leftrightarrow \alpha \leftrightarrow r'$, respectively. In either option, we get $d(r, r') = 2$.
3. Assume $\alpha \neq \alpha', r \not\leftrightarrow \alpha', r' \not\leftrightarrow \alpha$, and $\alpha \leftrightarrow \alpha'$. Again, we have the elements r, r', α , and α' are pairwise distinct. Also, we have the path $r \leftrightarrow \alpha \leftrightarrow \alpha' \leftrightarrow r'$. Therefore, $d(r, r') = 3$.
4. Assume $\alpha \neq \alpha', r \not\leftrightarrow \alpha', r' \not\leftrightarrow \alpha$, and $\alpha \not\leftrightarrow \alpha'$. Thus, $\alpha\alpha' \notin \text{Ann}_R(M)$ and $r \neq \alpha\alpha' \neq r'$. Hence, we have the path $r \leftrightarrow \alpha\alpha' \leftrightarrow r'$. So, we get $d(r, r') = 2$.

The above argument yields that $\Gamma(R, M)$ is connected with diameter at most 3.

Now, we talk about the completeness of $\Gamma(R, M)$. □

Theorem 20. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. Then, $\Gamma(R, M)$ is complete if and only if $\Gamma(R/\text{Ann}_R(M))$ is complete and $Z(R, M) \subseteq (\text{Ann}_R(M))^{1/2}$.*

Proof. Suppose $\Gamma(R, M)$ is complete. By (4) and (6) of Lemma 14, the graph $\Gamma(R/\text{Ann}_R(M))$ is complete. Let $a, a' \in \text{Ann}_R(M)$ and $r \in Z(R, M)$. Then $a + r \leftrightarrow a' + r$. Thus, $(a + r)(a' + r) \in \text{Ann}_R(M)$, which implies $r^2 \in \text{Ann}_R(M)$. So, $r \in (\text{Ann}_R(M))^{1/2}$. For the converse, suppose $\Gamma(R/\text{Ann}_R(M))$ is complete and $Z(R, M) \subseteq (\text{Ann}_R(M))^{1/2}$. Let $r, r' \in Z(R, M)$. If $r \neq r' \pmod{\text{Ann}_R(M)}$, then $[r]$ and $[r']$ are distinct vertices of $\Gamma(R/\text{Ann}_R(M))$. So, $[r] \leftrightarrow [r']$ in $\Gamma(R/\text{Ann}_R(M))$. By (5) of Lemma 14, we have $r \leftrightarrow r'$. If $r = r' \pmod{\text{Ann}_R(M)}$, we conclude from (2) of Lemma 14, that $r \leftrightarrow r'$. As a result, $\Gamma(R, M)$ is complete. □

Corollary 21. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. The following are equivalent:*

1. $\Gamma(R, M)$ is complete.
2. $\Gamma(R/\text{Ann}_R(M))$ is complete and $Z(R, M) \subseteq (\text{Ann}_R(M))^{1/2}$.
3. $R/\text{Ann}_R(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $[r][r'] = [0]$, for every $[r], [r'] \in Z(R/\text{Ann}_R(M))^*$.

Proof. Apply Theorem 20 and Theorem 2.8 of [5]. □

Recall that a commutative ring without a multiplicative identity (unit) is called a ring of square zero if the product of any two elements in the ring is zero. In contrast of the past corollary and theorem, the next theorem displays a full algebraic characterization for the completeness of $\Gamma(R, M)$.

Theorem 22. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. Then $\Gamma(R, M)$ is complete if and only if $\Theta = Z(R, M) \cup \text{Ann}_R(M)$ is a subring of R such that $\frac{\Theta}{\text{Ann}_R(M)}$ is a ring of square zero (and so $Z(R, M) \subseteq (\text{Ann}_R(M))^{1/2}$).*

Proof. Assume $\Gamma(R, M)$ is a complete graph. Firstly, we show that $\Theta = Z(R, M) \cup \text{Ann}_R(M)$ is closed under the multiplication by the elements of R . Let $r \in R$ and $\theta \in \Theta$. If $\theta \in \text{Ann}_R(M)$, then $r\theta \in \text{Ann}_R(M) \subset \Theta$ because $\text{Ann}_R(M)$ is an ideal of R . Let $\theta \in Z(R, M)$. If $r \in Z(R, M)$ and $r = \theta$, then $r\theta = \theta^2 \in \text{Ann}_R(M) \subset \Theta$ since $Z(R, M) \subset (\text{Ann}_R(M))^{1/2}$ by Theorem 20. If $r \in Z(R, M)$ and $r \neq \theta$, then by the completeness of $\Gamma(R, M)$, $r \leftrightarrow \theta$. Thus, $r\theta \in \text{Ann}_R(M) \subset \Theta$. Secondly, we show that Θ is closed under addition. Let $r, r' \in \Theta$. If $r, r' \in \text{Ann}_R(M)$, then $r + r' \in \text{Ann}_R(M) \subset \Theta$ since $\text{Ann}_R(M)$ is an ideal. If $r \in Z(R, M)$ and $r' \in \text{Ann}_R(M)$, then $r + r' \in Z(R, M) \subset \Theta$ because r and $r + r'$ are distinct elements outside $\text{Ann}_R(M)$ such that $r(r + r') = r^2 + rr' \in \text{Ann}_R(M)$. Now, let us assume $r, r' \in Z(R, M)$. In the first case, assume $r = r'$. Then, $r + r' = 2r$. If $2r \in \text{Ann}_R(M)$, we get $2r \in \Theta$. Assume $2r \notin \text{Ann}_R(M)$. If $2r = r$, then we have $2r \in Z(R, M) \subset \Theta$. Suppose $2r \neq r$. Let $r'' \in Z(R, M)$ such that $rr'' \in \text{Ann}_R(M)$. If $2r = r''$, again $2r \in Z(R, M) \subset \Theta$. If $2r \neq r''$, then $2rr'' \in \text{Ann}_R(M)$ and hence $2r \in Z(R, M) \subset \Theta$. In the second case, suppose $r \neq r'$. Since $\Gamma(R, M)$ is complete, we have $r \leftrightarrow r'$. Then r and $r + r'$ are distinct elements outside $\text{Ann}_R(M)$ such that $r(r + r') = rr' + r^2 \in \text{Ann}_R(M)$. In other words, $r + r' \in Z(R, M) \subset \Theta$. As a result, we get that Θ is closed under addition and the multiplication by the elements of R . Therefore, Θ is a subring of R . Next, let $[r], [r'] \in \frac{\Theta}{\text{Ann}_R(M)}$. If $[r] = [r']$, then r and r' exist in the same coset. Let $0 \neq a \in \text{Ann}_R(M)$. Then r and $r + a$ are distinct elements in $Z(R, M)$. Since $\Gamma(R, M)$ is complete, we have $r \leftrightarrow r + a$. This means, $r^2 + ar = r(r + a) \in \text{Ann}_R(M)$, which implies $r^2 \in \text{Ann}_R(M)$ or equivalently saying $[r]^2 = [0] = \text{Ann}_R(M)$. If $[r] \neq [r']$, then r and r' are distinct elements inside $Z(R, M)$. Again the completeness of $\Gamma(R, M)$ gives that $r \leftrightarrow r'$. So, $rr' \in \text{Ann}_R(M)$ or equivalently speaking $[r][r'] = [0]$. As a consequence, $\frac{\Theta}{\text{Ann}_R(M)}$ is a ring of square zero.

For the converse, assume Θ is a subring of R such that $Z(R, M) \subseteq (\text{Ann}_R(M))^{1/2}$. Let $r, r' \in Z(R, M)$ be distinct elements. Since Θ is a subring, we obtain that $-r' \in \Theta$. But $-r' \notin \text{Ann}_R(M)$ because $\text{Ann}_R(M)$ is an ideal. So, $-r' \in Z(R, M)$. In the first case, Let us assume $r = -r' \neq r'$, then $rr' = -r'^2 \in \text{Ann}_R(M)$, which implies $r \leftrightarrow r'$. In the second case, let us assume $r = \pm r' \pmod{\text{Ann}_R(M)}$, then $r = \pm r' + a$ for some $a \in \text{Ann}_R(M) - 0$. Thus, r and r' are distinct elements outside $\text{Ann}_R(M)$ such that $rr' = \pm r'^2 + r'a \in \text{Ann}_R(M)$. Thus $r \leftrightarrow r'$. In the third case, let us assume $r \neq \pm r' \pmod{\text{Ann}_R(M)}$. Then $r + r' \notin \text{Ann}_R(M)$. Since Θ is a subring, we have $r + r' \in \Theta$. Consequently, $r + r' \in Z(R, M)$. We have r and $r + r'$ are distinct elements of $Z(R, M)$. Again, $\frac{\Theta}{\text{Ann}_R(M)}$ being a

ring of square zero yields that $[r^2 + rr'] = [r][r + r'] = [0]$. So, we obtain that $r^2 + rr' \in \text{Ann}_R(M)$. Noticing that $r^2 \in \text{Ann}_R(M)$ because $\frac{\Theta}{\text{Ann}_R(M)}$ is a ring of square zero, we conclude that $rr' \in \text{Ann}_R(M)$, which means that $r \leftrightarrow r'$. To this point, we have demonstrated that $\Gamma(R, M)$ is a complete graph. $\square$

We turn now to talk when $\Gamma(R, M)$ is a star graph.

Theorem 23. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. If $|Z(R, M)| \geq 3$, then $\Gamma(R, M)$ is not a star graph.*

Proof. Assume, for contrary, that $\Gamma(R, M)$ is a star graph with center $x \in Z(R, M)$. Since $\text{Ann}_R(M) \neq 0$, by Lemma 14, $x + a$ is adjacent to every other vertex in $\Gamma(R, M)$, for every $0 \neq a \in \text{Ann}_R(M)$. Since $|Z(R, M)| \geq 3$, we shall obtain a cycle of length 3 or 4 inside $\Gamma(R, M)$, which contradicts that $\Gamma(R, M)$ is a star graph. $\square$

Corollary 24. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. The following statements are equivalent:*

1. $\Gamma(R, M)$ is a star graph.
2. $|Z(R, M)| = 2$.
3. $\Gamma(R, M) = K_2$.

Proof. The proof follows directly from Theorem 23. $\square$

3.2 Cycles, Girth, and Independence Number

In this subsection, we shall study the cycles of $\Gamma(R, M)$ and estimate the value of the girth. We introduce a method to compute the girth under certain conditions. Analogously, the independent sets and the independence number are investigated.

Theorem 25. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. If $|Z(R, M)| \geq 3$, then $\Gamma(R, M)$ has a cycle except in the following situations:*

1. $|Z(R, M)| = 2$.
2. $|Z(R, M)| = 3$ but $Z(R, M) \cap (\text{Ann}_R(M))^{1/2} = \emptyset$.

Moreover, when a cycle exists, then $g(\Gamma(R, M)) \leq 4$.

Proof. We shall firstly show that the graph has cycles in the following cases:

1. $|Z(R, M)| = 3$ and $Z(R, M) \cap (\text{Ann}_R(M))^{1/2} \neq \emptyset$.

2. $Z(R, M) \cap (Ann_R(M))^{1/2} \neq \emptyset$ and $|Ann_R(M)| \geq 3$ (i.e., $Ann_R(M) \not\cong \mathbb{Z}_2$).
3. $|Ann_R(M)| = 2$, $Z(R, M) \cap (Ann_R(M))^{1/2} \neq \emptyset$, and $|Z(R/Ann_R(M))| \geq 2$.
4. $|Ann_R(M)| \geq 2$, $Z(R, M) \cap (Ann_R(M))^{1/2} = \emptyset$, and $|Z(R/Ann_R(M))| \geq 2$.

Afterward, we show that in the remaining cases that the graph has no cycles.

1. Assume $|Z(R, M)| = 3$ and $Z(R, M) \cap (Ann_R(M))^{1/2} \neq \emptyset$. Then $Z(R, M) = [r]$ for some $r \notin Ann_R(M)$. To see that, assume $Z(R, M) \neq [r]$, by (6) of Lemma 14, there exists $r' \in Z(R, M)$ such that $[r] \neq [r']$. Since $Ann_R(M) \neq 0$, we have $|[r]| = |[r']| \geq 2$. Thus, again from (6) of Lemma 14, we get $|Z(R, M)| \geq 4$, which is a contradiction. Now, by the assumption, $[r] \cap (Ann_R(M))^{1/2} \neq \emptyset$. So, $[r]$ is a clique of order 3 and hence it forms a triangle.
2. Suppose $Z(R, M) \cap (Ann_R(M))^{1/2} \neq \emptyset$ and $|Ann_R(M)| \geq 3$. Let $r \in Z(R, M) \cap (Ann_R(M))^{1/2}$. Then, $[r] \in Z(R, M) \cap (Ann_R(M))^{1/2}$. Now, by Lemma 14, $[r]$ is a clique with at least 3 vertices and hence it contains a triangle.
3. Assume $|Ann_R(M)| = 2$, $Z(R, M) \cap (Ann_R(M))^{1/2} \neq \emptyset$, and $|Z(R/Ann_R(M))| \geq 2$. Then, there exists $r \in Z(R, M)$ such that $[r] \subseteq (Ann_R(M))^{1/2}$. On the other hand, there exists $r' \in Z(R, M)$ such that $r \neq r' \pmod{Ann_R(M)}$. Let $0 \neq a \in Ann_R(M)$. By Lemma 14, We have r, r' , and $r + a$ are distinct elements in $Z(R, M)$ that form a triangle.
4. Assume $|Ann_R(M)| \geq 2$, $Z(R, M) \cap (Ann_R(M))^{1/2} = \emptyset$, and $|Z(R/Ann_R(M))| \geq 2$. Let $r, r' \in Z(R, M)$ such that $[r] \neq [r']$, and $0 \neq a \in Ann_R(M)$. By Lemma 14, each of $[r]$ and $[r']$ are independent sets. We have $r, r', r' + a$, and $r + a$ are distinct elements in $Z(R, M)$. By pigeonhole principle, any three vertices taken from these four vertices must have two vertices that share the same coset. Therefore, we can't form a triangle from any three vertices taken from these four vertices. However, we have the cycle $r \leftrightarrow r' \leftrightarrow r + a \leftrightarrow r' + a \leftrightarrow r$ of length 4.
5. We show in the remaining cases that the graph has no cycles. If $|Z(R, M)| = 2$, then we can't have a cycle. If $|Ann_R(M)| = 2$, $Z(R, M) \cap (Ann_R(M))^{1/2} = \emptyset$, and $|Z(R/Ann_R(M))| = 1$, then once again $|Z(R, M)| = 2$ and hence no cycles are found. If $|Z(R, M)| = 3$ but $Z(R, M) \cap (Ann_R(M))^{1/2} = \emptyset$, then $Z(R, M) = [r]$ is an independent set by Lemma 14. So, no cycle is found.

Finally, the above argument tells that when a cycle is available, then $g(\Gamma(R, M)) \leq 4$. $\square$

Corollary 26. *Let R be a commutative ring, and M an unfaithful left R -module such that $Ann_R(M)$ is not a prime ideal. Then, $g(\Gamma(R, M)) \leq g(\Gamma(R/Ann_R(M)))$. Moreover, if $g(\Gamma(R/Ann_R(M))) = 3$, then $g(\Gamma(R, M)) = 3$.*

Proof. Apply Proposition 15 and Theorem 25. $\square$

In the rest of the subsection, we study the independent sets and the independence number of $\Gamma(R, M)$. Consider the natural epimorphism $f : R \rightarrow R/Ann_R(M)$ defined by $f(r) = [r]$. By Lemma 14, this homomorphism induces a well-defined onto graph function: $\bar{f} : \Gamma(R, M) \rightarrow \Gamma(R/Ann_R(M))$, $\bar{f}(r) = \bar{r} = [r]$. If W is a subset of vertices (or a subgraph) of $\Gamma(R, M)$, then we denote $\bar{f}(W)$ by $\bar{W}$. It is not difficult to see that if W is independent in $\Gamma(R, M)$, so is $\bar{W}$ in $\Gamma(R/Ann_R(M))$. The converse of the last statement is not necessarily true.

Lemma 27. *Let R be a commutative ring, and M an unfaithful left R -module such that $Ann_R(M)$ is not a prime ideal. Let T be an independent set of $\Gamma(R, M)$. Then*

1. If $r \in T \cap (Ann_R(M))^{1/2}$, then $|[r] \cap T| = 1$. In other words, $T \cap (Ann_R(M))^{1/2}$ is an independent set of $\Gamma(R, M)$ such that $\overline{T \cap (Ann_R(M))^{1/2}}$ is an independent set in $\Gamma(R/Ann_R(M))$ with the same cardinality of $T \cap (Ann_R(M))^{1/2}$.
2. If T is a maximal independent set, and $r \in T - (Ann_R(M))^{1/2}$, then $[r] \subseteq T$.

Proof. 1. Suppose $r \in T \cap (Ann_R(M))^{1/2}$. By Lemma 14, $[r]$ is a complete subgraph of $\Gamma(R, M)$. So, T contains exactly one element of $[r]$. The rest follows from Lemma 14.

2. Assume T is a maximal independent set, and $r \in T - (Ann_R(M))^{1/2}$. Again, in view of Lemma 14, $[r]$ and hence $[r] \cup T$ are independent sets. So, $T \subseteq T \cup [r]$. The maximality of T yields that either $T = T \cup [r]$ or $\Gamma(R, M) = T \cup [r]$. However, $\Gamma(R, M)$ is not an independent set because $Ann_R(M)$ is not a prime ideal. Thus, $\Gamma(R, M) \neq T \cup [r]$ and hence $T = T \cup [r]$. Consequently, $[r] \subseteq T$. $\square$

The following theorem gives an estimation of the independence number of $\Gamma(R, M)$ in terms of the independence number of $\Gamma(R/Ann_R(M))$.

Theorem 28. *Let R be a commutative ring, and M an unfaithful left R -module such that $Ann_R(M)$ is not a prime ideal. If $\Gamma(R, M)$ is incomplete, then*

$$\alpha(\Gamma(R/Ann_R(M))) \leq \alpha(\Gamma(R, M)) \leq |Ann_R(M)| \cdot \alpha(\Gamma(R/Ann_R(M))).$$

Proof. Assume $\Gamma(R, M)$ is an incomplete graph. Then, there are two vertices r and r' such that $r \not\sim r'$. By Lemma 14 and Proposition 15, if $\bar{T} = \{[r_k], k \in K\}$, where K is an indexing set, is an independent set in $\Gamma(R/Ann_R(M))$ such that $|\bar{T}| = \alpha(\Gamma(R/Ann_R(M)))$, then $T = \{r_k, k \in K\}$ is an independent set of $\Gamma(R, M)$. Thus, we obtain $\alpha(\Gamma(R/Ann_R(M))) \leq \alpha(\Gamma(R, M))$. On the other hand, let T be a

maximal independent set of $\Gamma(R, M)$. By Lemma 27, we have

$$\begin{aligned}
 |T| &= |(T \cap (\text{Ann}_R(M))^{1/2}) \cup (T \cap (R - (\text{Ann}_R(M))^{1/2}))| \\
 &= |(T \cap (\text{Ann}_R(M))^{1/2})| + |(T \cap (R - (\text{Ann}_R(M))^{1/2}))| \\
 &= \left| \overline{(T \cap (\text{Ann}_R(M))^{1/2})} \right| + \sum_{[r] \in \overline{(T \cap (R - (\text{Ann}_R(M))^{1/2}))}} |[r]| \\
 &= \left| \overline{(T \cap (\text{Ann}_R(M))^{1/2})} \right| + \sum_{[r] \in \overline{(T \cap (R - (\text{Ann}_R(M))^{1/2}))}} |\text{Ann}_R(M)| \\
 &= \left| \overline{(T \cap (\text{Ann}_R(M))^{1/2})} \right| + \left| \overline{T \cap (R - (\text{Ann}_R(M))^{1/2})} \right| \cdot |\text{Ann}_R(M)| \\
 &\leq \left| \overline{(T \cap (\text{Ann}_R(M))^{1/2})} \right| \cdot |\text{Ann}_R(M)| + \left| \overline{T \cap (R - (\text{Ann}_R(M))^{1/2})} \right| \cdot |\text{Ann}_R(M)| \\
 &= \left(\left| \overline{(T \cap (\text{Ann}_R(M))^{1/2})} \right| + \left| \overline{T \cap (R - (\text{Ann}_R(M))^{1/2})} \right| \right) \cdot |\text{Ann}_R(M)| \\
 &= \left| \overline{\left((T \cap (\text{Ann}_R(M))^{1/2}) \cup (T \cap (R - (\text{Ann}_R(M))^{1/2})) \right)} \right| \cdot |\text{Ann}_R(M)| \\
 &= \left| \overline{\left((T \cap (\text{Ann}_R(M))^{1/2}) \cup (T \cap (R - (\text{Ann}_R(M))^{1/2})) \right)} \right| \cdot |\text{Ann}_R(M)| \\
 &= |\overline{T}| \cdot |\text{Ann}_R(M)| \\
 &\leq \alpha(\Gamma(R/\text{Ann}_R(M))) \cdot |\text{Ann}_R(M)|.
 \end{aligned}$$

As a result, we obtain that $\alpha(\Gamma(R, M)) \leq \alpha(\Gamma(R/\text{Ann}_R(M))) \cdot |\text{Ann}_R(M)|$. $\square$

Corollary 29. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. Assume that $\Gamma(R, M)$ is incomplete, then*

1. *if $\text{Ann}_R(M)$ is semiprime, then $\alpha(\Gamma(R, M)) = |\text{Ann}_R(M)| \cdot \alpha(\Gamma(R/\text{Ann}_R(M)))$.*
2. *If $Z(R, M) \subset (\text{Ann}_R(M))^{1/2}$, then $\alpha(\Gamma(R, M)) = \alpha(\Gamma(R/\text{Ann}_R(M)))$.*

Proof. 1. Suppose $\text{Ann}_R(M)$ is semiprime. Then $Z(R, M) \cap (\text{Ann}_R(M))^{1/2} = \emptyset$. It follows from the same argument as in the proof of Theorem 28 that $\alpha(\Gamma(R, M)) = |\text{Ann}_R(M)| \cdot \alpha(\Gamma(R/\text{Ann}_R(M)))$.

2. Suppose $Z(R, M) \subset (\text{Ann}_R(M))^{1/2}$. Then $Z(R, M) \cap (R - (\text{Ann}_R(M))^{1/2}) = \emptyset$. A similar argument to that in the proof of Theorem 28 leads to that $\alpha(\Gamma(R, M)) = \alpha(\Gamma(R/\text{Ann}_R(M)))$. $\square$

3.3 Clique Number, Chromatic Number, and Domination Number

As the title of this section speaks, we study the cliques, coloring, and the dominating sets of $\Gamma(R, M)$. We give estimations to the clique number, the chromatic number,

and the domination number in terms of the counterparts of $\Gamma(R/Ann_R(M))$. In addition, we provide, under certain circumstances, a manner to extract the value of each number.

Lemma 30. *Let R be a commutative ring, and M an unfaithful left R -module such that $Ann_R(M)$ is not a prime ideal. Given a clique Λ of the incomplete graph $\Gamma(R, M)$, we have the following statements are true:*

1. If $r \in \Lambda \cap (R - (Ann_R(M))^{1/2})$, then $|[r] \cap \Lambda| = 1$. That is, $\Lambda \cap (R - (Ann_R(M))^{1/2})$ is a clique of $\Gamma(R, M)$ such that $\overline{\Lambda \cap (R - (Ann_R(M))^{1/2})}$ is a clique of $\Gamma(R/Ann_R(M))$ with the same cardinality of $\Lambda \cap (R - (Ann_R(M))^{1/2})$.
2. If Λ is a maximal clique of $\Gamma(R, M)$, and $r \in \Lambda \cap (Ann_R(M))^{1/2}$, then $[r] \subseteq \Lambda$.

Proof. 1. Let $r \in \Lambda \cap (R - (Ann_R(M))^{1/2})$. By Lemma 14, $[r]$ is an independent subgraph of $\Gamma(R, M)$. Hence, Λ shall contain exactly one element from $[r]$. The rest follows by applying Lemma 14.

2. Assume Λ is a maximal clique, and $r \in \Lambda \cap (Ann_R(M))^{1/2}$. As stated in Lemma 14, $[r] \cup \Lambda$ is a clique. But $\Lambda \subseteq \Lambda \cup [r]$, and Λ is a maximal clique. Thus, $\Lambda \cup [r] = \Lambda$ or $\Lambda \cup [r] = \Gamma(R, M)$. Since $\Gamma(R, M)$ is incomplete, then $\Lambda \cup [r] \neq \Gamma(R, M)$. So, we get $\Lambda = \Lambda \cup [r]$, which implies $[r] \subseteq \Lambda$. □

Theorem 31. *Let R be a commutative ring, and M an unfaithful left R -module such that $Ann_R(M)$ is not a prime ideal. Then*

$$\omega(\Gamma(R/Ann_R(M))) \leq \omega(\Gamma(R, M)) \leq |Ann_R(M)| \cdot \omega(\Gamma(R/Ann_R(M))).$$

Proof. Without loss of generality, assume $\Gamma(R, M)$ is an incomplete graph. According to Lemma 14 and Proposition 15, if $\bar{\Lambda} = \{[r_k], k \in K\}$, where K is an indexing set, is a clique in $\Gamma(R/Ann_R(M))$ such that $|\bar{\Lambda}| = \omega(\Gamma(R/Ann_R(M)))$, then $\Lambda = \{r_k, k \in K\}$, is a clique of $\Gamma(R, M)$. Thus, we obtain $\omega(\Gamma(R/Ann_R(M))) \leq$

$\omega(\Gamma(R, M))$. Next, let Λ be a maximal clique of $\Gamma(R, M)$. By Lemma 30, we have

$$\begin{aligned}
|\Lambda| &= |(\Lambda \cap (\text{Ann}_R(M))^{1/2}) \cup (\Lambda \cap (R - (\text{Ann}_R(M))^{1/2}))| \\
&= |(\Lambda \cap (R - (\text{Ann}_R(M))^{1/2}))| + |(\Lambda \cap (\text{Ann}_R(M))^{1/2})| \\
&= \left| \overline{(\Lambda \cap (R - (\text{Ann}_R(M))^{1/2}))} \right| + \sum_{[r] \in \overline{\Lambda \cap (\text{Ann}_R(M))^{1/2}}} |[r]| \\
&= \left| \overline{(\Lambda \cap R - (\text{Ann}_R(M))^{1/2})} \right| + \sum_{[r] \in \overline{\Lambda \cap (\text{Ann}_R(M))^{1/2}}} |\text{Ann}_R(M)| \\
&= \left| \overline{(\Lambda \cap (R - (\text{Ann}_R(M))^{1/2}))} \right| + \left| \overline{\Lambda \cap (\text{Ann}_R(M))^{1/2}} \right| \cdot |\text{Ann}_R(M)| \\
&\leq \left(\left| \overline{(\Lambda \cap (R - (\text{Ann}_R(M))^{1/2}))} \right| + \left| \overline{\Lambda \cap (\text{Ann}_R(M))^{1/2}} \right| \right) \cdot |\text{Ann}_R(M)| \\
&= \left(\left| \overline{(\Lambda \cap (R - (\text{Ann}_R(M))^{1/2}))} \right| + \left| \overline{\Lambda \cap (\text{Ann}_R(M))^{1/2}} \right| \right) \cdot |\text{Ann}_R(M)| \\
&= \left| \overline{((\Lambda \cap (R - (\text{Ann}_R(M))^{1/2})) \cup (\Lambda \cap (\text{Ann}_R(M))^{1/2}))} \right| \cdot |\text{Ann}_R(M)| \\
&= \left| \overline{((\Lambda \cap (R - (\text{Ann}_R(M))^{1/2})) \cup (\Lambda \cap (\text{Ann}_R(M))^{1/2}))} \right| \cdot |\text{Ann}_R(M)| \\
&= |\overline{\Lambda}| \cdot |\text{Ann}_R(M)| \\
&\leq \omega(\Gamma(R/\text{Ann}_R(M))) \cdot |\text{Ann}_R(M)|.
\end{aligned}$$

As a result, we obtain that $\omega(\Gamma(R, M)) \leq \omega(\Gamma(R/\text{Ann}_R(M))) \cdot |\text{Ann}_R(M)|$. $\square$

Corollary 32. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal. Assume that $\Gamma(R, M)$ is incomplete, then*

1. *If $\text{Ann}_R(M)$ is semiprime, then $\omega(\Gamma(R, M)) = \omega(\Gamma(R/\text{Ann}_R(M)))$.*
2. *If $Z(R, M) \subset (\text{Ann}_R(M))^{1/2}$, then*

$$\omega(\Gamma(R, M)) = |\text{Ann}_R(M)| \cdot \omega(\Gamma(R/\text{Ann}_R(M))).$$

Proof. 1. Suppose $\text{Ann}_R(M)$ is semiprime. Then $Z(R, M) \cap (\text{Ann}_R(M))^{1/2} = \emptyset$. It follows from the same argument of the proof of Theorem 31 that $\omega(\Gamma(R, M)) = \omega(\Gamma(R/\text{Ann}_R(M)))$.

2. Suppose $Z(R, M) \subset (\text{Ann}_R(M))^{1/2}$. Then $Z(R, M) \cap (R - (\text{Ann}_R(M))^{1/2}) = \emptyset$. A similar argument to that in the proof of Theorem 31 leads to that $\omega(\Gamma(R, M)) = |\text{Ann}_R(M)| \cdot \omega(\Gamma(R/\text{Ann}_R(M)))$. $\square$

Remark 33. *According to Corollaries 29 and 32, we can compute the independence and clique numbers of the graph $\Gamma(R/\text{Ann}_R(M))$ by using the counterparts of its extension $\Gamma(R, M)$. This reveals the significance of $\Gamma(R, M)$.*

For the chromatic number of $\Gamma(R, M)$, we introduce the following theorem.

Theorem 34. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is a semiprime ideal but not a prime ideal of R . Then*

$$\chi(\Gamma(R, M)) = \chi(\Gamma(R/\text{Ann}_R(M))).$$

Proof. Let $\{\bar{S}_1, \dots, \bar{S}_k\}$ be a family of coloring classes of $\Gamma(R/\text{Ann}_R(M))$, where k equals the chromatic number of $\Gamma(R/\text{Ann}_R(M))$. Let $S_i = \bar{f}^{-1}(\bar{S}_i)$ for every $i = 1, \dots, k$. By Lemma 14, $S_i = \bigcup_{[r] \in \bar{S}_i} [r]$ is an independent set of $\Gamma(R, M)$, for every $i = 1, \dots, k$. Thus, $\{S_1, \dots, S_k\}$ is a family of coloring classes of $\Gamma(R, M)$. Therefore, $\chi(\Gamma(R/\text{Ann}_R(M))) = k \geq \chi(\Gamma(R, M))$. Now, let $\{S_1, \dots, S_m\}$ be a family of coloring classes of $\Gamma(R, M)$. Since $\text{Ann}_R(M)$ is semiprime, by Lemma 14, $[r]$ is an independent set for every vertex r in $\Gamma(R, M)$. Fix $i \in \{1, \dots, m\}$. If $x \in S_i$, then there exist $j \in \{1, \dots, m\}$ such that $j \neq i$, and $y \in S_j$ such that $x \leftrightarrow y$. Since $x + a \leftrightarrow y + a'$ for every $a, a' \in \text{Ann}_R(M)$, we can move all the elements of $[x]$ to inside S_i to obtain a new family of coloring classes consisting of at most m classes. Repeating this procedure to each $i \in \{1, \dots, m\}$ and to every element $x \in S_i$, we eventually obtain a new family $\mathcal{S}$ of coloring classes with at most m classes with the property that $[x] \cap S = \emptyset$ or $[x] \cap S = [x]$ for every $S \in \mathcal{S}$ and $x \in Z(R, M)$. Consider the family $\{\bar{S} : S \in \mathcal{S}\}$, where $\bar{S} = f(S)$. By Lemma 14, this is a family of coloring classes of $\Gamma(R/\text{Ann}_R(M))$. Thus $m \geq |\mathcal{S}| \geq \chi(\Gamma(R/\text{Ann}_R(M)))$. Taking the minimum on m , we get $\chi(\Gamma(R, M)) \geq \chi(\Gamma(R/\text{Ann}_R(M)))$. As a conclusion, we obtain $\chi(\Gamma(R, M)) = \chi(\Gamma(R/\text{Ann}_R(M)))$. $\square$

In the final part of this subsection, we turn to the dominating sets and the domination number of $\Gamma(R, M)$.

Theorem 35. *Let R be a commutative ring, and M an unfaithful left R -module such that $\text{Ann}_R(M)$ is not a prime ideal of R . Then,*

$$\gamma(\Gamma(R/\text{Ann}_R(M))) \leq \gamma(\Gamma(R, M)) \leq \gamma(\Gamma(R/\text{Ann}_R(M))) \cdot |\text{Ann}_R(M)|.$$

Further, if $Z(R, M) \subseteq (\text{Ann}_R(M))^{1/2}$, then $\gamma(\Gamma(R/\text{Ann}_R(M))) = \gamma(\Gamma(R, M))$. However, if $Z(R, M) \cap (\text{Ann}_R(M))^{1/2} = \emptyset$, then

$$\gamma(\Gamma(R/\text{Ann}_R(M))) \leq \gamma(\Gamma(R, M)) \leq 2 \cdot \gamma(\Gamma(R/\text{Ann}_R(M))).$$

Proof. Let $D = \{r_i, i \in I\}$, where I is an indexing set, be a dominating set of $\Gamma(R, M)$ such that $|D| = \gamma(\Gamma(R, M))$. It is not difficult to see, by the virtue of Lemma 14, that $\bar{D} = \{[r_i], i \in I\}$ is a dominating set of $\Gamma(R/\text{Ann}_R(M))$ such that $|D| \geq |\bar{D}| \geq \gamma(\Gamma(R/\text{Ann}_R(M)))$. Thus, we get $\gamma(\Gamma(R, M)) \geq \gamma(\Gamma(R/\text{Ann}_R(M)))$. On the other hand, Let $\bar{D} = \{[r_i], i \in I\}$, where I is an indexing set, be a dominating set of $\Gamma(R/\text{Ann}_R(M))$ such that $|\bar{D}| = \gamma(\Gamma(R/\text{Ann}_R(M)))$. It is simple to see that $\bigcup_{i \in I} [r_i]$ is a dominating set of $\Gamma(R, M)$. So,

$$\gamma(\Gamma(R/\text{Ann}_R(M))) \cdot |\text{Ann}_R(M)| = |\bar{D}| \cdot |\text{Ann}_R(M)| = |D| \geq \gamma(\Gamma(R, M)).$$

Next, suppose that $Z(R, M) \subseteq (Ann_R(M))^{1/2}$. By Lemma 14, $[r]$ is a clique for every vertex $r \in Z(R, M)$. Therefore any dominating set D of $\Gamma(R, M)$, can be replaced by a dominating set D' of $\Gamma(R, M)$ with less than or equal cardinality and such that it consists of pairwise distinct elements modulo $Ann_R(M)$. The procedure runs through the following steps:

1. If D is already consisting of pairwise distinct elements modulo $Ann_R(M)$. Then $D = D'$ and hence $|D| = |D'|$.
2. Assume there exist $r, r' \in D$ such that $[r] = [r']$. Since

$$Z(R, M) \subseteq (Ann_R(M))^{1/2},$$

we have $r \leftrightarrow r'$. Also, every element adjacent to r' is adjacent to r . Hence, we move r' to outside D to obtain a dominating set D_1 such that $|D_1| = |D| - 1$. We repeat the procedure until D contains only pairwise distinct elements modulo $Ann_R(M)$.

Consequently, if $\overline{D} = \{[r_i] : i \in I\}$, where I is an indexing set, is a dominating set of $\Gamma(R/Ann_R(M))$ whose cardinality is equal to $\gamma(\Gamma(R/Ann_R(M)))$, then according the above procedure, we can replace the dominating set $D = \bigcup_{i \in I} [r_i]$ of $\Gamma(R, M)$ by

another dominating set D' where $|D'| = |\overline{D}|$. Therefore,

$$\gamma(\Gamma(R/Ann_R(M))) = |\overline{D}| = |D'| \geq \gamma(\Gamma(R, M)).$$

We conclude that $\gamma(\Gamma(R/Ann_R(M))) = \gamma(\Gamma(R, M))$.

Now, assume $Z(R, M) \cap (Ann_R(M))^{1/2} = \emptyset$. By Lemma 14, $[r]$ is an independent set for each vertex r in $\Gamma(R, M)$. Again, any dominating set D of $\Gamma(R, M)$, can be replaced by a dominating set D' of $\Gamma(R, M)$ with less than or equal cardinality and such that it consists of pairwise distinct elements modulo $Ann_R(M)$. The procedure runs through the following steps:

1. If D is already consisting of pairwise distinct elements modulo $Ann_R(M)$. Then $D = D'$ and hence $|D| = |D'|$.
2. Assume there exist $r, r' \in D$ such that $[r] = [r']$. Now $r \not\leftrightarrow r'$, but it is still true that every element adjacent to r' is also adjacent to r . By the definition of $Z(R, M)$, there exists $r'' \neq r \pmod{Ann_R(M)}$ such that $r' \leftrightarrow r''$. If $r'' \in D$, then we move r' outside D . The new set $D_1 = D - \{r'\}$ is a dominating set with less cardinality than D . If $r'' \notin D$, then we swap the positions of r' and r'' . The new set $D_1 = (D \cup \{r''\}) - \{r'\}$ is a dominating set with the same cardinality of D . We repeat this procedure until all elements of D are pairwise distinct modulo $Ann_R(M)$.

To this point, let $\overline{D} = \{[r_i] : i \in I\}$, where I is an indexing set, be a dominating set of $\Gamma(R/Ann_R(M))$ whose cardinality is equal to $\gamma(\Gamma(R/Ann_R(M)))$. After

applying the procedure above to the dominating set $D = \bigcup_{i \in I} [r_i]$ of $\Gamma(R, M)$, we replace every coset $[r_i]D$ with one element or two elements in D' . So, the final resulting dominating set D' has a cardinality less than or equal twice the cardinality of $\overline{D}$, and contains pairwise distinct elements modulo $\text{Ann}_R(M)$. Therefore, we have

$$\gamma(\Gamma(R, M)) \leq |D'| \leq 2|\overline{D}| = 2 \cdot \gamma(\Gamma(R/\text{Ann}_R(M)))$$

as claimed. $\square$

4. APPLICATIONS

This section is devoted to applications of the results established in the previous section.

Application 36. Let $R = \mathbb{Z}_{pq} = \mathbb{Z}_p \oplus \mathbb{Z}_q$ where q and p are distinct prime numbers. Let $M = \mathbb{Z}_p[x]$. Then $\text{Ann}_R(M) = \mathbb{Z}_q$ which is a maximal ideal of R and hence prime. We have $Z(R, M) = \{(r, 0) : r \in \mathbb{Z}_p - 0\}$. In fact, $\Gamma(R, M)$ is a null graph.

Application 37. Let $R = \mathbb{Z}_{p^3}$ where p is a prime number. Let $M = Rp$. Then $\text{Ann}_R(M) = Rp^2$ and $Z(R, M) = Rp - Rp^2$. We have $R/\text{Ann}_R(M) \cong \mathbb{Z}_{p^2}$ which is a ring of square zero. Also, $\text{Ann}_R(M) \cup Z(R, M) = Rp$ which is an ideal and hence a subring of R . By Theorem 22, $\Gamma(R, M)$ is a complete graph.

Application 38. Let $m = p_1 p_2 \dots p_n$, where $p_1, \dots, p_n$ are distinct prime numbers such that $p_1 < p_2 < \dots < p_n$, and $n \geq 5$. Let $R = \mathbb{Z}_m = \mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_n}$, and $M = \mathbb{Z}_{p_1 \dots p_k}$, where $2 < k < n - 1$. Now, $\text{Ann}_R(M) = \mathbb{Z}_{p_{k+1} \dots p_n} \not\cong \mathbb{Z}_2$ and $R/\text{Ann}_R(M) = M = \mathbb{Z}_{p_1 \dots p_k} = \mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k}$, which is not an integral domain. Thus $\text{Ann}_R(M)$ is not prime.

- Because $\text{Ann}_R(M)$ is semiprime, and according to Theorem 25, $\Gamma(R, M)$ has a cycle. Since $g(\Gamma(R/\text{Ann}_R(M))) = 3$ (we have the triangle $r_1 \leftrightarrow r_2 \leftrightarrow r_3 \leftrightarrow r_1$, where $r_i \in \mathbb{Z}_{p_i}$ for $i = 1, 2, 3$), we get by Corollary 26 that $g(\Gamma(R, M)) = 3$.
- Once again, because $\text{Ann}_R(M)$ is semiprime, it follows from Corollary 32 that $\omega(\Gamma(R, M)) = \omega(\Gamma(R/\text{Ann}_R(M)))$. We show that $\omega(\Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})) = k$. We have $Z(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})^* = \{(r_1, \dots, r_k) : r_1 \in \mathbb{Z}_{p_1}, \dots, r_k \in \mathbb{Z}_{p_k}, \text{ and at least one of } r_1, \dots, r_k \text{ is } 0 \text{ but not all of them are zeros}\}$. By the pigeonhole principle (the pigeons are the elements of $Z(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})^*$ and the holes are the sets $\mathbb{Z}_{p_1}, \dots, \mathbb{Z}_{p_k}$, any collection of $k + 1$ elements of $Z(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})^*$ includes at least two elements whose i th coordinates (for some $i \in \{1, \dots, k\}$) are not equal to zero, and hence they are not adjacent in $\Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})$. This means any $k + 1$ vertices of $\Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})$ can't form a clique. We conclude that $\omega(\Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})) \leq k$. Now, set $e_i \in \Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})$ to be the k -tuple whose coordinates are 0 except the i th

coordinate which is equal to 1. The set $\Lambda = \{e_i : i = 1, \dots, k\}$ form a clique. As a result, $\omega(\Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})) = k$. From the argument above, we obtain $\omega(\Gamma(R, M)) = k$.

- By Corollary 29, since $\text{Ann}_R(M)$ is semiprime, we have

$$\alpha(\Gamma(R, M)) = |\text{Ann}_R(M)| \cdot \alpha(\Gamma(R/\text{Ann}_R(M))).$$

We now compute $\alpha(\Gamma(R/\text{Ann}_R(M)))$. Let T be a maximal independent set of $\Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})$. We claim that there is $i \in \{1, \dots, k\}$ such that the i th coordinate of every vertex in T is not zero. Assume, for contrary, that for every $i \in \{1, \dots, k\}$, there is an element $r_i \in T$ such that the i th coordinate of r_i is 0. By induction on i , where $i \in \{1, \dots, k\}$, the set $\{r_1, \dots, r_i\}$ is independent if there exists $j > i$ such that the j th coordinate of each of $r_1, \dots, r_i$ is nonzero. It follows that for $i = k$, the set $\{r_1, \dots, r_k\}$ is independent if there is $j > k$ such that the j th coordinate of each of $r_1, \dots, r_k$ is nonzero, which is impossible. Thus, there is $i \in \{1, \dots, k\}$ such that the i th coordinate of every vertex in T is not zero. So, the maximal independent sets have the form $(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_i}^* \oplus \dots \oplus \mathbb{Z}_{p_k}) - (\mathbb{Z}_{p_1}^* \oplus \dots \oplus \mathbb{Z}_{p_{i-1}}^* \oplus \mathbb{Z}_{p_{i+1}}^* \oplus \dots \oplus \mathbb{Z}_{p_k}^*)$, for every $i \in \{1, \dots, k\}$. This implies $(\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \dots \oplus \mathbb{Z}_{p_{k-1}} \oplus \mathbb{Z}_{p_k}^*) - (\mathbb{Z}_{p_1}^* \oplus \mathbb{Z}_{p_2}^* \oplus \dots \oplus \mathbb{Z}_{p_{k-1}}^*)$ is a maximal independent set with the largest number of vertices. So, $\alpha(\Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})) = (p_k - 1) \cdot p_{k-1} \cdots p_2 \cdot p_1 - ((p_{k-1} - 1) \cdots (p_2 - 1) \cdot (p_1 - 1))$. Finally, $\alpha(\Gamma(R, M)) = p_n \cdots p_{k+1} \cdot [(p_k - 1) \cdot p_{k-1} \cdots p_2 \cdot p_1 - ((p_1 - 1)(p_2 - 1) \cdots (p_{k-1} - 1))] = |R| - \frac{|R|}{p_k} - \prod_{i=1}^{k-1} (p_i - 1) \cdot \prod_{i=k+1}^n p_i$.

- Let $\mathcal{F} = \{S_1, \dots, S_k\}$ be the family of sets of vertices defined by $S_i = \mathbb{Z}_{p_i}^* \oplus \mathbb{Z}_{p_{i-1}} \oplus \dots \oplus \mathbb{Z}_{p_1}$, where $i = 1, \dots, k$. We have by the above item, $|S_1| < |S_2| < \dots < |S_k|$, and each S_i is an independent set such that every vertex in S_i is adjacent to every vertex outside S_i . So, $\mathcal{F}$ is a family of k coloring classes of $\Gamma(R/\text{Ann}_R(M)) = \Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})$. The construction of $\mathcal{F}$ shows that we cannot construct a family of coloring classes with less than k classes (to obtain a family with a less number of classes, we have to merge some classes of $\mathcal{F}$ together. However, this merger produces non-independent sets). Therefore, by Theorem 34, we get $\chi(\Gamma(R, M)) = \chi(\Gamma(R/\text{Ann}_R(M))) = k$.
- By Theorem 35, we have

$$\gamma(\Gamma(R/\text{Ann}_R(M))) \leq \gamma(\Gamma(R, M)) \leq 2\gamma(\Gamma(R/\text{Ann}_R(M))).$$

We claim that $\gamma(\Gamma(R/\text{Ann}_R(M))) = k$. We have $\dot{D} = \{e_1, e_2, \dots, e_k\}$ is a dominating set. To see this, let $r = (z_1, \dots, z_k)$ be a vertex of $\Gamma(R/\text{Ann}_R(M)) = \Gamma(\mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_k})$. Then, $z_i = 0$ for some $i \in \{1, \dots, k\}$. Thus, $r \leftrightarrow e_i$. Moreover, any proper subset of $\dot{D}$ cannot be a dominating set. For example, $\dot{D} - \{e_1\}$ is not a dominating set because $(0, 1, 1, \dots, 1) \not\leftrightarrow e_i$, for every $e_i \neq e_1$. Similar proof is carried out for any proper subset of $\dot{D}$. We conclude that D is a minimal dominating set. Thus $\gamma(\Gamma(R/\text{Ann}_R(M))) \leq k$.

We prove that any set D of vertices of $\Gamma(R/\text{Ann}_R(M))$ with cardinality less than k is not a dominating set. For such a set D , there exists $i \in \{1, \dots, k\}$ such that the i th coordinate of all elements of D is zero. Now, consider the element $v = (z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_k)$ where all the coordinates are different from 0 and chosen in a way such that $v \notin D$ (this is possible because $|D| < k$). It is easy to see that $v \not\sim r$ for every $r \in D$. We conclude that $\gamma(\Gamma(R/\text{Ann}_R(M))) = k$. As a result we have

$$k \leq \gamma(\Gamma(R, M)) \leq 2k.$$

REFERENCES

1. M. AHMED AND F. MOH'D: *A new intersection-graph type for modules*. Comm. Algebra, **52** (2024), no. 5, 2065–2078.
2. M. AHMED AND F. MOH'D: *The Graded Annihilating Submodule Graph*. AKCE Int. J. Graphs Comb., **22** (2025), no. 3, 270–278.
3. T. ALRAQAD, H. SABER, R. ABU-DAWWAS: *Intersection Graph of Graded ideals of graded rings*. AIMS Math., **6** (2022), no. 10, 10355–10368.
4. F.W. ANDERSON AND K.R. FULLER: *Rings and Categories of Modules: Graduate Texts in Mathematics*. **13**, 2nd Ed., Springer-Verlag, New York, 1992.
5. D. F. ANDERSON AND P. S. LIVINGSTON: *The zero-divisor graph of a commutative ring*. J. Algebra **217** (1999), no. 2, 434–447.
6. I. BECK: *Coloring of commutative rings.*, J. Algebra **116** (1988), 208–226.
7. J. A. BONDY AND U. S. R. MURTY: *Graph Theory*. Splett moduleer-Verlag, London, 2011.
8. F. MOH'D: *A Generalization of Beck's Zero-divisor Graph: The Graph of Type N Associated with a $\mathbb{Z}_2$ -Graded Ring*. Discrete Math. Algorithms Appl., published online 2025. (<https://doi.org/10.1142/S1793830925501113>).
9. F. MOH'D, M. AHMED: *A Simple-Intersection Graph of a Ring Approach to Solving Coloring Optimization Problems.*, Commun. Comb. Optim., **10** (2025), no. 2, 423–442.
10. F. MOH'D AND M. AHMED: *Ring graph induced by a ring endomorphism*. Open Math., **23** (2025), no. 1, pp. 20250197.
11. F. MOH'D AND M. AHMED: *Simple-intersection graphs of rings.*, AIMS Math., **8**, no. 1, 1040–1054.
12. Y. TIAN AND L. LI: *Comments on the Clique Number of Zero-Divisor Graphs of $\mathbb{Z}_n$.*, Journal of Mathematics, **2022**, Article ID 6591317.

Fida Moh'd

Department of Basic Sciences,
Princess Sumaya University for Technology,
Amman, Jordan
E-mail: f.mohammad@psut.edu.jo
<https://orcid.org/0000-0003-4897-8684>

(Received 24.11.2024.)

(Revised 18.03.2026.)

Mamoon Ahmed

Department of Basic Sciences,
Princess Sumaya University for Technology,
Amman, Jordan
E-mail: *m.ahmed@psut.edu.jo*
<https://orcid.org/0000-0001-8368-6922>