

SIMPLEX-AVERAGED JENSEN INTERPOLATION, CONVEX ORDER, AND DIVIDED DIFFERENCES

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We combine classical Jensen interpolation with simplex averaging in convex order. Explicit subset reductions yield averaged measure chains, divided-difference inequalities and cyclic node-reduction bounds through the Hermite–Genocchi formula. For averaged Jensen gaps, we distinguish Dragomir’s coordinate-extremum factors from the sharper, optimal upper constant 1. The construction extends from uniform simplex weights to arbitrary positive Dirichlet parameters.

1. INTRODUCTION

1.1 Scope and motivation

Refinements of Jensen’s inequality have a long history; for broad background on convexity and Jensen-type inequalities we refer to [6, 24]. The classical interpolation due to Pečarić [25] fits into an older Popoviciu-type line of Jensen refinements; see Popoviciu’s foundational work [27], the treatment in [23, Ch. 1], and the related Jensen analyses of Beesack and Pečarić [5]. In the monograph [26, pp. 173–176] this interpolation appears as a chain of intermediate inequalities between the two

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extremes of Jensen’s inequality. In the setting of divided differences, Farwig and Zwick [11], Zwick [32], and Lupaş [20] obtained inequalities for n -convex functions by exploiting the Hermite–Genocchi integral representation; see also de Boor’s survey [8] and the recent divided-difference Jensen-type developments [2, 3]. The strategy of integrating a pointwise divided-difference inequality over a structured family of admissible weights also appears in the Jensen–Steffensen setting, where the averaging domain is the polytope of valid Jensen–Steffensen weights rather than the probability simplex; see [1].

In this paper we adopt the language of *convex order* (also called convex stochastic order or convex ordering of probability measures) as a unifying and transparent tool for formulating these results and their extensions. These three terms refer to the same order; we use *convex order* throughout. This choice separates the barycentric algebra from the analytic content of convexity, and it makes the monotonicity structure of the interpolation chain visible at a glance. For background on convex order, majorization, and stochastic comparison we refer to [21, 29, 30]; the bridge with classical majorization goes back to Hardy, Littlewood and Pólya [12].

A second theme of the paper is averaging over the simplex of all probability weights. In the uniform case this is exactly the Dirichlet distribution $\text{Dir}(1, \dots, 1)$; see [19, Ch. 49]. The constants that appear in averaged Jensen-gap comparisons can then be interpreted as Dirichlet moments of coordinate extrema, a point of view closely related to the classical “broken-stick” picture [13]. This naturally suggests replacing the uniform simplex law by a general Dirichlet law $\text{Dir}(\alpha_0, \dots, \alpha_n)$.

1.2 Main contributions and principal results

The principal contribution is the combination of three established ingredients: Jensen interpolation, averaging over probability weights, and the Hermite–Genocchi formula. We do not claim the classical interpolation, convex-order characterization, or Dirichlet moments as new results. The measure formulation makes their interaction explicit and provides a common construction for the following consequences.

First, averaging Pečarić’s interpolation measures produces the chain

$$\tilde{\mu}_n \preceq_{\text{cx}} \tilde{\mu}_{n-1} \preceq_{\text{cx}} \cdots \preceq_{\text{cx}} \tilde{\mu}_0, \quad \tilde{\mu}_0 = \nu_{\text{unif}} := \frac{1}{n+1} \sum_{i=0}^n \delta_{x_i},$$

whose other endpoint is the law of $T_{\mathbf{x}}(U) = \sum_i U_i x_i$ for $U \sim \text{Dir}(1, \dots, 1)$. The subset-reduction identity gives an explicit formula for every intermediate measure (Theorem 2.2). For $g = f^{(n)}$, this becomes a hierarchy of symmetrised divided-difference bounds; averaging the cyclic refinement similarly gives consecutive-block bounds (Remark 2.4 and Corollary 2.5).

Second, we distinguish the averaged pointwise comparison of Jensen gaps from the stronger comparison obtained by concavity in the weights. The former yields the factors $1/(n+1)$ and H_{n+1} , while the latter replaces the upper factor by 1. This distinction clarifies what averaging Dragomir’s estimate does, and what

it does not establish about sharpness. Third, the Dirichlet extension preserves the convex-order chain and gives explicit subset coefficients for arbitrary positive parameters (Theorem 3.8). It thus separates the geometry of the nodes from the averaging law, within one construction rather than a series of unrelated inequalities.

1.3 Organisation of the paper

The remainder of this section collects the prerequisites: divided differences and the Hermite–Genocchi formula (Subsection 1.4), the convex order and the measure-theoretic reformulation of the Jensen interpolation functionals (Subsection 1.5), a cyclic refinement of Jensen’s inequality due to Horváth, Khan and Pečarić together with its later permutation-based extensions [15, 16, 18] (Subsection 1.6), and Dragomir’s bounds for the normalised Jensen functional and their more recent Jensen-gap descendants [9, 17] (Subsection 1.7).

Section 2 develops the simplex-integrated Jensen chain and its consequences for divided differences and cyclic node reduction. Section 3 then isolates the Jensen gap as a separate deficit functional, beginning with the uniform simplex law $\text{Dir}(1, \dots, 1)$ and continuing to a general Dirichlet average. Finally, Section 4 summarizes the construction and possible future directions; sharpness is discussed at the end of Section 3.

1.4 Divided differences and the Hermite–Genocchi formula

Throughout, $n \geq 1$. We identify the coordinate simplex $\Delta_m \subset \mathbb{R}^m$ with its $m + 1$ barycentric coordinates, the last coordinate being one minus the sum of the others. We set $\Delta_0 = \{(1)\}$ and interpret integration over it as evaluation at its single point.

Let $I \subset \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$. For pairwise distinct points $a_0, \dots, a_n \in I$ the *divided difference* of order n is defined recursively (see [4, 8]) by

$$f[a_i] = f(a_i) \quad (i = 0, \dots, n),$$

and

$$f[a_0, \dots, a_n] = \frac{f[a_1, \dots, a_n] - f[a_0, \dots, a_{n-1}]}{a_n - a_0}.$$

If $f \in C^n(I)$, its divided differences of order at most n extend continuously to repeated nodes. In particular,

$$f[\underbrace{a, \dots, a}_{m+1}] = \frac{f^{(m)}(a)}{m!} \quad (0 \leq m \leq n).$$

Recall that f is called *n-convex* if every divided difference of order n is nonnegative. This viewpoint goes back to Popoviciu [27]; see also [26, 28] for general convexity background and [11, 20, 32] for early divided-difference inequalities in this framework.

The connection between divided differences and integration over simplices is provided by the following classical formula; see [4, Sec. 3.2, Theorem 3.3] and [8].

Theorem 1.1 (Hermite–Genocchi [4, Sec. 3.2, Theorem 3.3]; see also [8]). *Assume that $f \in C^n(I)$ and let $a_0, \dots, a_n \in I$ (possibly repeated). Then*

$$(1) \quad f[a_0, \dots, a_n] = \int_{\Delta_n} f^{(n)} \left(\sum_{j=0}^n u_j a_j \right) du_0 \cdots du_{n-1},$$

where

$$\Delta_n = \left\{ (u_0, \dots, u_{n-1}) : u_j \geq 0, \sum_{j=0}^{n-1} u_j \leq 1 \right\}, \quad u_n = 1 - \sum_{j=0}^{n-1} u_j.$$

1.5 Jensen interpolation and convex order

The classical Jensen inequality admits a family of intermediate refinements; for general background see [6, 24]. The interpolation chain studied systematically by Popoviciu and later extended by Pečarić [25] and Pečarić, Proschan and Tong [26] is the specific discrete refinement that we recast below in the language of convex order.

Convex order compares probability measures with a common barycenter by integration against convex functions; see Definition 1.3 below. Its martingale characterization is due to Strassen [30]; see also [21, 29]. It describes a mean-preserving increase in spread.

All measures considered below are supported on the finite-dimensional convex hull $K = \text{conv}\{x_0, \dots, x_n\}$. We equip its affine hull with its usual Euclidean topology and use the induced Borel structure; barycenters are taken in this affine hull. Functions integrated against these measures are assumed Borel measurable and integrable.

Let C be a convex subset of a real vector space V and let $g : C \rightarrow \mathbb{R}$ be convex. Fix points $\mathbf{x} = (x_0, \dots, x_n) \in C^{n+1}$ and weights $\mathbf{p} = (p_0, \dots, p_n)$ with $p_i \geq 0$ and $\sum_{i=0}^n p_i = 1$. For $k = 0, 1, \dots, n$ define

$$(2) \quad g_{k,n}(\mathbf{x}, \mathbf{p}) := \frac{1}{\binom{n}{k}} \sum_{0 \leq i_0 < \dots < i_k \leq n} \left(\sum_{j=0}^k p_{i_j} \right) g \left(\frac{\sum_{j=0}^k p_{i_j} x_{i_j}}{\sum_{j=0}^k p_{i_j}} \right).$$

A summand with $\sum_{j=0}^k p_{i_j} = 0$ is defined to be zero. In particular,

$$g_{0,n}(\mathbf{x}, \mathbf{p}) = \sum_{i=0}^n p_i g(x_i), \quad g_{n,n}(\mathbf{x}, \mathbf{p}) = g \left(\sum_{i=0}^n p_i x_i \right).$$

For each $k \in \{0, 1, \dots, n\}$ denote by

$$\mathcal{I}_k := \{ I \subset \{0, 1, \dots, n\} : |I| = k + 1 \}$$

the family of all $(k+1)$ -subsets. For $I \in \mathcal{I}_k$ set

$$s_I := \sum_{i \in I} p_i, \quad b_I := \begin{cases} \frac{\sum_{i \in I} p_i x_i}{s_I}, & s_I > 0, \\ x_0, & s_I = 0, \end{cases}$$

(where the value of b_I is irrelevant if $s_I = 0$, since it carries zero mass).

Each functional $g_{k,n}$ can be represented as an integral against a finitely supported probability measure. Define Λ_k on $\mathcal{I}_k$ by

$$\Lambda_k := \sum_{I \in \mathcal{I}_k} \frac{s_I}{\binom{n}{k}} \delta_I.$$

Indeed,

$$\sum_{I \in \mathcal{I}_k} \frac{s_I}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{I \in \mathcal{I}_k} \sum_{i \in I} p_i = \frac{1}{\binom{n}{k}} \sum_{i=0}^n p_i \# \{I \in \mathcal{I}_k : i \in I\} = \frac{1}{\binom{n}{k}} \sum_{i=0}^n p_i \binom{n}{k} = 1.$$

Let $B_k : \mathcal{I}_k \rightarrow C$ be the barycenter map $B_k(I) = b_I$ and define the push-forward measure on C ,

$$\mu_k := (B_k)_\# \Lambda_k.$$

Equivalently, μ_k is the finitely supported probability measure on C given by

$$(3) \quad \mu_k = \sum_{I \in \mathcal{I}_k} \frac{s_I}{\binom{n}{k}} \delta_{b_I}.$$

Then, for every convex $g : C \rightarrow \mathbb{R}$,

$$(4) \quad g_{k,n}(\mathbf{x}, \mathbf{p}) = \int_C g d\mu_k.$$

In particular,

$$\mu_0 = \sum_{i=0}^n p_i \delta_{x_i}, \quad \mu_n = \delta_m, \quad m := \sum_{i=0}^n p_i x_i.$$

Proposition 1.2 (Common barycenter of the interpolation chain). *With $m = \sum_{i=0}^n p_i x_i$, one has*

$$\int_C t d\mu_k(t) = m \quad (k = 0, \dots, n).$$

Proof. Using (3), we obtain

$$\int_C t d\mu_k(t) = \frac{1}{\binom{n}{k}} \sum_{I \in \mathcal{I}_k} s_I b_I = \frac{1}{\binom{n}{k}} \sum_{I \in \mathcal{I}_k} \sum_{i \in I} p_i x_i.$$

Interchanging the order of summation gives

$$\int_C t d\mu_k(t) = \frac{1}{\binom{n}{k}} \sum_{i=0}^n p_i x_i \# \{I \in \mathcal{I}_k : i \in I\} = \frac{1}{\binom{n}{k}} \sum_{i=0}^n p_i x_i \binom{n}{k} = \sum_{i=0}^n p_i x_i = m.$$

□

We record the formal definition for later use.

Definition 1.3 (Convex order). For probability measures ν, η on C with the same barycenter, we write $\nu \preceq_{\text{cx}} \eta$ if

$$\int_C \varphi d\nu \leq \int_C \varphi d\eta$$

for every convex $\varphi : C \rightarrow \mathbb{R}$ for which the integrals are finite.

The Jensen interpolation chain now takes a particularly clean form in this language.

Theorem 1.4 (Jensen interpolation chain: measure form). *With μ_k defined by (3), the inequalities, required for every convex $g : C \rightarrow \mathbb{R}$,*

$$g_{n,n}(\mathbf{x}, \mathbf{p}) \leq g_{n-1,n}(\mathbf{x}, \mathbf{p}) \leq \cdots \leq g_{1,n}(\mathbf{x}, \mathbf{p}) \leq g_{0,n}(\mathbf{x}, \mathbf{p})$$

are equivalent to the convex-order chain

$$(5) \quad \mu_n \preceq_{\text{cx}} \mu_{n-1} \preceq_{\text{cx}} \cdots \preceq_{\text{cx}} \mu_1 \preceq_{\text{cx}} \mu_0.$$

Proof. For $1 \leq k \leq n$ and $I \in \mathcal{I}_k$ with $s_I > 0$, one has

$$\sum_{\substack{J \subset I \\ |J|=k}} s_J = k s_I, \quad b_I = \sum_{\substack{J \subset I \\ |J|=k}} \frac{s_J}{k s_I} b_J.$$

Jensen's inequality therefore gives $s_I g(b_I) \leq k^{-1} \sum_{J \subset I, |J|=k} s_J g(b_J)$. After summing over I , each J occurs $n - k + 1$ times. Since

$$\frac{n - k + 1}{k \binom{n}{k}} = \frac{1}{\binom{n}{k-1}},$$

we obtain $g_{k,n} \leq g_{k-1,n}$. Zero-mass subsets contribute zero. By (4), validity for every convex g is exactly (5). This also gives a direct proof of the classical chain; see [23, 25, 26] for its original functional formulation and [14] for a related integral refinement. $\square$

For real nodes, the next result gives a stronger domination by a convex combination of the two endpoint measures. This additional Popoviciu bound uses the order structure of the real line; see [26, p. 173] for its functional formulation. We include a direct proof.

Lemma 1.5 (Popoviciu/Jensen interpolation as convex order). *Assume that $C \subset \mathbb{R}$ is an interval. Let $g : C \rightarrow \mathbb{R}$ be convex. For $k = 0, 1, \dots, n$ set*

$$\nu_k := \frac{n-k}{n} \mu_0 + \frac{k}{n} \mu_n = \frac{n-k}{n} \sum_{i=0}^n p_i \delta_{x_i} + \frac{k}{n} \delta_m.$$

Then the Popoviciu–Jensen interpolation inequality

$$(6) \quad g_{k,n}(\mathbf{x}, \mathbf{p}) \leq \frac{n-k}{n} g_{0,n}(\mathbf{x}, \mathbf{p}) + \frac{k}{n} g_{n,n}(\mathbf{x}, \mathbf{p})$$

holds for every such g and is equivalent to the convex-order domination

$$(7) \quad \mu_k \preceq_{\text{cx}} \nu_k.$$

Proof. The cases $k = 0, n$ are identities, so assume $1 \leq k \leq n-1$. Affine functions give equality by the common-barycenter property. It suffices to consider $g(x) = (x-t)_+$: on the finite set of nodes and barycenters in the inequality, every convex function agrees with a convex piecewise-linear interpolant, which is an affine function plus a nonnegative combination of such functions. Here $(r)_+ := \max\{r, 0\}$.

Set $z_i = p_i(x_i - t)$, $P = \sum_i (z_i)_+$ and $M = \sum_i (-z_i)_+$. Then

$$\binom{n}{k} g_{k,n}(\mathbf{x}, \mathbf{p}) = \Phi(z) := \sum_{|I|=k+1} \left(\sum_{i \in I} z_i \right)_+.$$

If $P, M > 0$, the vector z is the convex combination of $Pe_i - Me_j$ with weights $(z_i/P)(-z_j/M)$, where $z_i > 0$, $z_j < 0$, and e_i are the standard coordinate vectors. Convexity of Φ and counting subsets containing i but not j , or both, give

$$\Phi(z) \leq \binom{n-1}{k} P + \binom{n-1}{k-1} (P-M)_+.$$

The same estimate follows directly if $P = 0$ or $M = 0$. Dividing by $\binom{n}{k}$ proves (6), since $P = g_{0,n}(\mathbf{x}, \mathbf{p})$ and $(P-M)_+ = g_{n,n}(\mathbf{x}, \mathbf{p})$. Finally, (4) and Definition 1.3 give the equivalence with (7). $\square$

This convex-order formulation makes explicit that the underlying node-reduction principle is Popoviciu-type; compare [23, 27].

The Popoviciu–Jensen interpolation admits a natural interpretation in terms of the Jensen gap, which we now record.

Remark 1.6 (Jensen gap and contraction under node reduction). *For a convex function $\varphi : C \rightarrow \mathbb{R}$ and a probability measure μ on C with finite barycenter, define the Jensen gap by*

$$J_\varphi(\mu) := \int_C \varphi d\mu - \varphi\left(\int_C t d\mu\right).$$

In particular, $J_\varphi(\mu) \geq 0$ for convex φ by Jensen's inequality.

For the stronger contraction estimate below, assume that $C \subset \mathbb{R}$ is an interval. For $k = 0, 1, \dots, n$, let μ_k be as in (3). By Proposition 1.2, all these measures have the same barycenter

$$m = \sum_{i=0}^n p_i x_i.$$

Moreover, (6) is equivalent to

$$\int_C \varphi d\mu_k \leq \frac{n-k}{n} \int_C \varphi d\mu_0 + \frac{k}{n} \varphi(m) \quad (\forall \varphi \text{ convex}),$$

and hence, after subtracting $\varphi(m)$ from both sides,

$$J_\varphi(\mu_k) \leq \frac{n-k}{n} J_\varphi(\mu_0) \quad (\forall \varphi \text{ convex}).$$

Thus reduction to $(k+1)$ -subsets decreases the convex spread, measured by the Jensen gap, with the uniform contraction factor $(n-k)/n$. This deficit-functional viewpoint is close in spirit to later Jensen-gap refinements such as [9, 10, 17]. Moreover, since $\varphi(m) \leq \int_C \varphi d\mu_0$ by Jensen, (6) also implies $\mu_k \preceq_{\text{cx}} \mu_0$ for every k . The chain (5), which remains valid for vector-valued nodes, also shows that node reduction acts as a mean-preserving contraction in convex order.

1.6 A cyclic refinement of Jensen's inequality

In many applications one does not average over all $(k+1)$ -subsets but only over the $n+1$ cyclically consecutive blocks. The following result of Horváth, Khan and Pečarić [18] provides such a refinement; see also [15, 16] for permutation-based extensions and integral analogues. We state it here and give its measure interpretation, which will be used in Section 2.

Throughout this subsection, indices are understood cyclically modulo $n+1$. For $i \in \{0, \dots, n\}$ and $j \in \mathbb{Z}$ we write

$$i \oplus j \in \{0, \dots, n\} \quad \text{for the unique index such that } i \oplus j \equiv i + j \pmod{n+1}.$$

Theorem 1.7 (Horváth–Khan–Pečarić [18]). *Let C be a convex subset of a real vector space and let $f : C \rightarrow \mathbb{R}$ be convex. Let $v_0, \dots, v_n \in C$ and let $p_0, \dots, p_n \geq 0$ satisfy $\sum_{i=0}^n p_i = 1$. Fix $k \in \{0, \dots, n\}$ and let $\lambda_0, \dots, \lambda_k \geq 0$ satisfy $\sum_{j=0}^k \lambda_j = 1$. For $i = 0, \dots, n$ set*

$$P_i := \sum_{j=0}^k \lambda_j p_{i \oplus j}.$$

(If $P_i = 0$ we interpret the corresponding term $P_i f(\dots)$ as 0.) Then

$$f\left(\sum_{i=0}^n p_i v_i\right) \leq \sum_{i=0}^n P_i f\left(\frac{\sum_{j=0}^k \lambda_j p_{i \oplus j} v_{i \oplus j}}{P_i}\right) \leq \sum_{i=0}^n p_i f(v_i).$$

The cyclic refinement also fits into the convex-order framework.

Remark 1.8 (Measure interpretation). *Define*

$$\mu_0 := \sum_{i=0}^n p_i \delta_{v_i}, \quad m := \sum_{i=0}^n p_i v_i.$$

For the fixed coefficients $(\lambda_0, \dots, \lambda_k)$ in Theorem 1.7, set

$$y_i := \frac{\sum_{j=0}^k \lambda_j p_{i \oplus j} v_{i \oplus j}}{P_i} \quad (P_i > 0), \quad \mu_\lambda := \sum_{i=0}^n P_i \delta_{y_i}.$$

If $P_i = 0$, set $y_i = v_0$; the corresponding summand has zero mass. A direct computation shows that μ_λ has the same barycenter as μ_0 :

$$\int_C t d\mu_\lambda(t) = \sum_{i=0}^n P_i y_i = \sum_{i=0}^n \sum_{j=0}^k \lambda_j p_{i \oplus j} v_{i \oplus j} = \sum_{j=0}^k \lambda_j \sum_{i=0}^n p_i v_i = m.$$

With this notation, Theorem 1.7 can be written succinctly as

$$\int_C f d\delta_m \leq \int_C f d\mu_\lambda \leq \int_C f d\mu_0 \quad (\forall f \text{ convex}),$$

that is,

$$\delta_m \preceq_{\text{cx}} \mu_\lambda \preceq_{\text{cx}} \mu_0$$

in the sense of Definition 1.3. For a proof and further cyclic refinements, see [18]; for higher-order cyclic variants obtained by approximation-theoretic methods, see [22].

1.7 Dragomir's bounds for the Jensen gap under change of weights

The results of the preceding subsections compare the Jensen gap for a *fixed* probability vector across different levels of node reduction. A complementary question is: how does the Jensen gap change when one replaces one probability vector by another on the *same* set of nodes? The following result of Dragomir [9] gives a sharp answer in terms of the pointwise ratio of the two weight vectors. It is also the starting point for later Jensen-gap refinements such as [17]; for a complementary refinement with f -divergence applications see [10].

Theorem 1.9 (Dragomir [9]). *Let C be a convex subset of a real vector space and let $\varphi : C \rightarrow \mathbb{R}$ be convex. Fix points $x_0, \dots, x_n \in C$ and let $\mathbf{p} = (p_0, \dots, p_n)$, $\mathbf{q} = (q_0, \dots, q_n)$ be two probability vectors (i.e. $p_i, q_i \geq 0$, $\sum_{i=0}^n p_i = \sum_{i=0}^n q_i = 1$) with $q_i > 0$ for every $i \in \{0, \dots, n\}$. Define the discrete probability measures*

$$\mu := \sum_{i=0}^n p_i \delta_{x_i}, \quad \nu := \sum_{i=0}^n q_i \delta_{x_i},$$

and the corresponding Jensen gaps (cf. Remark 1.6)

$$J_\varphi(\mu) = \sum_{i=0}^n p_i \varphi(x_i) - \varphi\left(\sum_{i=0}^n p_i x_i\right), \quad J_\varphi(\nu) = \sum_{i=0}^n q_i \varphi(x_i) - \varphi\left(\sum_{i=0}^n q_i x_i\right).$$

Then

$$(8) \quad \min_{0 \leq i \leq n} \frac{p_i}{q_i} J_\varphi(\nu) \leq J_\varphi(\mu) \leq \max_{0 \leq i \leq n} \frac{p_i}{q_i} J_\varphi(\nu).$$

Proof. See [9], Theorem 1 (stated there for n points indexed from 1). $\square$

A natural and important special case arises when the reference measure is the uniform (arithmetic-mean) distribution on the nodes.

Corollary 1.10 (Uniform reference measure). *Let φ , $x_0, \dots, x_n$, $\mathbf{p}$, and μ be as in Theorem 1.9 and set*

$$\nu_{\text{unif}} := \frac{1}{n+1} \sum_{i=0}^n \delta_{x_i}, \quad \bar{x} := \frac{1}{n+1} \sum_{i=0}^n x_i.$$

Then

$$\begin{aligned} (n+1) \min_{0 \leq i \leq n} p_i \left(\frac{1}{n+1} \sum_{i=0}^n \varphi(x_i) - \varphi(\bar{x}) \right) \\ \leq J_\varphi(\mu) \\ \leq (n+1) \max_{0 \leq i \leq n} p_i \left(\frac{1}{n+1} \sum_{i=0}^n \varphi(x_i) - \varphi(\bar{x}) \right). \end{aligned}$$

In particular, the Jensen gap $J_\varphi(\mu)$ is controlled by the arithmetic-mean Jensen gap $J_\varphi(\nu_{\text{unif}})$ up to the multiplicative factor $(n+1) \max_{0 \leq i \leq n} p_i \geq 1$.

Proof. Take $q_i = \frac{1}{n+1}$ for every i in (8). Then $p_i/q_i = (n+1)p_i$ and $J_\varphi(\nu_{\text{unif}}) = \frac{1}{n+1} \sum_{i=0}^n \varphi(x_i) - \varphi(\bar{x})$. $\square$

2. SIMPLEX REDUCTION AND A SIMPLEX-INTEGRATED JENSEN CHAIN

In this section we average the pointwise inequalities from the Jensen interpolation chain over the simplex Δ_n of probability weights. This turns discrete Jensen-type bounds into identities and estimates for divided differences through the Hermite–Genocchi formula, in a form comparable with [2, 11, 20, 32]. It is also the positive-weight counterpart of the Jensen–Steffensen approach in [1], where the integration is carried out over the polytope of admissible Jensen–Steffensen weights rather than over the full probability simplex.

For $m \geq 1$ denote the standard simplex

$$\Delta_m := \left\{ (w_0, \dots, w_{m-1}) : w_j \geq 0, \sum_{j=0}^{m-1} w_j \leq 1 \right\}, \quad w_m := 1 - \sum_{j=0}^{m-1} w_j,$$

and let σ_m be the normalized Lebesgue probability measure on Δ_m ,

$$d\sigma_m(w) = m! dw_0 \cdots dw_{m-1}.$$

For $m = 0$, σ_0 is the unit mass on Δ_0 .

The first step is a change-of-variables lemma that reduces an integral over Δ_n involving a barycentric evaluation on a subset of nodes to an integral over the lower-dimensional simplex Δ_k .

Lemma 2.1 (Simplex reduction). *Let $C \subset V$ be convex, let $x_0, \dots, x_n \in C$, and fix $k \in \{0, \dots, n\}$. Let $I = \{i_0 < \dots < i_k\} \subset \{0, \dots, n\}$ and define, for $u \in \Delta_n$,*

$$s_I(u) := \sum_{j=0}^k u_{i_j}, \quad b_I(u) := \frac{\sum_{j=0}^k u_{i_j} x_{i_j}}{s_I(u)} \quad (s_I(u) > 0).$$

Then for every Borel function $h : C \rightarrow \mathbb{R}$ whose composition in the right-hand integral below is integrable,

$$(9) \quad \int_{\Delta_n} s_I(u) h(b_I(u)) d\sigma_n(u) = \frac{k+1}{n+1} \int_{\Delta_k} h\left(\sum_{j=0}^k v_j x_{i_j}\right) d\sigma_k(v).$$

Proof. The calculation below applied first to $|h|$ justifies absolute integrability and the use of Fubini's theorem. The case $k = n$ is immediate: then $I = \{0, \dots, n\}$, hence $s_I(u) \equiv 1$ and $b_I(u) = \sum_{j=0}^n u_j x_j$, so (9) is an identity.

Assume now that $0 \leq k \leq n-1$. Choose an index $m \in \{0, \dots, n\} \setminus I$. By relabeling the indices $0, \dots, n$ (simultaneously relabeling x_i and the coordinates u_i), we may assume that $m = n$ and $I = \{0, 1, \dots, k\}$. Write

$$s := s_I(u) = u_0 + \dots + u_k, \quad b_I(u) = \frac{\sum_{j=0}^k u_j x_j}{s} \quad (s > 0).$$

(The set $\{s = 0\}$ contributes nothing since the integrand is multiplied by s .)

Fix nonnegative $u_0, \dots, u_k$ with $s \leq 1$. The remaining coordinates range over the simplex

$$\left\{ (u_{k+1}, \dots, u_{n-1}) : u_\ell \geq 0, \sum_{\ell=k+1}^{n-1} u_\ell \leq 1-s \right\},$$

whose volume equals $\frac{(1-s)^{n-k-1}}{(n-k-1)!}$. Hence, by Fubini,

$$\begin{aligned} & \int_{\Delta_n} s h(b_I(u)) du_0 \cdots du_{n-1} \\ &= \frac{1}{(n-k-1)!} \int_{\{u_0, \dots, u_k \geq 0, s \leq 1\}} s(1-s)^{n-k-1} h\left(\frac{\sum_{j=0}^k u_j x_j}{s}\right) du_0 \cdots du_k. \end{aligned}$$

Now perform the change of variables

$$u_j = s v_j \quad (j = 0, \dots, k), \quad (v_0, \dots, v_{k-1}) \in \Delta_k, \quad v_k = 1 - \sum_{j=0}^{k-1} v_j,$$

with $s \in [0, 1]$. Then

$$du_0 \cdots du_k = s^k dv_0 \cdots dv_{k-1} ds, \quad \frac{\sum_{j=0}^k u_j x_j}{s} = \sum_{j=0}^k v_j x_j.$$

Therefore the integral factorizes as

$$\int_{\Delta_n} s h(b_I(u)) du = \frac{1}{(n-k-1)!} \int_0^1 s^{k+1} (1-s)^{n-k-1} ds \int_{\Delta_k} h\left(\sum_{j=0}^k v_j x_j\right) dv.$$

The one-dimensional integral is a Beta integral:

$$\int_0^1 s^{k+1} (1-s)^{n-k-1} ds = B(k+2, n-k) = \frac{(k+1)!(n-k-1)!}{(n+1)!}.$$

Hence

$$\int_{\Delta_n} s h(b_I(u)) du = \frac{(k+1)!}{(n+1)!} \int_{\Delta_k} h\left(\sum_{j=0}^k v_j x_j\right) dv.$$

Multiplying both sides by $n!$ and using $d\sigma_k(v) = k! dv$ gives (9). $\square$

We now apply the simplex reduction to the entire Jensen interpolation chain, obtaining a family of simplex-averaged probability measures that are monotone in convex order.

Theorem 2.2 (A simplex-integrated Jensen chain in measure form). *Let $C \subset V$ be convex and let $x_0, \dots, x_n \in C$. For each $u \in \Delta_n$ set $\mathbf{p}(u) := (u_0, \dots, u_n)$ and define, for $k = 0, \dots, n$, the finitely supported probability measure on C*

$$\begin{aligned} \mu_k(u) &:= \sum_{I \in \mathcal{I}_k} \frac{s_I(u)}{\binom{n}{k}} \delta_{b_I(u)}, \\ s_I(u) &:= \sum_{i \in I} u_i, \\ b_I(u) &:= \frac{\sum_{i \in I} u_i x_i}{s_I(u)} \quad (s_I(u) > 0). \end{aligned}$$

with the convention that terms with $s_I(u) = 0$ contribute 0. Define the simplex-averaged probability measures

$$\tilde{\mu}_k(A) := \int_{\Delta_n} \mu_k(u)(A) d\sigma_n(u), \quad A \subset C \text{ Borel}.$$

Then the measures $\tilde{\mu}_k$ have common barycenter

$$\bar{x} := \frac{1}{n+1} \sum_{i=0}^n x_i,$$

their endpoints are

$$\tilde{\mu}_0 = \nu_{\text{unif}} := \frac{1}{n+1} \sum_{i=0}^n \delta_{x_i}, \quad \tilde{\mu}_n = (T_{\mathbf{x}})_{\#} \sigma_n, \quad T_{\mathbf{x}}(u) = \sum_{i=0}^n u_i x_i,$$

and they satisfy the convex-order chain

$$(10) \quad \tilde{\mu}_n \preceq_{\text{cx}} \tilde{\mu}_{n-1} \preceq_{\text{cx}} \cdots \preceq_{\text{cx}} \tilde{\mu}_0.$$

Moreover, for every convex $g : C \rightarrow \mathbb{R}$ for which the integrals are finite,

$$(11) \quad \int_C g d\tilde{\mu}_k = \int_{\Delta_n} g_{k,n}(\mathbf{x}, \mathbf{p}(u)) d\sigma_n(u) = \frac{1}{\binom{n+1}{k+1}} \sum_{I \in \mathcal{I}_k} \int_{\Delta_k} g \left(\sum_{j=0}^k v_j x_{i_j} \right) d\sigma_k(v).$$

Proof. For each fixed $u \in \Delta_n$, Theorem 1.4 applied with weights $\mathbf{p}(u)$ yields

$$\int_C g d\mu_n(u) \leq \int_C g d\mu_{n-1}(u) \leq \cdots \leq \int_C g d\mu_0(u) \quad (\forall g \text{ convex}).$$

Integrating these inequalities over Δ_n gives (10) by Definition 1.3.

Since each $\mu_k(u)$ is a probability measure, so is $\tilde{\mu}_k$. By Proposition 1.2, the barycenter of $\mu_k(u)$ equals

$$T_{\mathbf{x}}(u) = \sum_{i=0}^n u_i x_i.$$

Therefore

$$\int_C t d\tilde{\mu}_k(t) = \int_{\Delta_n} T_{\mathbf{x}}(u) d\sigma_n(u) = \sum_{i=0}^n \left(\int_{\Delta_n} u_i d\sigma_n(u) \right) x_i = \frac{1}{n+1} \sum_{i=0}^n x_i = \bar{x},$$

by symmetry of σ_n . The formula for $\tilde{\mu}_n$ is immediate from $\mu_n(u) = \delta_{T_{\mathbf{x}}(u)}$. Similarly,

$$\tilde{\mu}_0(A) = \int_{\Delta_n} \sum_{i=0}^n u_i \delta_{x_i}(A) d\sigma_n(u) = \sum_{i=0}^n \delta_{x_i}(A) \int_{\Delta_n} u_i d\sigma_n(u) = \frac{1}{n+1} \sum_{i=0}^n \delta_{x_i}(A),$$

so $\tilde{\mu}_0 = \nu_{\text{unif}}$.

To prove (11), note that by construction,

$$\int_C g d\tilde{\mu}_k = \int_{\Delta_n} \left(\int_C g d\mu_k(u) \right) d\sigma_n(u) = \int_{\Delta_n} g_{k,n}(\mathbf{x}, \mathbf{p}(u)) d\sigma_n(u),$$

using (4) pointwise in u . Expanding the definition (2) and applying Lemma 2.1 termwise gives

$$\begin{aligned} \int_{\Delta_n} g_{k,n}(\mathbf{x}, \mathbf{p}(u)) d\sigma_n(u) &= \frac{1}{\binom{n}{k}} \sum_{I \in \mathcal{I}_k} \int_{\Delta_n} s_I(u) g(b_I(u)) d\sigma_n(u) \\ &= \frac{1}{\binom{n}{k}} \sum_{I \in \mathcal{I}_k} \frac{k+1}{n+1} \int_{\Delta_k} g\left(\sum_{j=0}^k v_j x_{i_j}\right) d\sigma_k(v) \\ &= \frac{1}{\binom{n+1}{k+1}} \sum_{I \in \mathcal{I}_k} \int_{\Delta_k} g\left(\sum_{j=0}^k v_j x_{i_j}\right) d\sigma_k(v), \end{aligned}$$

which is (11). $\square$

Example 2.3 (Three equally spaced nodes). Take $n = 2$ and $(x_0, x_1, x_2) = (0, 1, 2)$. Write $\mathcal{U}[a, b]$ for the uniform probability law on $[a, b]$. Formula (11) gives

$$\tilde{\mu}_0 = \frac{1}{3}(\delta_0 + \delta_1 + \delta_2), \quad \tilde{\mu}_1 = \frac{1}{3}(\mathcal{U}[0, 1] + \mathcal{U}[0, 2] + \mathcal{U}[1, 2]) = \mathcal{U}[0, 2].$$

The last endpoint is the law of $U_1 + 2U_2$, where $(U_0, U_1, U_2) \sim \text{Dir}(1, 1, 1)$. Changing variables from (u_1, u_2) to (t, u_2) , with $t = u_1 + 2u_2$, and integrating the constant simplex density 2 gives its density

$$\rho(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2 - t, & 1 < t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus all three measures have mean 1, and the chain compares the triangular law, the uniform interval law, and the uniform three-node law. For $g(t) = t^2$,

$$\int g d\tilde{\mu}_2 = \int_0^1 t^3 dt + \int_1^2 t^2(2-t) dt = \frac{7}{6}, \quad \int g d\tilde{\mu}_1 = \frac{4}{3}, \quad \int g d\tilde{\mu}_0 = \frac{5}{3}.$$

This verifies the strict inequalities $7/6 < 4/3 < 5/3$ directly. The averaged random-weight Jensen gap is $5/3 - 7/6 = 1/2$, whereas $J_g(\nu_{\text{unif}}) = 5/3 - 1 = 2/3$, so their ratio is $3/4$. Taking $f(t) = t^4/12$, for which $f'' = g$, gives the same three values for the divided-difference chain in Remark 2.4 below.

When the convex function is the n -th derivative of a sufficiently smooth function, the Hermite–Genocchi formula identifies the simplex integrals as divided differences, yielding the following monotonicity result.

Remark 2.4 (A chain for divided differences). Assume $C = [\alpha, \beta] \subset \mathbb{R}$ and $f \in C^n([\alpha, \beta])$ with $f^{(n)}$ convex. Applying Theorem 2.2 with $g = f^{(n)}$ and using the Hermite–Genocchi formula (1) for $\phi = f^{(n-k)}$ (so $\phi^{(k)} = f^{(n)}$) yields

$$\int_{\Delta_k} f^{(n)}\left(\sum_{j=0}^k v_j x_{i_j}\right) d\sigma_k(v) = k! f^{(n-k)}[x_{i_0}, \dots, x_{i_k}].$$

Hence (11) becomes

$$\int_C f^{(n)} d\tilde{\mu}_k = \frac{k!}{\binom{n+1}{k+1}} \sum_{0 \leq i_0 < \dots < i_k \leq n} f^{(n-k)}[x_{i_0}, \dots, x_{i_k}].$$

In particular, the chain (10) implies the monotonicity

$$B_n \leq B_{n-1} \leq \dots \leq B_0,$$

where

$$B_k := \frac{k!}{\binom{n+1}{k+1}} \sum_{0 \leq i_0 < \dots < i_k \leq n} f^{(n-k)}[x_{i_0}, \dots, x_{i_k}] \quad (k = 0, \dots, n).$$

Thus

$$n! f[x_0, \dots, x_n] \leq \dots \leq \frac{1}{n+1} \sum_{i=0}^n f^{(n)}(x_i).$$

Compare this monotone chain with the earlier divided-difference bounds of [11, 20, 32], with Wulbert's average-value inequalities for convex functions [31], and with the recent Jensen-type inequalities for divided differences in [2, 3].

The cyclic refinement of Theorem 1.7 can similarly be integrated over Δ_n to produce bounds involving only cyclically consecutive blocks of nodes.

Corollary 2.5 (A cyclic node-reduction bound for divided differences). *Assume that $[\alpha, \beta] \subset \mathbb{R}$, $x_0, \dots, x_n \in [\alpha, \beta]$ (possibly repeated), and that $f \in C^n([\alpha, \beta])$ with $f^{(n)}$ convex on $[\alpha, \beta]$. Fix $k \in \{0, \dots, n\}$ and interpret indices cyclically modulo $n+1$ (i.e. $i \oplus j$). Then*

(12)

$$n! f[x_0, \dots, x_n] \leq \frac{k!}{n+1} \sum_{i=0}^n f^{(n-k)}[x_i, x_{i \oplus 1}, \dots, x_{i \oplus k}] \leq \frac{1}{n+1} \sum_{i=0}^n f^{(n)}(x_i).$$

Proof. For $u \in \Delta_n$, apply Theorem 1.7 with the nodes $v_i = x_i$, the weights $p_i = u_i$, and the uniform cyclic coefficients

$$\lambda_0 = \dots = \lambda_k = \frac{1}{k+1}.$$

This yields, for the convex function $g = f^{(n)}$,

$$g\left(\sum_{j=0}^n u_j x_j\right) \leq \sum_{i=0}^n \frac{s_i(u)}{k+1} g(b_i(u)) \leq \sum_{i=0}^n u_i g(x_i),$$

where

$$s_i(u) := \sum_{j=0}^k u_{i \oplus j}, \quad b_i(u) := \frac{\sum_{j=0}^k u_{i \oplus j} x_{i \oplus j}}{s_i(u)} \quad (s_i(u) > 0).$$

Integrating over Δ_n with respect to σ_n gives

$$(13) \quad \int_{\Delta_n} f^{(n)} \left(\sum_{j=0}^n u_j x_j \right) d\sigma_n(u) \leq \frac{1}{k+1} \sum_{i=0}^n \int_{\Delta_n} s_i(u) f^{(n)}(b_i(u)) d\sigma_n(u) \\ \leq \int_{\Delta_n} \sum_{i=0}^n u_i f^{(n)}(x_i) d\sigma_n(u).$$

For the left-hand side, the Hermite–Genocchi formula (1) yields

$$\int_{\Delta_n} f^{(n)} \left(\sum_{j=0}^n u_j x_j \right) d\sigma_n(u) = n! f[x_0, \dots, x_n].$$

For the middle term, Lemma 2.1 applied to the cyclic block $I_i = \{i, i \oplus 1, \dots, i \oplus k\}$ gives

$$\int_{\Delta_n} s_i(u) f^{(n)}(b_i(u)) d\sigma_n(u) = \frac{k+1}{n+1} \int_{\Delta_k} f^{(n)} \left(\sum_{j=0}^k v_j x_{i \oplus j} \right) d\sigma_k(v).$$

Hence

$$\frac{1}{k+1} \sum_{i=0}^n \int_{\Delta_n} s_i(u) f^{(n)}(b_i(u)) d\sigma_n(u) = \frac{1}{n+1} \sum_{i=0}^n \int_{\Delta_k} f^{(n)} \left(\sum_{j=0}^k v_j x_{i \oplus j} \right) d\sigma_k(v).$$

Applying Hermite–Genocchi to $\phi = f^{(n-k)}$ at the nodes $(x_i, x_{i \oplus 1}, \dots, x_{i \oplus k})$ yields

$$\int_{\Delta_k} f^{(n)} \left(\sum_{j=0}^k v_j x_{i \oplus j} \right) d\sigma_k(v) = k! f^{(n-k)}[x_i, x_{i \oplus 1}, \dots, x_{i \oplus k}].$$

For the right-hand side, symmetry of σ_n gives

$$\int_{\Delta_n} \sum_{i=0}^n u_i f^{(n)}(x_i) d\sigma_n(u) = \sum_{i=0}^n f^{(n)}(x_i) \int_{\Delta_n} u_i d\sigma_n(u) = \frac{1}{n+1} \sum_{i=0}^n f^{(n)}(x_i).$$

Substituting these evaluations into (13) gives (12). $\square$

Remark 2.6 (Approximation and numerical analysis). *Divided differences are the coefficients of Newton interpolation and enter its remainder formula; see [4, 8]. Under the convexity assumption on $f^{(n)}$, Remark 2.4 gives computable upper bounds for $n!f[x_0, \dots, x_n]$ using lower-order divided differences of derivatives, while Jensen’s inequality gives the lower bound $f^{(n)}(\bar{x})$. The cyclic bound uses only $n+1$ blocks at each level, in place of all $\binom{n+1}{k+1}$ subsets. These are potential tools for bounding interpolation coefficients when derivative information is available; an error estimate*

must also account for the nodal product in the remainder. For $n = 1$, the averaged endpoints give the classical Hermite–Hadamard bounds for the integral mean of a convex function, and the averaged Jensen gap is its normalized trapezoidal-rule deficit. In higher orders, the positive integral representation connects the construction with the B-spline interpretation of divided differences described in [8].

3. DIRICHLET AVERAGES, JENSEN GAPS, AND SHARPNESS

We first evaluate the minimum and maximum coordinate moments for the uniform law and average Dragomir’s comparison. We then use concavity in the weights to improve its upper bound, translate the estimates into divided differences, and extend both the reduction formula and the chain to general Dirichlet parameters. The final remarks discuss limiting regimes and sharpness.

The normalized simplex law is $\sigma_n = \text{Dir}(1, \dots, 1)$ [19, Ch. 49]. Its coordinate extrema are order statistics [7]; the harmonic-number formula for the maximum belongs to the classical broken-stick literature [13]. Combined with Dragomir’s theorem [9], their moments give the factors $1/(n+1)$ and H_{n+1} below.

For $u = (u_0, \dots, u_n) \in \Delta_n$ write

$$u_{\min} := \min_{0 \leq i \leq n} u_i, \quad u_{\max} := \max_{0 \leq i \leq n} u_i.$$

Lemma 3.1 (Uniform Dirichlet minimum). *If $U \sim \text{Dir}(1, \dots, 1)$, then*

$$\mathbb{E}(U_{\min}) = \frac{1}{(n+1)^2}.$$

Proof. Using the tail formula $\mathbb{E}[Y] = \int_0^\infty \sigma_n(Y \geq t) dt$ for the nonnegative random variable $Y = U_{\min}$, we compute $\sigma_n(u_{\min} \geq t)$. The event $\{u_{\min} \geq t\}$ means $u_i \geq t$ for every $i = 0, \dots, n$. Setting $u_i = t + r_i$ yields $r_i \geq 0$ with $\sum_{i=0}^n r_i = 1 - (n+1)t$, which is the simplex of total mass $1 - (n+1)t$. Its n -dimensional volume scales as $(1 - (n+1)t)^n$ relative to Δ_n , so

$$\sigma_n(u_{\min} \geq t) = (1 - (n+1)t)^n, \quad 0 \leq t \leq \frac{1}{n+1}.$$

Substituting $s = 1 - (n+1)t$ yields

$$\mathbb{E}(U_{\min}) = \int_0^{1/(n+1)} (1 - (n+1)t)^n dt = \frac{1}{n+1} \int_0^1 s^n ds = \frac{1}{(n+1)^2}.$$

□

Lemma 3.2 (Uniform Dirichlet maximum). *If $U \sim \text{Dir}(1, \dots, 1)$, then*

$$\mathbb{E}(U_{\max}) = \frac{H_{n+1}}{n+1},$$

where $H_m := \sum_{j=1}^m \frac{1}{j}$ is the m -th harmonic number.

Proof. We use $\mathbb{E}[U_{\max}] = \int_0^1 \sigma_n(u_{\max} > t) dt$. For each index set $J \subset \{0, \dots, n\}$ with $|J| = j$, the event $\{u_i > t \ \forall i \in J\}$ has probability $(1 - jt)_+^n$ by the same scaling argument as in Lemma 3.1. By inclusion-exclusion,

$$\sigma_n(u_{\max} > t) = \sum_{j=1}^{n+1} (-1)^{j+1} \binom{n+1}{j} (1 - jt)_+^n.$$

Integrating termwise,

$$\int_0^{1/j} (1 - jt)^n dt = \frac{1}{j} \int_0^1 s^n ds = \frac{1}{j(n+1)},$$

so

$$\mathbb{E}(U_{\max}) = \frac{1}{n+1} \sum_{j=1}^{n+1} (-1)^{j+1} \binom{n+1}{j} \frac{1}{j}.$$

To evaluate the alternating sum, start from the identity

$$1 - (1 - x)^{n+1} = \sum_{j=1}^{n+1} (-1)^{j+1} \binom{n+1}{j} x^j.$$

Dividing by x and integrating over $[0, 1]$ gives

$$\sum_{j=1}^{n+1} (-1)^{j+1} \binom{n+1}{j} \frac{1}{j} = \int_0^1 \frac{1 - (1 - x)^{n+1}}{x} dx = \sum_{m=0}^n \int_0^1 (1 - x)^m dx = H_{n+1},$$

which proves the claim. $\square$

Theorem 3.3 (Uniform Dirichlet averaged Dragomir comparison). *Let $C \subset V$ be convex and let $x_0, \dots, x_n \in C$. Let $U \sim \text{Dir}(1, \dots, 1)$ and set*

$$\mu_0(U) := \sum_{i=0}^n U_i \delta_{x_i}, \quad \nu_{\text{unif}} := \frac{1}{n+1} \sum_{i=0}^n \delta_{x_i}.$$

Then, for every convex $\varphi : C \rightarrow \mathbb{R}$,

$$\frac{1}{n+1} J_{\varphi}(\nu_{\text{unif}}) \leq \mathbb{E} J_{\varphi}(\mu_0(U)) = \int_C \varphi d\tilde{\mu}_0 - \int_C \varphi d\tilde{\mu}_n \leq H_{n+1} J_{\varphi}(\nu_{\text{unif}}).$$

Proof. Corollary 1.10 applied with the random weights $\mathbf{p} = U = (U_0, \dots, U_n)$ gives, almost surely,

$$(n+1) U_{\min} J_{\varphi}(\nu_{\text{unif}}) \leq J_{\varphi}(\mu_0(U)) \leq (n+1) U_{\max} J_{\varphi}(\nu_{\text{unif}}).$$

Taking expectations and using Lemma 3.1 and Lemma 3.2 yields

$$\frac{1}{n+1} J_{\varphi}(\nu_{\text{unif}}) \leq \mathbb{E} J_{\varphi}(\mu_0(U)) \leq H_{n+1} J_{\varphi}(\nu_{\text{unif}}).$$

The identity in the middle follows from Theorem 2.2: since $\tilde{\mu}_0 = \nu_{\text{unif}}$ and $\tilde{\mu}_n = (T_{\mathbf{x}})_{\#}\sigma_n$, one has

$$\mathbb{E}J_{\varphi}(\mu_0(U)) = \frac{1}{n+1} \sum_{i=0}^n \varphi(x_i) - \int_{\Delta_n} \varphi\left(\sum_{i=0}^n u_i x_i\right) d\sigma_n(u) = \int_C \varphi d\tilde{\mu}_0 - \int_C \varphi d\tilde{\mu}_n.$$

□

Proposition 3.4 (Concavity of the averaged Jensen gap). *With the notation of Theorem 3.3, one also has*

$$(14) \quad \mathbb{E}J_{\varphi}(\mu_0(U)) \leq J_{\varphi}(\nu_{\text{unif}}).$$

More generally, the map

$$F(u) := J_{\varphi}\left(\sum_{i=0}^n u_i \delta_{x_i}\right) = \sum_{i=0}^n u_i \varphi(x_i) - \varphi\left(\sum_{i=0}^n u_i x_i\right) \quad (u \in \Delta_n)$$

is concave on Δ_n .

Proof. The first term in F is affine in u , while the second term is convex in u because φ is convex and the map $u \mapsto \sum_{i=0}^n u_i x_i$ is affine. Hence F is concave on Δ_n .

If $U \sim \text{Dir}(1, \dots, 1)$, then $\mathbb{E}(U_i) = \frac{1}{n+1}$ for every i . Therefore Jensen's inequality for the concave function F gives

$$\mathbb{E}J_{\varphi}(\mu_0(U)) = \mathbb{E}F(U) \leq F(\mathbb{E}U) = F\left(\frac{1}{n+1}, \dots, \frac{1}{n+1}\right) = J_{\varphi}(\nu_{\text{unif}}),$$

which is (14). □

Remark 3.5 (Explicit form and divided-difference consequences). *Since $\tilde{\mu}_0 = \nu_{\text{unif}}$ and $\tilde{\mu}_n = (T_{\mathbf{x}})_{\#}\sigma_n$,*

$$\mathbb{E}J_{\varphi}(\mu_0(U)) = \frac{1}{n+1} \sum_{i=0}^n \varphi(x_i) - \int_{\Delta_n} \varphi\left(\sum_{j=0}^n u_j x_j\right) d\sigma_n(u).$$

When $C = [\alpha, \beta] \subset \mathbb{R}$ and $\varphi = f^{(n)}$ is the convex n -th derivative of some $f \in C^n([\alpha, \beta])$, the Hermite–Genocchi formula gives

$$\int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^n u_j x_j\right) d\sigma_n(u) = n! f[x_0, \dots, x_n].$$

Consequently,

$$\begin{aligned} \frac{1}{n+1} \sum_{i=0}^n f^{(n)}(x_i) - J_{f^{(n)}}(\nu_{\text{unif}}) &\leq n! f[x_0, \dots, x_n] \\ &\leq \frac{1}{n+1} \sum_{i=0}^n f^{(n)}(x_i) - \frac{1}{n+1} J_{f^{(n)}}(\nu_{\text{unif}}). \end{aligned}$$

The lower estimate uses Proposition 3.4, whereas Theorem 3.3 also yields the weaker but structurally informative alternative

$$\frac{1}{n+1} \sum_{i=0}^n f^{(n)}(x_i) - H_{n+1} J_{f^{(n)}}(\nu_{\text{unif}}) \leq n! f[x_0, \dots, x_n].$$

In this way the averaged Jensen deficit becomes a direct companion to the recent divided-difference Jensen inequalities in [2, 3] and to average-value inequalities for convex functions such as Wulbert's result [31].

3.1 Dirichlet distribution and simplex reduction

Let $\alpha \in (0, \infty)^{n+1}$, say $\alpha = (\alpha_0, \dots, \alpha_n)$. Write

$$\alpha_+ := \sum_{i=0}^n \alpha_i.$$

For singleton parameter vectors, $\text{Dir}(\alpha_i)$ denotes the unit mass on Δ_0 . For $n \geq 1$, we denote by $\sigma_\alpha = \text{Dir}(\alpha_0, \dots, \alpha_n)$ the Dirichlet probability measure on Δ_n , with density

$$d\sigma_\alpha(u) = \frac{\Gamma(\alpha_+)}{\prod_{i=0}^n \Gamma(\alpha_i)} \prod_{i=0}^n u_i^{\alpha_i-1} du_0 \cdots du_{n-1}.$$

Dirichlet laws form the natural parametric family of probability measures on the simplex [19, Ch. 49]; see also [7] for their connection with order-statistics techniques. Their first moments satisfy

$$\mathbb{E}_{\sigma_\alpha}(U_i) = \frac{\alpha_i}{\alpha_+}, \quad \text{Var}_{\sigma_\alpha}(U_i) = \frac{\alpha_i(\alpha_+ - \alpha_i)}{\alpha_+^2(\alpha_+ + 1)}.$$

For $I = \{i_0 < \dots < i_k\} \in \mathcal{I}_k$ set

$$\alpha_I := (\alpha_{i_0}, \dots, \alpha_{i_k}), \quad \gamma_I := \sum_{j=0}^k \alpha_{i_j}.$$

The following reduction formula is the Dirichlet analogue of Lemma 2.1.

Lemma 3.6 (Dirichlet simplex reduction). *Let $C \subset V$ be convex, let $x_0, \dots, x_n \in C$, and fix $k \in \{0, \dots, n\}$. Let $I = \{i_0 < \dots < i_k\} \subset \{0, \dots, n\}$ and define, for $u \in \Delta_n$,*

$$s_I(u) := \sum_{j=0}^k u_{i_j}, \quad b_I(u) := \frac{\sum_{j=0}^k u_{i_j} x_{i_j}}{s_I(u)} \quad (s_I(u) > 0).$$

Then for every Borel function $h : C \rightarrow \mathbb{R}$ whose composition in the right-hand integral below is integrable,

$$(15) \quad \int_{\Delta_n} s_I(u) h(b_I(u)) d\sigma_\alpha(u) = \frac{\gamma_I}{\alpha_+} \int_{\Delta_k} h\left(\sum_{j=0}^k v_j x_{i_j}\right) d\sigma_{\alpha_I}(v).$$

Proof. The calculation below applied first to $|h|$ justifies absolute integrability and the use of Fubini's theorem. The case $k = n$ is immediate, since then $I = \{0, \dots, n\}$, $\gamma_I = \alpha_+$, and $\sigma_{\alpha_I} = \sigma_\alpha$.

Assume $0 \leq k \leq n - 1$. By relabeling we may suppose that $I = \{0, 1, \dots, k\}$. Write

$$s := u_0 + \dots + u_k, \quad \gamma := \gamma_I = \alpha_0 + \dots + \alpha_k, \quad \beta := \alpha_+ - \gamma.$$

Set $c_\alpha := \Gamma(\alpha_+)/\prod_{i=0}^n \Gamma(\alpha_i)$. For fixed $(u_0, \dots, u_k)$ with $u_j \geq 0$ and $s \leq 1$, the remaining coordinates range over an $(n - k - 1)$ -simplex of total mass $1 - s$. Substituting $u_\ell = (1 - s)w_\ell$ for $\ell = k + 1, \dots, n$ yields the Jacobian $(1 - s)^{n-k-1}$ and therefore

$$\prod_{\ell=k+1}^n u_\ell^{\alpha_\ell-1} = (1 - s)^{\beta-(n-k)} \prod_{\ell=k+1}^n w_\ell^{\alpha_\ell-1}.$$

Hence the integration over the remaining coordinates contributes the factor

$$(1 - s)^{\beta-1} \frac{\prod_{\ell=k+1}^n \Gamma(\alpha_\ell)}{\Gamma(\beta)}.$$

Next substitute $u_j = sv_j$ for $j = 0, \dots, k$, with $(v_0, \dots, v_{k-1}) \in \Delta_k$, $v_k = 1 - \sum_{j=0}^{k-1} v_j$, and $s \in [0, 1]$. Then

$$du_0 \dots du_k = s^k dv_0 \dots dv_{k-1} ds, \quad \prod_{j=0}^k u_j^{\alpha_j-1} = s^{\gamma-(k+1)} \prod_{j=0}^k v_j^{\alpha_j-1}.$$

Therefore

$$\begin{aligned} & \int_{\Delta_n} s h(b_I(u)) d\sigma_\alpha(u) \\ &= c_\alpha \frac{\prod_{\ell=k+1}^n \Gamma(\alpha_\ell)}{\Gamma(\beta)} \int_0^1 s^\gamma (1 - s)^{\beta-1} ds \int_{\Delta_k} h\left(\sum_{j=0}^k v_j x_j\right) \prod_{j=0}^k v_j^{\alpha_j-1} dv. \end{aligned}$$

The one-dimensional integral is a Beta integral:

$$\int_0^1 s^\gamma (1 - s)^{\beta-1} ds = B(\gamma + 1, \beta) = \frac{\Gamma(\gamma + 1)\Gamma(\beta)}{\Gamma(\alpha_+ + 1)} = \frac{\gamma \Gamma(\gamma)\Gamma(\beta)}{\alpha_+ \Gamma(\alpha_+)}.$$

After cancellation, the prefactor becomes

$$\frac{\gamma}{\alpha_+} \frac{\Gamma(\gamma)}{\prod_{j=0}^k \Gamma(\alpha_j)},$$

which is precisely $\frac{\gamma_I}{\alpha_+}$ times the normalizing constant of $\sigma_{\alpha_I} = \text{Dir}(\alpha_{i_0}, \dots, \alpha_{i_k})$. This gives (15). $\square$

Remark 3.7 (Recovery of the Lebesgue reduction). *If $\alpha_i = 1$ for every i , then $\sigma_\alpha = \sigma_n$, $\gamma_I = k + 1$, and $\sigma_{\alpha_I} = \sigma_k$. Hence (15) reduces to Lemma 2.1.*

3.2 Dirichlet-averaged extensions

Averaging preserves convex order for any probability law on the simplex. The common barycenter depends on its mean, and the averaged Dragomir factors depend on its coordinate-ratio extrema. The explicit subset formula below additionally uses the Dirichlet reduction in Lemma 3.6. Since Dirichlet laws form the natural parametric family on the simplex [19, Ch. 49], the general extension takes the following form.

Theorem 3.8 (Dirichlet-averaged Jensen chain). *Let $C \subset V$ be convex and fix $x_0, \dots, x_n \in C$. Let $\alpha = (\alpha_0, \dots, \alpha_n) \in (0, \infty)^{n+1}$ and write*

$$q_i := \frac{\alpha_i}{\alpha_+} \quad (i = 0, \dots, n), \quad \alpha_+ = \sum_{i=0}^n \alpha_i.$$

Let σ_α be the Dirichlet probability measure on Δ_n and define, for $k = 0, \dots, n$,

$$\tilde{\mu}_k^{(\alpha)}(A) := \int_{\Delta_n} \mu_k(u)(A) d\sigma_\alpha(u), \quad A \subset C \text{ Borel.}$$

Then the measures $\tilde{\mu}_k^{(\alpha)}$ are probability measures with common barycenter

$$m_\alpha := \sum_{i=0}^n q_i x_i,$$

their endpoints are

$$\tilde{\mu}_0^{(\alpha)} = \nu_\alpha := \sum_{i=0}^n q_i \delta_{x_i}, \quad \tilde{\mu}_n^{(\alpha)} = (T_{\mathbf{x}})_\# \sigma_\alpha,$$

and they satisfy the convex-order chain

$$\tilde{\mu}_n^{(\alpha)} \preceq_{\text{cx}} \tilde{\mu}_{n-1}^{(\alpha)} \preceq_{\text{cx}} \dots \preceq_{\text{cx}} \tilde{\mu}_0^{(\alpha)}.$$

Moreover, for every convex $g : C \rightarrow \mathbb{R}$ for which the integrals are finite,

$$(16) \quad \int_C g d\tilde{\mu}_k^{(\alpha)} = \frac{1}{\binom{n}{k}} \sum_{I=\{i_0 < \dots < i_k\} \in \mathcal{I}_k} \frac{\gamma_I}{\alpha_+} \int_{\Delta_k} g\left(\sum_{j=0}^k v_j x_{i_j}\right) d\sigma_{\alpha_I}(v).$$

Proof. For each fixed $u \in \Delta_n$, the pointwise chain

$$\mu_n(u) \preceq_{\text{cx}} \mu_{n-1}(u) \preceq_{\text{cx}} \dots \preceq_{\text{cx}} \mu_0(u)$$

is given by Theorem 1.4. Integrating against the probability measure σ_α gives the asserted convex-order chain.

Since each $\mu_k(u)$ is a probability measure, so is $\tilde{\mu}_k^{(\alpha)}$. As in the proof of Theorem 2.2, Proposition 1.2 shows that the barycenter of $\mu_k(u)$ equals $T_{\mathbf{x}}(u) = \sum_{i=0}^n u_i x_i$. Therefore

$$\int_C t d\tilde{\mu}_k^{(\alpha)}(t) = \sum_{i=0}^n \left(\int_{\Delta_n} u_i d\sigma_{\alpha}(u) \right) x_i = \sum_{i=0}^n q_i x_i = m_{\alpha},$$

because $\mathbb{E}_{\sigma_{\alpha}}(U_i) = \alpha_i/\alpha_+$ for $U \sim \text{Dir}(\alpha)$.

Expanding the definition of $\mu_k(u)$ and applying Lemma 3.6 termwise yields

$$\begin{aligned} \int_C g d\tilde{\mu}_k^{(\alpha)} &= \int_{\Delta_n} \left(\int_C g d\mu_k(u) \right) d\sigma_{\alpha}(u) \\ &= \frac{1}{\binom{n}{k}} \sum_{I=\{i_0 < \dots < i_k\} \in \mathcal{I}_k} \int_{\Delta_n} s_I(u) g(b_I(u)) d\sigma_{\alpha}(u) \\ &= \frac{1}{\binom{n}{k}} \sum_{I=\{i_0 < \dots < i_k\} \in \mathcal{I}_k} \frac{\gamma_I}{\alpha_+} \int_{\Delta_k} g\left(\sum_{j=0}^k v_j x_{i_j}\right) d\sigma_{\alpha_I}(v), \end{aligned}$$

which is (16).

For $k=0$, the only sets are the singletons $I = \{i\}$, for which $\gamma_I = \alpha_i$ and the integral over Δ_0 reduces to evaluation at x_i . Hence

$$\tilde{\mu}_0^{(\alpha)} = \sum_{i=0}^n \frac{\alpha_i}{\alpha_+} \delta_{x_i} = \nu_{\alpha}.$$

For $k=n$, one has $\tilde{\mu}_n^{(\alpha)} = (T_{\mathbf{x}})_{\#}\sigma_{\alpha}$ by the identity $\mu_n(u) = \delta_{T_{\mathbf{x}}(u)}$. $\square$

Theorem 3.9 (Dirichlet-averaged Dragomir comparison). *Let $C \subset V$ be convex, fix $x_0, \dots, x_n \in C$, and let $\alpha_i > 0$ for $i = 0, \dots, n$. Set*

$$\alpha_+ := \sum_{i=0}^n \alpha_i, \quad q_i := \frac{\alpha_i}{\alpha_+}, \quad \nu_{\alpha} := \sum_{i=0}^n q_i \delta_{x_i}.$$

Use the averaged measures of Theorem 3.8. Let $U \sim \text{Dir}(\alpha_0, \dots, \alpha_n)$ and define

$$\mu_0(U) := \sum_{i=0}^n U_i \delta_{x_i}, \quad c_-(\alpha) := \mathbb{E} \left[\min_{0 \leq i \leq n} \frac{U_i}{q_i} \right], \quad c_+(\alpha) := \mathbb{E} \left[\max_{0 \leq i \leq n} \frac{U_i}{q_i} \right].$$

Then, for every convex $\varphi : C \rightarrow \mathbb{R}$,

$$(17) \quad c_-(\alpha) J_{\varphi}(\nu_{\alpha}) \leq \mathbb{E} J_{\varphi}(\mu_0(U)) = \int_C \varphi d\tilde{\mu}_0^{(\alpha)} - \int_C \varphi d\tilde{\mu}_n^{(\alpha)} \leq c_+(\alpha) J_{\varphi}(\nu_{\alpha}).$$

Moreover,

$$(18) \quad \mathbb{E} J_{\varphi}(\mu_0(U)) \leq J_{\varphi}(\nu_{\alpha}).$$

For the uniform choice $\alpha = (1, \dots, 1)$, one recovers

$$c_-(\alpha) = \frac{1}{n+1}, \quad c_+(\alpha) = H_{n+1}.$$

Proof. Apply Theorem 1.9 to the random probability vector $\mathbf{p} = U$ and the deterministic reference vector $\mathbf{q} = (q_0, \dots, q_n)$. Almost surely,

$$\min_{0 \leq i \leq n} \frac{U_i}{q_i} J_\varphi(\nu_\alpha) \leq J_\varphi(\mu_0(U)) \leq \max_{0 \leq i \leq n} \frac{U_i}{q_i} J_\varphi(\nu_\alpha).$$

Taking expectations yields the inequalities in (17). The identity in the middle follows exactly as in Theorem 3.3, using $\tilde{\mu}_0^{(\alpha)} = \nu_\alpha$ and $\tilde{\mu}_n^{(\alpha)} = (T_{\mathbf{x}})_\# \sigma_\alpha$.

For the upper comparison (18), use the concave map

$$F(u) = \sum_{i=0}^n u_i \varphi(x_i) - \varphi\left(\sum_{i=0}^n u_i x_i\right)$$

from Proposition 3.4. Since $\mathbb{E}(U_i) = q_i$, Jensen's inequality gives

$$\mathbb{E}J_\varphi(\mu_0(U)) = \mathbb{E}F(U) \leq F(\mathbb{E}U) = J_\varphi(\nu_\alpha).$$

Finally, if $\alpha = (1, \dots, 1)$, then $q_i = \frac{1}{n+1}$ and

$$c_-(\alpha) = (n+1)\mathbb{E}(U_{\min}) = \frac{1}{n+1}, \quad c_+(\alpha) = (n+1)\mathbb{E}(U_{\max}) = H_{n+1},$$

by Lemma 3.1 and Lemma 3.2. $\square$

Remark 3.10 (Symmetric Dirichlet family and limiting regimes). *In the symmetric case $\alpha_0 = \dots = \alpha_n = \alpha > 0$, one has $\nu_\alpha = \nu_{\text{unif}}$ and*

$$c_-(\alpha) = (n+1)\mathbb{E}_\alpha(U_{\min}), \quad c_+(\alpha) = (n+1)\mathbb{E}_\alpha(U_{\max}),$$

where $U \sim \text{Dir}(\alpha, \dots, \alpha)$ and $\mathbb{E}_\alpha$ denotes expectation with respect to this law. The uniform case $\alpha = 1$ recovers

$$c_-(1) = \frac{1}{n+1}, \quad c_+(1) = H_{n+1},$$

by Lemma 3.1 and Lemma 3.2. Since

$$\text{Var}(U_i) = \frac{n}{(n+1)^2((n+1)\alpha+1)},$$

the symmetric Dirichlet family concentrates at the barycenter $(1/(n+1), \dots, 1/(n+1))$ as $\alpha \rightarrow \infty$, so that $U_{\min}, U_{\max} \rightarrow 1/(n+1)$ in probability and hence

$$c_-(\alpha) \rightarrow 1, \quad c_+(\alpha) \rightarrow 1.$$

By contrast, as $\alpha \rightarrow 0^+$ the symmetric Dirichlet law concentrates near the vertices of Δ_n , so one obtains the extremal regime

$$c_-(\alpha) \rightarrow 0, \quad c_+(\alpha) \rightarrow n + 1.$$

Indeed, $\mathbb{E} \sum_i U_i^2 = (\alpha+1)/((n+1)\alpha+1) \rightarrow 1$, and $\sum_i U_i^2 \leq U_{\max} \leq 1$, so $\mathbb{E} U_{\max} \rightarrow 1$ and $\mathbb{E} U_{\min} \leq \mathbb{E}(1 - U_{\max})/n \rightarrow 0$. Thus the harmonic-number upper factor is the special value attached to the Lebesgue case $\text{Dir}(1, \dots, 1)$. More generally, the symmetric Dirichlet family interpolates between a vertex-dominated and a barycenter-dominated averaging regime; compare [7, 13, 19].

Remark 3.11 (Sharpness discussion). The factor $c_+(\alpha)$ in Theorem 3.9 records the average of Dragomir's pointwise upper factor; it is not an optimal averaged constant. The upper constant 1 in (18) is optimal even for a fixed positive parameter vector and any fixed nonconstant real node configuration, when all convex functions are admitted. Indeed, set $b = \max_i x_i$, $q_i = \alpha_i/\alpha_+$, $Q = \sum_{x_i=b} q_i$, and

$$\varphi_\varepsilon(t) = \frac{(t - b + \varepsilon)_+}{\varepsilon}, \quad (r)_+ := \max\{r, 0\}.$$

For sufficiently small $\varepsilon > 0$, all nodes below b and the mean m_α lie below $b - \varepsilon$, so $J_{\varphi_\varepsilon}(\nu_\alpha) = Q$. Since $T_{\mathbf{x}}(U) < b$ almost surely and $0 \leq \varphi_\varepsilon(T_{\mathbf{x}}(U)) \leq 1$, dominated convergence gives

$$\frac{\mathbb{E} J_{\varphi_\varepsilon}(\mu_0(U))}{J_{\varphi_\varepsilon}(\nu_\alpha)} = 1 - \frac{\mathbb{E} \varphi_\varepsilon(T_{\mathbf{x}}(U))}{Q} \rightarrow 1.$$

The lower factor $c_-(\alpha)$ is also optimal if arbitrary vector-valued nodes are allowed. On the standard simplex, take its vertices $x_i = e_i$ and $\varphi(t) = -\min_i(t_i/q_i)$. Then $J_\varphi(\nu_\alpha) = 1$ and $J_\varphi(\mu_0(U)) = \min_i(U_i/q_i)$, attaining the lower bound. This does not determine the best lower factor for a fixed real node configuration.

For the quadratic test function $\varphi(t) = t^2$ on $\mathbb{R}$, the covariance identity

$$\text{Cov}(U_i, U_j) = \frac{q_i \delta_{ij} - q_i q_j}{\alpha_+ + 1}$$

instead gives the exact identity

$$\mathbb{E} J_\varphi(\mu_0(U)) = \frac{\alpha_+}{\alpha_+ + 1} J_\varphi(\nu_\alpha).$$

For $\alpha = (1, \dots, 1)$ this factor is $(n+1)/(n+2)$, as illustrated for $n = 2$ in Example 2.3. Thus the quadratic factor is exact for this test function but is not the optimal upper factor over all convex functions.

4. CONCLUDING REMARKS

The main conceptual contribution is a common construction that combines classical Jensen interpolation with averaging over weights and the Hermite–Genocchi representation. Convex order records the monotonicity of the entire family, the reduction identities identify its intermediate measures, and the divided-difference formulas make this structure available in approximation-theoretic notation. The Dirichlet version changes the averaging law while preserving this organization. The construction is also the positive-weight/simplex counterpart of the polytope-integration method in the Jensen–Steffensen setting [1].

Natural further questions concern optimal lower Jensen-gap constants for fixed real node configurations and Dirichlet laws, equality in the intermediate averaged comparisons, and the dependence of these comparisons on the concentration parameter. The cyclic formulas also suggest studying when a small family of node blocks gives useful interpolation bounds. These questions concern the strength and efficient use of the combined framework, beyond the classical ingredients from which it is built.

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