

LOWER BOUNDS ON THE NUMBER OF ENVY-FREE DIVISIONS

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We determine lower bounds for the number of envy-free divisions in two cases.

(A) In the classical *Stromquist-Woodall setting* (r “hungry players” and the “cake” is cut into r pieces), there always exist at least two different envy-free divisions. Examples are constructed showing that this bound is sharp.

(B) The cake is cut into $2p - 1$ pieces (for prime p) and p non-hungry players can take multiple pieces. Then at least $\binom{2p-1}{p-1} \cdot 2^{2-p}$ envy-free divisions are possible, even if they include an equidistribution of a measure on the cake.

1. INTRODUCTION

Given some resource, identified with the unit segment $I = [0, 1]$ (which is referred to as the *cake*) and a set of players, one of the goals of welfare economics is to divide the resource among them in an *envy-free* manner. Envy-freeness is the principle where every player feels that their share is at least as good as the share of any other player, and thus nobody feels envy.

A *cut* of the cake is a sequence of numbers $x = (x_1, \dots, x_r)$ where

$$(1) \quad \sum_{i=1}^r x_i = 1, \quad x_i \geq 0,$$

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so the set of all cuts is naturally identified as the standard $(r - 1)$ -dimensional simplex Δ^{r-1} .

The pieces of the cake arising from the cut (1) are the closed intervals (*tiles*) whose lengths are x_i .

After a cut x is chosen, each of the players expresses her individual *preferences*, over the tiles arising from this particular cut, by pointing to one (or more) intervals which they like more than the rest.

This idea is formalized in the following definition, where the individual preferences are interpreted as subsets of the simplex Δ^{r-1} .

Definition 1.1. ([10, Definition 2.2]) *The preferences of r players is a $(r \times r)$ -matrix of subsets $\mathcal{A} = (A_i^j)_{i \in [r]}^{j \in [r]}$ of the standard simplex Δ^{r-1} . The subsets are interpreted as preferences of the players in the following sense:*

$$(2) \quad x \in A_i^j \Leftrightarrow \text{in the cut } x \text{ the player } j \text{ prefers the tile } i.$$

Characteristic for the **Stromquist–Woodall preferences**, see [11] and [13], are the following conditions, which originally appeared already in the classical Knaster-Kuratowski-Mazurkiewicz theorem (KKM), [6].

- (**P_{cl}**) The preferences (the sets A_i^j) are *closed*.
- (**P_{cov}**) The preferences form a covering: $\bigcup_{i=1}^r A_i^j = \Delta^{r-1}$, $\forall j = 1, \dots, r$. This means that whatever a cut is, each player is expected to prefer at least one of the offered tiles.
- (**P_{hung}**) The players are hungry: they never prefer degenerate tiles. That is, A_i^j does not intersect the facet of Δ^{r-1} defined by $x_i = 0$.

A *division* of the interval $[0, 1]$ is, by definition, a pair (x, σ) consisting of a cut (1) of the cake, and an *allocation* $\sigma : [r] \rightarrow [r]$, that is, a bijection between (the labels of) the tiles and players.

A fundamental result [11, 13], was independently discovered and almost simultaneously proved by Stromquist and Woodall in 1980. It says that an envy-free division for Stromquist–Woodall preferences is always possible. In other words, for preferences satisfying the conditions (**P_{cl}**), (**P_{cov}**), (**P_{hung}**), there always exists a pair (x, σ) such that

$$(3) \quad x \in A_{\sigma(1)}^1 \cap A_{\sigma(2)}^2 \cap \dots \cap A_{\sigma(r)}^r.$$

1.1 Envy-free division with secretive or expelled players

The history of envy-free divisions is rich and interesting, see for example [1, 2, 3, 4, 5, 7, 8, 9, 10], etc. Here we summarize some of the central results, directly related to the subject of our paper.

- (a) D.R. Woodall proved (already in [13]) a refined version of the envy-free division-theorem with a *secretive player*.
- (b) Woodall's result was rediscovered by Asada et al. [1], with a much simpler proof, under the name *Strong colorful KKM theorem*.
- (c) Meunier and Su, applying the method of multilabeled Sperner lemmas, offered [7, Theorem 2.4] a unified point of view on envy-free division with secretive players, as well as the result where some of the players (unknown in advance) are "voted off the group".
- (d) Closely related are the problems of "envy-free division in the presence of a dragon", studied in [9], in a broader setting of non-hungry players.

The following theorems illustrate the role of *secretive* and *expelled* players.

Theorem 1.2. (Secretive player) [13, 1, 7, 9] *There are $r - 1$ players and the cake is divided into r pieces. If the preferences of players $\mathcal{A} = \{A_i^j\}_{i \in [r]}^{j \in [r-1]}$, satisfy classical KKM conditions, then there exists a cut $x \in \Delta^{r-1}$ and r bijections $\sigma_i : [r - 1] \rightarrow [r] \setminus \{i\}$, ($i \in [r]$), such that for each $i \in [r]$,*

$$x \in A_{\sigma_i(1)}^1 \cap A_{\sigma_i(2)}^2 \cap \cdots \cap A_{\sigma_i(r-1)}^{r-1}.$$

In simple words, there exists a cut such that whatever tile is taken by the secretive player r , the rest of the tiles can be allocated in an envy-free manner.

Theorem 1.3. (Expelled player) [7, 9] *There are $r + 1$ players and the cake is divided into r pieces. If the preferences of players $\{A_i^j\}_{i \in [r]}^{j \in [r+1]}$, satisfy classical KKM conditions, then there exists a cut $x \in \Delta^{r-1}$ and $r + 1$ bijections $\tau_j : [r] \rightarrow [r + 1] \setminus \{j\}$, ($j \in [r + 1]$), such that for each $j \in [r + 1]$,*

$$x \in A_1^{\tau_j(1)} \cap A_2^{\tau_j(2)} \cap \cdots \cap A_r^{\tau_j(r)}.$$

In simple words, there exists a cut such that, whoever is the expelled player, the tiles can be allocated to the remaining players in an envy-free manner.

Remark 1.4. The bijections σ_i and τ_j respectively go from players to tiles (in Theorem 1.2) and from tiles to players (in Theorem 1.3). They both describe the matchings in the corresponding bipartite graphs so, in principle, one can alternatively use their inverses $(\sigma_i)^{-1}$ and $(\tau_j)^{-1}$. This would lead to similar formulas, where upper and lower indices interchange places. Note however that σ_i and τ_j have, unlike their inverses, fixed domains, which is the reason why the present formulas are simpler and more natural.

1.2 The central new results in the paper

The problem of estimating the number of envy-free divisions doesn't seem to have received much attention in the existing literature. Here we analyze two problems

of this type, both fairly well-known and studied with the emphasis on the existence of a solution.

By shifting the emphasis to the number of solutions here we determine:

1. A sharp lower bound for the number of envy-free division in the classical Stromquist-Woodall setting, i.e. for players with standard KKM preferences.
2. Lower bounds for combined envy-free division and equipartition of a measure, in the spirit of [5].

(1) Our first result (Proposition 2.6) states that each envy-free division problem in the Stromquist-Woodall setting has at least two distinct solutions. This bound is sharp for each number of players, as shown by an example constructed in Theorem 2.8.

In Section 3 we focus on preferences which are *extremal* in the sense that they have precisely two distinct envy-free divisions. We show (Proposition 3.10) that extremal preferences have only one cut, allowing two different envy-free allocations of pieces.

These two allocations define a bipartite graph on the sets of players and pieces. As shown in Theorem 3.12, the only bipartite graph that arises in extremal preferences by this construction is a complete cycle (of size $2r$).

(2) In Section 4 we focus on a hybrid problem where the number of cuts (and tiles) is roughly twice as big as the number of players. This allows us (Theorem 4.16) to achieve both an envy-free division among the players and an equipartition of a measure, which is prescribed in advance. As before, the emphasis is not only on the existence of a solution but includes an estimate of the number of simultaneous envy-free divisions and measure equipartitions.

It is interesting that in this case the lower bound $\binom{2p-1}{p-1} \cdot 2^{2-p}$ is much larger, illustrating different behaviour of relatively similar problems of envy-free division.

2. THE EXISTENCE OF TWO DIFFERENT ENVY-FREE DIVISIONS

Relying on the “secretive player theorem” (Theorem 1.2), here we show (Proposition 2.6) that there always exist at least two different envy-free divisions. This result is somewhat unexpected and surprising, considering that Stromquist and Woodall established the existence of envy-free divisions almost half a century ago and our main tool, the secretive player theorem, was also proved by D.R. Woodall. Indeed, it appeared as Theorem 4 in [13], almost side by side with the original envy-free division theorem.

Proposition 2.6 is in general the best possible since, as demonstrated in Theorem 2.8, there are examples of preferences where the number of envy-free divisions is exactly two.

The following definition is just a useful reformulation of the preference relation (2), in the language of bipartite graphs.

Definition 2.5. Let $\mathcal{A} = (A_i^j)_{i \in \mathcal{T}}^{j \in \mathcal{P}}$ ($\mathcal{P} = \text{Players}, \mathcal{T} = \text{Tiles}$) be a matrix of preferences and let $x \in \Delta^{r-1}$ be a cut. The (bipartite) preference graph $\Gamma_x = \Gamma_x(\mathcal{A})$ is a subgraph of the complete bipartite graph $\text{Players} \times \text{Tiles}$, where $(j, i) \in \Gamma_x$ if and only if i is a tile preferred by j in the cut x .

Proposition 2.6. Suppose $\mathcal{A} = (A_i^j)_{i \in [r]}^{j \in [r]}$ are the Stromquist-Woodall preferences, meaning that they satisfy the Knaster-Kuratowski-Mazurkiewicz (KKM) conditions $(\mathbf{P}_{\text{cl}}), (\mathbf{P}_{\text{cov}}), (\mathbf{P}_{\text{hung}})$. Then there always exist at least two distinct envy-free divisions $(x_1, \sigma_1) \neq (x_2, \sigma_2)$ such that for each $i \in \{1, 2\}$,

$$x_i \in A_{\sigma_i(1)}^1 \cap A_{\sigma_i(2)}^2 \cap \cdots \cap A_{\sigma_i(r)}^r.$$

Proof. We use the *secretive player theorem* (Theorem 1.2). The proof can be summarized as follows.

Let us assume there is exactly one cut which gives an envy-free allocation, and prove that in this case this cut admits at least two envy-free allocations. Indeed, we may pretend that the player 1 is secretive. This means that each of the players $2, 3, \dots, r$ prefers at least two tiles that arise from this cut (otherwise the secretive player 1 would be able to take this tile, with contradicting consequences). Now let us declare that the player 2 is secretive. He may take any of the two tiles he prefers, and we have an allocation of the rest.

The proof above is essentially complete but it may be instructive to supply some details explaining, for example, what we mean when we say “let’s pretend that a player is secretive”, etc.

Here is a more formal argument, essentially based on the same ideas, but with more explicit use of Theorem 1.2.

Suppose (x, σ) is an envy-free division (which exists by Stromquist-Woodall theorem). Assume it is the only one, meaning that for any cut $x' \neq x$, there is no envy-free division associated with x' . It remains to show that there are at least two different envy-free divisions associated to the cut x , or equivalently, that there exist two different complete matchings in the associated preference graph Γ_x (Definition 2.5). Recall that $(j, i) \in \Gamma_x \subseteq \text{Players} \times \text{Tiles} \cong [r] \times [r]$ precisely if the player j prefers the tile i .

We claim that each player has at least two neighbors in Γ_x . In the opposite case let $e = (j, i) \in [r] \times [r]$ be the only edge adjacent to the player j . Choose a player k different from j . Theorem 1.2 says that we can *declare* any player secretive, say the player k . What we really want to say is that after removing the tile i and the edge (k, i) , then there must exist a perfect matching in the graph $\Gamma' := \Gamma \cap ([r] \setminus \{k\}) \times ([r] \setminus \{i\})$, which is a contradiction (since j has no neighbors in Γ').

For the completion of the proof choose and declare secretive an arbitrary player $k \in [r]$. Assuming $j_1 \neq j_2$ are two Γ_x -neighbors of k and applying Theorem 1.2 to $\Gamma \setminus \{(k, j_\nu)\}$, $\nu = 1, 2$, we obtain two distinct envy-free divisions (complete matchings) in the graph Γ_x , associated to the cut x . $\square$

The following example illustrates both the Proposition 2.6 and forthcoming Theorem 2.8. It may also provide additional insight into a geometric interpretation (visualization) of preferences, as closed subsets of the fundamental simplex Δ^{r-1} .

Example 2.7. For $r = 3$ there exist preferences that have exactly two different envy-free divisions, see Fig. 1.

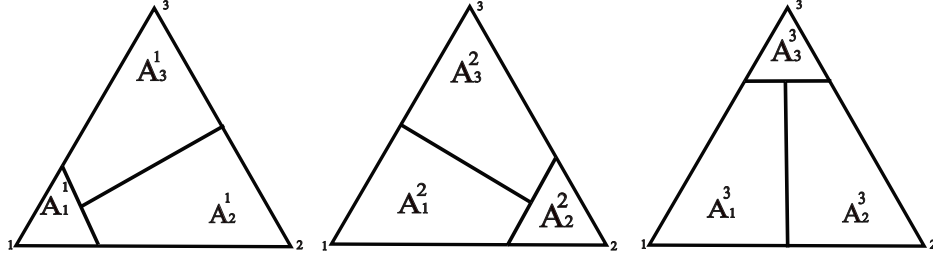


Figure 1: An example of preferences for three players.

Let us find, for the preferences exhibited in Figure 1, all envy-free divisions (x, σ) which, in agreement with (3), satisfy the relation

$$x \in A_{\sigma(1)}^1 \cap A_{\sigma(2)}^2 \cap A_{\sigma(3)}^3.$$

There are 6 possible permutation σ and, by inspection, we find out that the only common point (admissible cut) is the barycenter $O = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of the triangle Δ^2 , while the corresponding envy-free divisions are:

$$A_2^1 \cap A_3^2 \cap A_1^3 = \{O\} = A_3^1 \cap A_1^2 \cap A_2^3.$$

The following theorem generalizes Example 2.7 to any number of players.

Theorem 2.8. *For any $r \geq 2$ there exist preferences that admit exactly two different envy-free divisions.*

Proof. In the case of two players we choose a point $0 < \alpha < 1$ and define

$$A_1^1 = A_1^2 = [0, \alpha] \quad \text{and} \quad A_2^1 = A_2^2 = [\alpha, 1].$$

The corresponding envy-free divisions are

$$A_1^1 \cap A_2^2 = \{\alpha\} = A_2^1 \cap A_1^2.$$

Let us turn now to the general case $r \geq 3$. We construct a set (matrix) $(A_i^j)_{i \in [r]}^{j \in [r]}$ of preferences with exactly two envy-free divisions, which have one and the same cut of the cake $[0, 1]$. In other words, the cut is the same but the tiles can be allocated in an envy-free fashion in two different ways.

Choose a descending sequence of “small numbers” $\{\varepsilon_i\}_{i=1}^{r-2}$, which satisfy the inequality

$$1/r > \varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_{r-2} > 0.$$

First define the preferences $A_i^1 \subset \Delta^{r-1}$ of the first player:

$$A_1^1 := \{x \in \Delta^{r-1} \mid x_1 \geq 1 - \varepsilon_1\} \text{ which we shorten as } \{x_1 \geq 1 - \varepsilon_1\}$$

$$A_2^1 := \{x_1 \leq 1 - \varepsilon_1, x_1 + x_2 \geq 1 - \varepsilon_2\}$$

$$A_3^1 := \{x_1 \leq 1 - \varepsilon_1, x_1 + x_2 \leq 1 - \varepsilon_2, x_1 + x_2 + x_3 \geq 1 - \varepsilon_3\}$$

.....

$$A_{r-2}^1 := \{x_1 \leq 1 - \varepsilon_1, \dots, \sum_{i=1}^{r-3} x_i \leq 1 - \varepsilon_{r-3}, \sum_{i=1}^{r-2} x_i \geq 1 - \varepsilon_{r-2}\}$$

At the last two steps we change the rule and set:

$$A_{r-1}^1 := \{x \in \Delta^{r-1} \mid x_{r-1} \geq x_r\} \setminus \text{Int}\left(\bigcup_{i=1}^{r-2} A_i^1\right)$$

$$A_r^1 := \{x \in \Delta^{r-1} \mid x_r \geq x_{r-1}\} \setminus \text{Int}\left(\bigcup_{i=1}^{r-2} A_i^1\right)$$

The preferences A_i^j for $j \geq 2$ are defined by the substitution

$$(4) \quad A_i^1 \longrightarrow A_{i+j-1}^j, \quad x_i \rightarrow x_{i+j-1}$$

or more explicitly,

$$A_j^j := \{x_j \geq 1 - \varepsilon_1\}$$

$$A_{j+1}^j := \{x_j \leq 1 - \varepsilon_1, x_j + x_{j+1} \geq 1 - \varepsilon_2\}$$

.....

$$A_{j+r-3}^j := \{x_j \leq 1 - \varepsilon_1, \dots, \sum_{i=j}^{j+r-4} x_i \leq 1 - \varepsilon_{r-3}, \sum_{i=j}^{j+r-3} x_i \geq 1 - \varepsilon_{r-2}\}$$

$$A_{j+r-2}^j := \{x \in \Delta^{r-1} \mid x_{j+r-2} \geq x_{j+r-1}\} \setminus \text{Int}\left(\bigcup_{i=1}^{r-2} A_{i+j-1}^j\right)$$

$$A_{j+r-1}^j := \{x \in \Delta^{r-1} \mid x_{j+r-1} \geq x_{j+r-2}\} \setminus \text{Int}\left(\bigcup_{i=1}^{r-2} A_{i+j-1}^j\right)$$

The substitution (4) means that there is an action of the cyclic group of order r on the set of preferences. For this reason the indices in A_i^j and x_i are always understood modulo r .

We continue by showing that for each envy-free division (x, σ) of the constructed set of preferences $\mathcal{A} = (A_i^j)$, the corresponding cut x must be the barycentric cut $x = b_0 = (\frac{1}{r}, \frac{1}{r}, \dots, \frac{1}{r})$. Assuming the opposite, let $x = (x_1, x_2, \dots, x_r) \in \Delta^{r-1} \setminus \{b_0\}$ be a different cut of the cake, which admits an envy-free division.

Let $m = \max\{x_i\}$ and let k_0 be an index such that $x_{k_0-1} < m = x_{k_0}$. In light of the cyclic group action (4), we can assume that $k_0 = 1$ and $x_r < m = x_1$. Also note that, by assumption, $m > \frac{1}{r} > \varepsilon_1$.

Case 1. This is a warming-up case which highlights the idea of the proof. Let us first analyze the case $m = x_1 > x_2$.

From the viewpoint of the player 2, none of the tiles $x_2, x_3, \dots, x_{r-1}$ is preferred. Indeed, this follows from the inequalities

$$\begin{aligned} x_2 &< 1 - \varepsilon_1 \\ x_2 + x_3 &< 1 - \varepsilon_2 \\ &\dots\dots\dots \\ x_2 + x_3 + \dots + x_{r-1} &< 1 - \varepsilon_{r-2} \end{aligned}$$

which are an immediate consequence of $1 = x_1 + x_2 + \dots + x_r$ and the assumption $m = x_1 > \varepsilon_1$. It follows that the player 2 chooses one (or both) of the remaining tiles x_1 or x_r , as prescribed by preferences A_r^2 and $A_{r+1}^2 = A_1^2$. Finally, since $x_1 > x_r$, the tile x_1 is the only preferred choice of player 2.

The situation with player 3 is similar. In light of the inequalities

$$\begin{aligned} x_3 &< 1 - \varepsilon_1 \\ x_3 + x_4 &< 1 - \varepsilon_2 \\ &\dots\dots\dots \\ x_3 + x_4 + \dots + x_r &< 1 - \varepsilon_{r-2} \end{aligned}$$

the player 3 may choose only x_2 or x_1 , which means that x_1 is her only choice. But this is a contradiction, since we have two different players whose only preferred choice is the same tile x_1 .

Case 2. Now turn to the general case, that is, suppose that for some $k > 1$

$$x_r < x_1 = x_2 = \dots = x_k > x_{k+1}.$$

Then by similar analysis as in *Case 1*, we obtain that player 2 prefers only the tile x_1 . The situation with player 3 is slightly different, she may now prefer either of the tiles x_1 or x_2 . Similarly, the player 4 prefers tiles x_2 and x_3 etc. This doesn't

change until the player $k+1$ who may choose tiles x_k and x_{k-1} . Finally, the player $k+2$ may choose only the tile x_k .

The conclusion is that the players $\{2, 3, \dots, k+1, k+2\}$ may only be matched with the tiles from the set $\{x_1, x_2, \dots, x_k\}$, which is a contradiction (violation of Hall's matching condition).

It remains to show that the barycentric cut $b_0 = (\frac{1}{r}, \frac{1}{r}, \dots, \frac{1}{r})$ is associated with exactly two envy-free divisions. This is straightforward if we convince ourselves that the associated *preference graph*

$$\Gamma_{b_0} \subseteq \text{Players} \times \text{Tiles} \cong [r] \times [r]$$

(Definition 2.5) is a cycle of length $2r$. The details are left to the reader. $\square$

3. EXTREMAL PREFERENCES

Definition 3.9. Preferences $\mathcal{A} = (A_i^j)_{i \in [r]}^{j \in [r]}$ are called *extremal* if they admit exactly two envy-free divisions.

Proposition 3.10. If $\mathcal{A} = (A_i^j)_{i \in [r]}^{j \in [r]}$ is an extremal collection of preferences then the following is satisfied:

1. There exists exactly one cut x which allows an envy-free allocation. In other words the two envy-free divisions differ only in how the pieces of the one and the same cut are allocated to the players.
2. For this unique cut, each player prefers at least two tiles.
3. For this unique cut, each tile is preferred by at least two players.

Proof. The case of two players is elementary, so we assume $r \geq 3$.

(1, 2) Assume there are two different cuts, say x and x' , which both admit an envy-free distribution of tiles among the players. Note that in this case each of the cuts necessarily has a unique associated envy-free allocation. Choose the player 1 and declare him secretive. We conclude that for each of the two cuts the player 1 prefers exactly one and the same tile, which appears in both x and x' . Otherwise one has two different allocations for one of the cuts. Now let us replace the player 1 by player 2, in the role of a secretive player. There exists a cut which guarantees an envy-free allocation for any choice of the secretive player. This cut is necessarily equal to one of x and x' ; let it be x . Since a secretive player can choose an arbitrary tile, we conclude that for the cut x each of the players prefers at least two tiles. A contradiction.

(3) Assume the tile i is preferred by a unique player r . Consider the following setting: there are $r+1$ players, the extra player $r+1$ has the same preferences as the player $r-1$. The cake is divided in r pieces. Then one of the players is

expelled. Theorem 1.3 states that there exists a cut which guarantees an envy-free division. Assume the player $r + 1$ is expelled, therefore the cut is unique, and it is x . If the player r is expelled, the tile i remains non-preferred by any of the players. A contradiction. $\square$

Remark 3.11. The proof of Proposition 3.10 repeatedly uses the ‘secret player’ and the ‘expelled player’ theorems (Theorems 1.2 and 1.3). Such proofs can be alternatively presented in the language of ‘preference graphs’ (Definition 2.5), as demonstrated in the proof of Proposition 2.6, see also the proof of Theorem 3.12.

Theorem 3.12. *For an extremal collection of preferences $\mathcal{A} = (A_i^j)$, and the unique cut x whose existence is established in Proposition 3.10, the associated preference graph $\Gamma_x = \Gamma_x(\mathcal{A})$ is a cycle.*

Proof. Each envy-free division is a perfect matching on Γ . So there are exactly two perfect matchings, call them *red* and *blue*. After taking their union, one imagines that some of the edges are colored red, some are colored blue, some are red-blue, and some remain uncolored.

Eliminating all uncolored edges yields a graph Γ' which splits into connected components. There are two possible types of connected components: (1) cycles with alternating red and blue edges, and (2) one-edge graphs that correspond to red-blue edges.

Two observations are:

1. Γ' has at least one cycle. Otherwise the red and blue perfect matchings coincide.
2. If Γ' has more than one cycle, Γ' supports at least 4 perfect matchings, since for two cycles one can combine “red-red”, “red-blue”, “blue-red” and “blue-blue” matchings on the cycles.

So Γ' has exactly one cycle; we call it the *alternating cycle*. Let us convince ourselves that Γ' has no red-blue edges. Assume there is a red-blue edge e . Since each of the vertices of Γ has valency at least 2, the edge e participates in a path P in the graph Γ such that in the path uncolored edges alternate with red-blue ones. There are two options:

1. The path P closes to a cycle. Then the cycle admits yet another perfect matching. A contradiction.
2. The path P terminates by its two endpoints at the alternating cycle. Then it is possible to create a new perfect matching on their union, see Figure 2.

It is an easy exercise to prove that $\Gamma = \Gamma'$, that is, we have one alternating cycle and no more other (uncolored) edges. $\square$

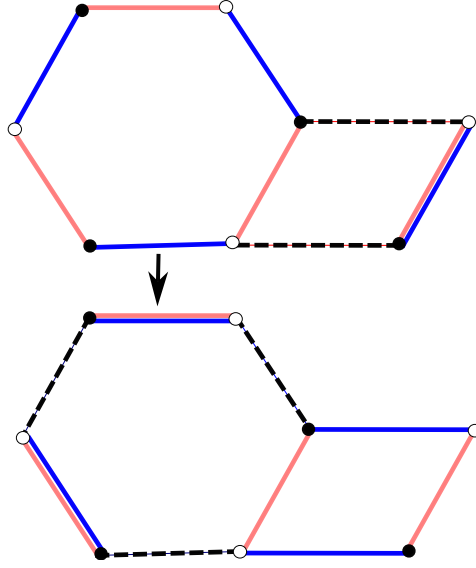


Figure 2: Generating new perfect matchings from the old.

Remark 3.13. *As in the proof of Proposition 2.6, some readers may find the argument above more readable, if it is expressed in the language of the associated preference graphs. (Definition 2.5).*

4. SIMULTANEOUS ENVY-FREE DIVISION AND A MEASURE EQUIPARTITION

In this section we assume that the number of players p is a prime. Let μ be a continuous measure on the segment (cake) $[0, 1]$. We aim not only at an envy-free division, but also at an equipartition of the measure. Clearly, $p - 1$ cuts are insufficient, so we propose the following scenario: the cake is cut into $2p - 1$ tiles that are allocated to p boxes, with some restrictions on the number of tiles in a box. The players express their preferences over the boxes (by inspection of the tiles inside the boxes).

These are the so called *new-style preferences* (B_i^j) , as described in [10, Section 3.2]. The tiles are allocated to (labelled) boxes (by an allocation function $\alpha : [2p - 1] \rightarrow [p]$) and the preferences are expressed by the rule:

$$(x, \alpha) \in B_i^j \Leftrightarrow \text{in the cut } x \text{ and the allocation } \alpha \text{ the player } j \\ \text{prefers the content } \alpha^{-1}(i) \text{ of the box } i.$$

We refer the reader for a detailed exposition in [10], briefly pinpointing important details:

1. The preferences (B_i^j) are now subsets of the configuration space $\mathcal{C}$ (described below).
2. In addition of being closed and covering, the preferences are also assumed to be *equivariant*. In simple words, a player does not care about the numbering of the boxes, his preferences depend only on what is contained inside.
3. We do not assume that the players are hungry.
4. It is assumed that degenerate tiles (if any) do not affect the preferences.

Definition 4.14. *We say that a division in the above setting is favourable if*

1. *It is envy-free (every player gets a preferred box).*
2. *The measure μ is equidistributed among the players.*
3. *Each player gets at most three tiles.*
4. *The number of players who get three tiles is at most one.*

Remark 4.15. A favourable division of the cake may not exist if the cake is cut into $2p - 2$ pieces. As a consequence, if the cake is cut into $2p - 1$ pieces, and a favourable division exists, it never contains degenerate tiles.

Here is a simple example (in the case $p = 3$ and $2p - 2 = 4$), illustrating this phenomenon. Suppose that the preference of player P_1 is to maximize the measure of the interval $[0, 1/4]$, the preference of player P_2 is to maximize the measure of the interval $[1/4, 1/2]$, and the preference of player P_3 is to maximize the measure of the interval $[1/2, 3/4]$. The measure μ , which is supposed to be equidistributed among the players, is the uniform measure on the interval $[3/4, 1]$. Then a favourable division is possible only if the number of pieces is at least $2p - 1 = 5$, and the number of cuts is at least 4.

Indeed, in order to equidistribute μ one needs at least two cuts, and it is not difficult to check that the unique remaining cut never gives envy-free division.

Theorem 4.16. *Let p be a prime number and the preferences B_i^j are closed and covering. Let μ be a continuous measure such that there does not exist a favourable division of $I = [0, 1]$ with $2p - 2$ tiles. Then there exist at least $\binom{2p-1}{p-1} \cdot 2^{2-p}$ favourable divisions with $2p - 1$ tiles.*

Proof. The proof follows the usual *Configuration space/Test map*-scheme [14], modeled on examples described in [10].

Consider the following configuration space $\mathcal{C}$ (a generalized chessboard complex) on the $[p] \times [2p - 1]$ chessboard. The vertices of $\mathcal{C}$ are the elements of the chessboard $[p] \times [2p - 1]$, while the simplices correspond to the rook placements which have at most one rook in each of $2p - 1$ columns. In addition to that, in each row of the chessboard a simplex may have at most one rook in columns $1, 2, \dots, p - 1$

and at most one rook in columns $p, p+1, \dots, 2p-2$. (As a consequence in each row there are at most three rooks.)

Informally speaking, the first $p-1$ columns of the chessboard are colored in red, the second $p-1$ columns are blue, and the last column is white. So the original chessboard $[p] \times [2p-1]$ decomposes into three chessboards, positioned side by side, and for each of them we obtain the corresponding chessboard complex.

From this description it easily follows that $\mathcal{C}$ is a join of standard chessboard complexes [10, 15]

$$(5) \quad \mathcal{C} \cong \Delta_{p,p-1} * \Delta_{p,p-1} * \Delta_{p,1}$$

where the vertices of $\Delta_{m,n}$ are elements of $[m] \times [n]$, and simplices correspond to non-attacking arrangements of rooks in the chessboard $[m] \times [n]$.

The columns of the chessboard correspond to tiles of the cake, so we imagine that given a cut, the first $p-1$ tiles are red, the second $p-1$ tiles are blue, and the last tile is white.

The associated test map

$$(6) \quad F = (f_1, f_2) : \Delta_{p,p-1} * \Delta_{p,p-1} * \Delta_{p,1} \rightarrow W_1 \oplus W_2$$

has two components,

$$f_1 : \mathcal{C} \rightarrow W_1 \text{ and } f_2 : \mathcal{C} \rightarrow W_2$$

where $W_1 \cong W_2 \cong \mathbb{R}^p/D$ is the $(p-1)$ -dimensional $\mathbb{Z}_p$ -representation, obtained by subtracting the diagonal D from $\mathbb{R}^p$. The component f_1 accounts for the envy-free division, while f_2 tests for an equipartition of μ^1

The test map (6) must have a zero, as a consequence of the fact (see [12]) that the degree of the following ($\mathbb{Z}_p$ -equivariant) map

$$\Delta_{p,p-1} * \Delta_{p,p-1} \rightarrow S(W_1 \oplus W_2) \cong S(W_1) * S(W_2)$$

is non-zero. By construction, each zero of the test map corresponds to a favourable division with additional restrictions:

$$(*) \quad \text{each player gets a box with all tiles of different colors.}$$

Secretive player theorem holds true: there exists a cut and an allocation to boxes, such that whatever box is taken by the secretive player p , the rest of the boxes can be allocated to the other players, and the division is favourable and satisfies (*). The proof of this fact is routine; it uses either Birkhoff polytope or Hall marriage lemma. We refer the reader to [9] for details of a proof for a different

¹The functions f_1 and f_2 arise in a routine way, in *envy-free and fair necklace splitting theorems*, see [5]. In short, f_1 records the values of the measure μ of the boxes, and f_2 comes from partition of unity related to the preferences .

configuration space. Therefore there exist at least two favourable divisions with additional restriction (*).

Now the idea is to look at a similar (and isomorphic) configuration space, obtained by a different choice of $(p - 1)$ red and $(p - 1)$ blue columns in the chessboard $[p] \times [2p - 1]$. Red (blue) columns correspond to the first (second) copy of $\Delta_{p,p-1}$ in (5). Each way of coloring gives at least two favourable division.

There are

$$\binom{2p-1}{p-1} \cdot p$$

ways to choose $p - 1$ red columns and one white. The remaining $p - 1$ columns are blue.

Note that each coloring produces at least two favourable division. However, each favourable division arises with different colorings.

There are two cases:

1. A favourable division has no box with three tiles. Therefore each box (except for one box) contains two tiles, and a unique box contains one tile. The number of colorings is

$$2^{p-1} + (p - 1)2^{p-1} = p2^{p-1}.$$

2. There is a box with three tiles, $p - 3$ boxes with two tiles, and 2 boxes with one tile. The number of colorings is

$$3 \cdot 2^{p-1}.$$

We conclude that for $p \geq 3$ the number of favourable divisions is at least

$$\frac{2 \binom{2p-1}{p-1} \cdot p}{p \cdot 2^{p-1}} = \frac{\binom{2p-1}{p-1}}{2^{p-2}}.$$

□

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