

DERIVATION OF CALCULATION FORMULAS FOR CARLITZ NUMBERS AND POLYNOMIALS WITH THEIR APPLICATIONS

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Motivation of this paper is to investigate the Carlitz numbers and polynomials by using generating function techniques. Using these functions, some identities and relations are obtained. By applying the differential operator to these functions, several calculation formulas involving these numbers and polynomials are derived. Furthermore, for the calculation of values and plotting of graphs of the Carlitz numbers and polynomials, some procedures and illustrations in Wolfram language are provided.

1. INTRODUCTION

The aim of this paper is to investigate some fundamental properties of the Carlitz numbers $B(n, k; s)$ and the Carlitz polynomials $B_n(t; s)$, which are respectively defined by

$$(1) \quad B(n, k; s) = \sum_{r=0}^k (-1)^r \binom{n+1}{r} \binom{s(k-r)}{n}$$

and

$$(2) \quad B_n(t; s) = \sum_{k=0}^n B(n, k; s) t^k,$$

2020 Mathematics Subject Classification. 05A15, 11B68, 11B73, 11B83, 26C05, 26A24.

Keywords and Phrases. Apostol-type numbers and polynomials, Carlitz numbers and polynomials, Generalized Fubini numbers, Multiple variables polynomials, Stirling numbers.

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where $k = 0, 1, \dots, n$ and $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$,

$$\binom{st}{n} = \frac{st(st-1)(st-2)\dots(st-n+1)}{n!} = \frac{(st)^n}{n!},$$

and $s \in \mathbb{R}$, set of real numbers (*cf.* [2, pp. 523-525, Eqs. (14.14) and (14.16)]). By using generating functions of the Carlitz numbers and polynomials, we also derive many novel relations involving the Apostol-type numbers and polynomials, the Carlitz numbers and polynomials, the generalized Fubini numbers, the multiple variables polynomials, the Stirling numbers, etc. The results of this paper are related to many special numbers and polynomials, which have attracted considerable attention from researchers. These numbers and polynomials, together with their generating functions, play an important role in various areas of mathematics. These numbers are often computed using binomial coefficients. Moreover, generalized binomial coefficients play a fundamental part in the construction of many special numbers. Similarly, polynomials whose coefficients involve these numbers provide a clearer understanding of their structural properties. This reveals strong relationships among special numbers, polynomials, and binomial coefficients. Our other aim in this paper is to apply differential operators to generating functions; some identities and derivative formulas pertaining to the Carlitz numbers and polynomials are also derived. In addition, many relations and explicit formulas are obtained.

In order to give our calculations formulas for the Carlitz numbers, we firstly begin by presenting the required notation and definitions. In our calculations, we assume that

$$0^r = \begin{cases} 1, & r = 0 \\ 0, & r \in \mathbb{N} \end{cases}$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$.

By using the following generalized binomials

$$\begin{aligned} \binom{st}{0} &= \binom{t}{0} = 1, \\ \binom{st}{1} &= \binom{s}{1} \binom{t}{1}, \\ \binom{st}{2} &= \binom{s}{2} \binom{t+1}{2} + \binom{s+1}{2} \binom{t}{2}, \end{aligned}$$

the following basic properties of the Carlitz numbers can be easily found:

$$\binom{st}{n} = \sum_{k=0}^n B(n, k; s) \binom{t+n-k}{n},$$

where $n \in \mathbb{N}_0$ (*cf.* [2, p. 523, Eq. (14.12)]). For $k > n$, from the generalized binomials, one has

$$B(n, k; s) = 0.$$

By the following initial conditions

$$B(0, 0; s) = 1,$$

for $n > 0$

$$B(n, 0; s) = 0,$$

and also for $k > n$

$$B(n, k; s) = 0,$$

a triangular recurrence relation is given by

$$(3) \quad (n+1) B(n+1, k; s) = (sk - n) B(n, k; s) + (s(n - k + 2) + n) B(n, k - 1; s)$$

where $k = 1, 2, \dots, n+1$ and $n \in \mathbb{N}_0$ (cf. [2, p. 524, Eq. (14.15)]). In particular, when $k > 0$, we have

$$B(0, k; s) = 0.$$

Generating function for the Carlitz numbers is given by

$$(4) \quad \frac{1-t}{1-t(1+u(1-t))^s} = \sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; s) t^k u^n,$$

where $k = 0, 1, \dots, n$ and $n \in \mathbb{N}_0$ (cf. [2, p. 527, Eq. (14.18)]).

For $t = 1$, the quotient on the left-hand side of Eq. (4) has the indeterminate form $\frac{0}{0}$. For fixed u with $su \neq 1$, the necessary limit process gives

$$\lim_{t \rightarrow 1} \frac{1-t}{1-t(1+u(1-t))^s} = \frac{1}{1-su}.$$

Using the method of Kucukoglu and Simsek [12], we also obtain

$$B_0(1; s) = 1 \quad \text{and} \quad B_n(1; s) = s^n, \quad n \geq 1.$$

Note that some formulas and relations involving the numbers $B(n, k; s)$ and the polynomials $B_n(t; s)$ were given in the book by Charalambides [2]. The book explains some of their properties and also includes several useful examples and identities. Kilar and Simsek [6] also studied probabilistic moment generating functions of the negative binomial and geometric distributions for the Carlitz polynomials.

For $|u| < 1$, the Stirling numbers of the first kind are defined by

$$(5) \quad \frac{(\ln(1+u))^r}{r!} = \sum_{n=0}^{\infty} S_1(n, r) \frac{u^n}{n!}$$

and

$$x^{\underline{n}} = \sum_{v=0}^n S_1(n, v) x^v$$

with $S_1(n, n) = 1$ for $n \geq 0$, $S_1(n, 0) = 0$ for $n > 0$, and $S_1(n, v) = 0$ for $v > n$ (cf. [1]-[4], [14, 15, 21]).

The Stirling numbers of the second kind are defined by

$$(6) \quad \frac{(e^u - 1)^r}{r!} = \sum_{n=0}^{\infty} S_2(n, r) \frac{u^n}{n!}$$

and

$$x^n = \sum_{v=0}^n S_2(n, v) x^v$$

with $S_2(n, n) = 1$ for $n \geq 0$, $S_2(n, 0) = 0$ for $n > 0$, and $S_2(n, v) = 0$ for $v > n$ (cf. [1]-[4], [14, 15, 21]).

The generalized Fubini numbers of higher order are defined by

$$(7) \quad \left(\frac{e^u - 1}{k + 1 - ke^u} \right)^m = \sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{u^n}{n!},$$

where $|u| < \left| \ln \left(\frac{k}{k+1} \right) \right|$, $m \in \mathbb{N}$ and $k \in \mathbb{N}$ (cf. [7]). For $k = 0$, Eq. (7) reduces to

$$(e^u - 1)^m = \sum_{n=1}^{\infty} f_{n,0}^{(m)} \frac{u^n}{n!}.$$

Setting $m = 1$ in (7) yields the generalized Fubini numbers

$$f_{n,k}^{(1)} = f_{n,k}$$

(cf. [14, p. 398, Eq. (10.26)]), and their calculation formula

$$(8) \quad f_{n,k} = \sum_{v=1}^n k^{v-1} v! S_2(n, v)$$

(cf. [14, p. 399, Eq. (10.26)]).

When $k = m = 1$ in (7), one has

$$(9) \quad f_{n,1} = \sum_{v=0}^{n-1} \binom{n}{v} w_g(v),$$

where $w_g(n)$ denotes the Fubini numbers given by

$$\frac{1}{2 - e^u} = \sum_{n=0}^{\infty} w_g(n) \frac{u^n}{n!},$$

where $|u| < \ln 2$ (cf. [3, 4]).

The Fubini numbers play an important role in combinatorics because they count the number of weak orderings on a set of n elements. These numbers are connected with not only the Stirling numbers of the second kind but also the Apostol-Bernoulli numbers. These numbers demonstrate that they have significant applications in both theoretical and applied mathematics. In the literature, many authors have studied Fubini type numbers and derived numerous formulas for them (cf. [3]-[5], [7, 20, 21]).

The Peters-type Simsek numbers and polynomials of the second kind are respectively defined by

$$(10) \quad \frac{2}{\lambda^2 u + 2(\lambda - 1)} = \sum_{n=0}^{\infty} Y_{n,2}(\lambda) \frac{u^n}{n!},$$

where $|u| < \frac{2|1-\lambda|}{|\lambda|^2}$ and $\lambda \neq 0, 1$, and

$$(11) \quad \frac{2(1+\lambda u)^x}{\lambda^2 u + 2(\lambda - 1)} = \sum_{n=0}^{\infty} Y_{n,2}(x; \lambda) \frac{u^n}{n!},$$

where $|u| < \min \left\{ \frac{1}{|\lambda|}, \frac{2|1-\lambda|}{|\lambda|^2} \right\}$ and $\lambda \neq 0, 1$ (cf. [18, Eqs. (12) and (14)]; see also [8, 9, 17]). For $x = 0$ in (11), we obtain

$$Y_{n,2}(0; \lambda) = Y_{n,2}(\lambda).$$

Using (6), (7) and (10), the following relation is given below:

$$(12) \quad f_{n,k} = -\frac{1}{2k+1} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{v=0}^l \left(\frac{2k+1}{2k} \right)^v Y_{v,2} \left(\frac{2k}{2k+1} \right) S_2(l, v),$$

where $n, k \in \mathbb{N}$ (cf. [5, Corollary 15]).

The Apostol-Euler numbers and polynomials of higher order are respectively defined by

$$(13) \quad \left(\frac{2}{\lambda e^u + 1} \right)^r = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(\lambda) \frac{u^n}{n!}$$

and

$$(14) \quad \left(\frac{2}{\lambda e^u + 1} \right)^r e^{xu} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(x; \lambda) \frac{u^n}{n!}$$

where $|u| < |\ln(-\lambda)|$ (cf. [13, 22]). For $x = 0$ in (14), we have

$$\mathcal{E}_n^{(r)}(0; \lambda) = \mathcal{E}_n^{(r)}(\lambda).$$

For $r = 1$ in (13) and (14), one obtains the Apostol-Euler numbers and the Apostol-Euler polynomials, respectively:

$$\mathcal{E}_n^{(1)}(\lambda) = \mathcal{E}_n(\lambda) \quad \text{and} \quad \mathcal{E}_n^{(1)}(x; \lambda) = \mathcal{E}_n(x; \lambda)$$

(cf. [13, 22]).

The Apostol-type Peters numbers and polynomials are defined by

$$(15) \quad \frac{1}{\left(1 + \gamma(1+u)^\lambda\right)^\mu} = \sum_{n=0}^{\infty} \mathcal{A}_n(\lambda, \mu; \gamma) \frac{u^n}{n!}$$

and

$$(16) \quad \frac{(1+u)^x}{\left(1 + \gamma(1+u)^\lambda\right)^\mu} = \sum_{n=0}^{\infty} \mathcal{A}_n(x; \lambda, \mu; \gamma) \frac{u^n}{n!},$$

respectively (cf. [16, Eqs. (19) and (20)]).

When $x = 0$ in (16), one has

$$\mathcal{A}_n(0; \lambda, \mu; \gamma) = \mathcal{A}_n(\lambda, \mu; \gamma).$$

From (15) and (16), we get

$$(17) \quad \mathcal{A}_n(x; \lambda, \mu; \gamma) = \sum_{v=0}^n \binom{n}{v} x^v \mathcal{A}_{n-v}(\lambda, \mu; \gamma)$$

(cf. [16, Eq. (28)]).

Using (5), (14) and (16), the following relation is derived:

$$(18) \quad \mathcal{A}_n(x; \lambda, \mu; \gamma) = \frac{1}{2^\mu} \sum_{j=0}^n \lambda^j S_1(n, j) \mathcal{E}_j^{(\mu)}\left(\frac{x}{\lambda}; \gamma\right),$$

where $\lambda \neq 0$ (cf. [16, Eq. (23)]).

In [19], Simsek defined the following generating function for sequences of new classes of multiple variables polynomials

$$(19) \quad \frac{1}{1 + \sum_{j=1}^m P_j(x_j) u^j} = \sum_{n=0}^{\infty} \mathbb{Y}_n\left(P\left(\vec{X}_m\right)\right) u^n,$$

where

$$P\left(\vec{X}_m\right) = (P_1(x_1), P_2(x_2), \dots, P_m(x_m))$$

and

$$P_j(x_j) = \sum_{v=0}^d a_v x_j^v$$

is any polynomials in x_j , $m \in \mathbb{N}$, $d \in \mathbb{N}_0$ and $0 \leq j \leq m$ (cf. [19, Eq. (1)]).

Using (19), for $m = 2$ and $P(\vec{X}_2) = (P_1(x_1), P_2(x_2))$, the polynomials $\mathbb{Y}_n(P(\vec{X}_2))$ are computed by the following formula

$$\mathbb{Y}_n(P(\vec{X}_2)) = \sum_{v=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-v} \binom{n-v}{v} (P_1(x_1))^{n-2v} (P_2(x_2))^v,$$

where $\lfloor m \rfloor$ denotes the greatest integer less than or equal to m , that is, $\lfloor m \rfloor = \max\{n \in \mathbb{Z} : n \leq m\}$ (cf. [19, Eq. (7)]; see also [10]). Simsek [19] gave many relations for these polynomials and also derived explicit formulas for them.

The main contribution of this paper is the derivation of new identities and recurrence relations for the Carlitz numbers and polynomials by means of generating functions and differential operators. Some of the obtained results provide alternative derivations or reformulations of known identities. In addition, the connections established with some families of special numbers and polynomials yield alternative representations and further relations for the Carlitz numbers and polynomials.

The remainder of the paper is organized as follows.

In Section 2, some properties of the Carlitz numbers and polynomials are investigated with the aid of their generating functions. Using these functions many formulas and identities are obtained. Further, by applying differential operators to these functions, some relations pertaining to these numbers and polynomials are derived. In Section 3, some applications of the Carlitz numbers and polynomials are presented. Using implementations in the Wolfram Language, some graphical representations and special values of these numbers and polynomials are illustrated. Finally, a concluding section is included.

2. FORMULAS AND RECURRENCE RELATIONS FOR CARLITZ NUMBERS, CARLITZ POLYNOMIALS, AND SPECIAL NUMBERS

In this section, using the generating function techniques, properties of the Carlitz numbers and polynomials are investigated. Some calculation formulas and relations for the Carlitz numbers and polynomials, as well as for the Apostol-type Peters numbers and polynomials, the generalized Fubini numbers, the Peters-type Simsek numbers of the second kind, the Apostol-Euler numbers, the multiple variables polynomials, and the Stirling numbers, are derived. Further, by applying the differential operator to the generating functions of the Carlitz numbers, some recurrence formulas containing these numbers and polynomials are obtained.

Theorem 1. *Let $n \in \mathbb{N}_0$ and $t \neq 1$. Then we have*

$$(20) \quad \sum_{k=0}^n B(n, k; s) t^k = \frac{(1-t)^{n+1}}{n!} \mathcal{A}_n(s, 1; -t),$$

where $\mathcal{A}_n(s, 1; -t)$ denotes the Apostol-type Peters numbers.

Proof. For $t \neq 1$, substituting $\mu = 1$, $\lambda = s$ and $\gamma = -t$ into (15), and replacing u by $(1-t)u$, we obtain

$$\frac{1}{1-t(1+u(1-t))^s} = \sum_{n=0}^{\infty} \mathcal{A}_n(s, 1; -t) (1-t)^n \frac{u^n}{n!}.$$

Combining (4) with the previous equation, we obtain

$$\sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; s) t^k u^n = \sum_{n=0}^{\infty} \mathcal{A}_n(s, 1; -t) (1-t)^{n+1} \frac{u^n}{n!}.$$

Equating the coefficients of u^n in the above equation yields the desired formula. $\square$

Combining (20) with (2) yields the following corollary:

Corollary 2. *Let $n \in \mathbb{N}_0$ and $t \neq 1$. Then we have*

$$B_n(t; s) = \frac{(1-t)^{n+1}}{n!} \mathcal{A}_n(s, 1; -t),$$

where $B_n(t; s)$ denotes the Carlitz polynomials.

Substituting $x = 0$, $\mu = 1$, $\lambda = s$, and $\gamma = -t$ into (18), one has

$$\mathcal{A}_n(s, 1; -t) = \frac{1}{2} \sum_{j=0}^n s^j S_1(n, j) \mathcal{E}_j(-t).$$

Combining the above equation with Eq. (20), we obtain the following result:

Corollary 3. *Let $k = 0, 1, \dots, n$, $n \in \mathbb{N}_0$ and $t \neq 1$. Then we have*

$$\sum_{k=0}^n B(n, k; s) t^k = \frac{(1-t)^{n+1}}{2(n!)} \sum_{j=0}^n s^j S_1(n, j) \mathcal{E}_j(-t).$$

Using Corollary 2 and Eq. (17), we achieve the following result:

Corollary 4. *Let $n \in \mathbb{N}_0$ and $t \neq 1$. Then we have*

$$\mathcal{A}_n(x; s, 1; -t) = \sum_{j=0}^n \frac{n! x^j}{j! (1-t)^{n-j+1}} B_{n-j}(t; s).$$

Remark 5. In [16], Simsek gave many relations and formulas for the Apostol-type Peters numbers and polynomials involving the Stirling numbers of the first and second kinds, the Apostol-Euler polynomials, and the Apostol-Bernoulli polynomials. Using these relations together with Theorem 1 and Corollary 2, some identities can be derived for the Carlitz numbers and polynomials.

Setting $s = 1$ in (4), and performing some algebraic calculations, we get

$$(21) \quad \sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; 1) t^k u^n = \frac{1}{1 - ut}.$$

Thus,

$$\sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; 1) t^k u^n = \sum_{n=0}^{\infty} t^n u^n.$$

Equating the coefficients of u^n on both sides of the above equation yields the following result:

Remark 6. Let $n \in \mathbb{N}_0$. Then we have

$$(22) \quad \sum_{k=0}^n B(n, k; 1) t^k = t^n,$$

or equivalently,

$$B_n(t; 1) = t^n.$$

Further, setting $n = k$ in (22), one has

$$B(n, n; 1) = 1.$$

Theorem 7. Let $n \in \mathbb{N}$. Then we have

$$(23) \quad \sum_{m=0}^{n-1} \sum_{k=0}^m B(m, k; 1) S_2(n, m+1) (m+1)! t^k = f_{n,t}.$$

Proof. By replacing u with $e^u - 1$ in (21), we get

$$\sum_{m=0}^{\infty} \sum_{k=0}^m B(m, k; 1) t^k (e^u - 1)^{m+1} = \frac{e^u - 1}{1 + t - te^u}.$$

Using (6) and (7) together with the above relation and performing the necessary calculations, we obtain

$$\sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \sum_{k=0}^m B(m, k; 1) S_2(n, m+1) t^k (m+1)! \frac{u^n}{n!} = \sum_{n=1}^{\infty} f_{n,t} \frac{u^n}{n!}.$$

Equating the coefficients of $\frac{u^n}{n!}$ in the last equation yields the desired formula. $\square$

By combining (23) with (2) and (12), one obtains the following relation involving the numbers $Y_{n,2}(\lambda)$, the numbers $S_2(n, r)$, and the polynomials $B_n(t; s)$:

Corollary 8. Let $n \in \mathbb{N}$ and $t \neq 0, -\frac{1}{2}$. Then we have

$$\sum_{m=0}^{n-1} (m+1)! B_m(t; 1) S_2(n, m+1) = - \sum_{j=0}^{n-1} \binom{n}{j} \sum_{v=0}^j \frac{(2t+1)^{v-1}}{(2t)^v} S_2(j, v) Y_{v,2} \left(\frac{2t}{2t+1} \right).$$

Substituting $t = 1$ into (23), and joining final equation with (9), we achieve the following corollary:

Corollary 9. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{m=0}^{n-1} \sum_{k=0}^m B(m, k; 1) S_2(n, m+1) (m+1)! = \sum_{j=0}^{n-1} \binom{n}{j} w_g(j).$$

Remark 10. *By using Eq. (22), Eq. (23) reduces to the identity given in Eq. (8). Similarly, Corollaries 8 and 9 recover Eqs. (12) and (9), respectively.*

Theorem 11. *Let $n \in \mathbb{N}_0$. Then we have*

$$(n+1) B_{n+1}(t; s) = st(1-t) \left\{ \frac{\partial}{\partial t} B_n(t; s) \right\} + (t(s(n+1) + n) - n) B_n(t; s).$$

Proof. Summing both sides of the Eq. (3) from $k = 1$ to $n+1$, we get

$$\begin{aligned} \sum_{k=1}^{n+1} B(n+1, k; s) (n+1) t^k &= \sum_{k=1}^{n+1} B(n, k; s) (sk - n) t^k \\ &+ \sum_{k=1}^{n+1} B(n, k-1; s) (s(n-k+2) + n) t^k. \end{aligned} \quad (24)$$

The above equation yields

$$\begin{aligned} \sum_{k=1}^{n+1} B(n+1, k; s) (n+1) t^k &= s \sum_{k=1}^{n+1} k B(n, k; s) t^k - n \sum_{k=1}^{n+1} B(n, k; s) t^k \\ &+ \sum_{k=0}^{n+1} B(n, k; s) (s(n-k+1) + n) t^{k+1}. \end{aligned}$$

Since

$$\sum_{k=1}^n k B(n, k; s) t^k = t \left\{ \frac{\partial}{\partial t} B_n(t; s) \right\},$$

and using (2) and (24), the above equation reduces to the following expression:

$$\begin{aligned} (n+1) B_{n+1}(t; s) &= st \left\{ \frac{\partial}{\partial t} B_n(t; s) \right\} - n B_n(t; s) \\ &+ t(sn + s + n) B_n(t; s) - st^2 \left\{ \frac{\partial}{\partial t} B_n(t; s) \right\}. \end{aligned}$$

Therefore, the proof is complete. $\square$

Theorem 12. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_{n+1}(t; s) = \frac{st}{n+1} \sum_{r=0}^n \sum_{j=0}^r \binom{s-1}{n-r} B_j(t; s) B_{r-j}(t; s) (1-t)^{n-r}.$$

Proof. Differentiating Eq. (4) with respect to u , we obtain

$$(25) \quad \sum_{n=1}^{\infty} n \sum_{k=0}^n B(n, k; s) t^k u^{n-1} = \frac{(1-t)^2 st (1+u(1-t))^{s-1}}{(1-t(1+u(1-t))^s)^2}.$$

Combining (2) with the above equation, we get

$$\sum_{n=0}^{\infty} (n+1) B_{n+1}(t; s) u^n = \sum_{n=0}^{\infty} B_n(t; s) u^n \sum_{n=0}^{\infty} B_n(t; s) u^n st (1+u(1-t))^{s-1}.$$

From the above expression yields

$$\sum_{n=0}^{\infty} (n+1) B_{n+1}(t; s) u^n = st \sum_{n=0}^{\infty} \sum_{j=0}^n B_j(t; s) B_{n-j}(t; s) u^n \sum_{n=0}^{\infty} \binom{s-1}{n} (1-t)^n u^n.$$

Thus

$$\sum_{n=0}^{\infty} (n+1) B_{n+1}(t; s) u^n = st \sum_{n=0}^{\infty} \sum_{r=0}^n \sum_{j=0}^r \binom{s-1}{n-r} (1-t)^{n-r} B_j(t; s) B_{r-j}(t; s) u^n.$$

Equating the coefficients of u^n in the above equation yields the desired formula. $\square$

Theorem 13. Let $n \in \mathbb{N}$. Then we have

$$(26) \quad B_n(t; s) = \sum_{j=1}^n \binom{s}{j} t (1-t)^{j-1} B_{n-j}(t; s),$$

with $B_0(t; s) = 1$.

Proof. Using (2) and (25), we obtain

$$\begin{aligned} & \left(1 - t \sum_{n=0}^{\infty} \binom{s}{n} (1-t)^n u^n \right) \sum_{n=0}^{\infty} (n+1) B_{n+1}(t; s) u^n \\ &= st \sum_{n=0}^{\infty} B_n(t; s) u^n \sum_{n=0}^{\infty} \binom{s-1}{n} (1-t)^{n+1} u^n. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1) B_{n+1}(t; s) u^n &= t \sum_{n=0}^{\infty} \sum_{j=0}^n B_{j+1}(t; s) \binom{s}{n-j} (1-t)^{n-j} (j+1) u^n \\ &= st \sum_{n=0}^{\infty} \sum_{j=0}^n B_j(t; s) \binom{s-1}{n-j} (1-t)^{n+1-j} u^n. \end{aligned}$$

Equating the coefficients of u^n in the above equation, we obtain

$$\begin{aligned} B_{n+1}(t; s) &= \frac{ts}{n+1} \sum_{j=0}^n \binom{s-1}{n-j} (1-t)^{n+1-j} B_j(t; s) \\ &\quad + \frac{t}{n+1} \sum_{j=0}^n \binom{s}{n-j} (j+1) (1-t)^{n-j} B_{j+1}(t; s). \end{aligned}$$

In the second sum, the term for $j = n$ is $tB_{n+1}(t; s)$. Then after some calculations, we get

$$B_n(t; s) = \sum_{j=1}^n \binom{s}{j} t(1-t)^{j-1} B_{n-j}(t; s),$$

where $B_0(t; s) = 1$. Observe that the resulting polynomial identity also holds at $t = 1$. Thus, this gives the desired result. $\square$

Theorem 14. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} \frac{\partial}{\partial t} \{B_n(t; s)\} &= s \sum_{j=0}^{n-1} B_j(t; s) B_{n-1-j}(t; s) \\ &\quad + \sum_{m=2}^n \sum_{j=0}^{n-m} \binom{s}{m} B_j(t; s) B_{n-m-j}(t; s) (1-t)^{m-2} (1-tm). \end{aligned}$$

Proof. Differentiating the Eq. (4) with respect to t , we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial t} \left\{ \sum_{k=0}^n B(n, k; s) t^k \right\} u^n = \frac{(1+u(1-t))^s - 1 - sut(1-t)(1+u(1-t))^{s-1}}{(1-t(1+u(1-t))^s)^2}.$$

After performing the necessary computations, and merging the final equation with Eq. (2) yields

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \{B_n(t; s)\} u^n &= \left(\sum_{n=0}^{\infty} \binom{s}{n} (1-t)^n u^n - st \sum_{n=0}^{\infty} \binom{s-1}{n} (1-t)^{n+1} u^{n+1} - 1 \right) \\ &\quad \times \frac{1}{(1-t)^2} \sum_{n=0}^{\infty} B_n(t; s) u^n \sum_{n=0}^{\infty} B_n(t; s) u^n. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \{B_n(t; s)\} u^n &= \frac{1}{(1-t)^2} \sum_{n=0}^{\infty} \sum_{j=0}^n B_j(t; s) B_{n-j}(t; s) u^n \\ &\quad \times \sum_{n=1}^{\infty} \left(\binom{s}{n} - st \binom{s-1}{n-1} \right) (1-t)^n u^n. \end{aligned}$$

By using above equation, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \{B_n(t; s)\} u^n &= \frac{1}{(1-t)^2} \sum_{n=0}^{\infty} \sum_{j=0}^n B_j(t; s) B_{n-j}(t; s) u^n \\ &\times \sum_{n=0}^{\infty} (1-t(n+1)) \binom{s}{n+1} (1-t)^{n+1} u^{n+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \{B_n(t; s)\} u^n &= \frac{1}{(1-t)^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \sum_{j=0}^{n-1-m} B_j(t; s) B_{n-m-1-j}(t; s) \\ &\times (1-t(m+1)) \binom{s}{m+1} (1-t)^{m+1} u^n. \end{aligned}$$

Equating the coefficients of u^n in the last equation, after some calculations, we get the desired formula. $\square$

Now we give some relations involving the Carlitz numbers and polynomials, and the multiple variables polynomials.

Theorem 15. *Let $n \in \mathbb{N}_0$ and $s \in \mathbb{N}$. Then we have*

$$(27) \quad B_n(t; s) = \mathbb{Y}_n \left(-\binom{s}{1} t, -\binom{s}{2} t(1-t), \dots, -\binom{s}{s} t(1-t)^{s-1} \right).$$

Proof. From (4), we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; s) t^k u^n = \frac{1-t}{1-t \sum_{j=0}^s \binom{s}{j} (1-t)^j u^j}.$$

After some calculations in the equation above yields the following result:

$$\begin{aligned} &\frac{1}{1 - \binom{s}{1} t u - \binom{s}{2} t(1-t) u^2 - \dots - \binom{s}{s} t(1-t)^{s-1} u^s} \\ (28) \quad &= \sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; s) t^k u^n. \end{aligned}$$

By joining the above equation with (19), we obtain

$$\sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; s) t^k u^n = \sum_{n=0}^{\infty} \mathbb{Y}_n \left(-\binom{s}{1} t, -\binom{s}{2} t(1-t), \dots, -\binom{s}{s} t(1-t)^{s-1} \right) u^n.$$

By (2), the above equation can also be written as follows:

$$\sum_{n=0}^{\infty} B_n(t; s) u^n = \sum_{n=0}^{\infty} \mathbb{Y}_n \left(-\binom{s}{1} t, -\binom{s}{2} t(1-t), \dots, -\binom{s}{s} t(1-t)^{s-1} \right) u^n.$$

Equating the coefficients of u^n in the above equation yields the desired formula. $\square$

Remark 16. Combining (27) with (2), we obtain

$$\sum_{k=0}^n B(n, k; s) t^k = \mathbb{Y}_n \left(-\binom{s}{1} t, -\binom{s}{2} t(1-t), \dots, -\binom{s}{s} t(1-t)^{s-1} \right),$$

where $n \in \mathbb{N}_0$ and $s \in \mathbb{N}$.

In what follows, by using similar method in the work of Simsek [19], we give explicit formula for the Carlitz numbers and polynomials.

Substituting $s = 2$ into (28), we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; 2) t^k u^n = \frac{1}{1 - \binom{2}{1} t u - \binom{2}{2} t(1-t) u^2}.$$

From the above equation, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; 2) t^k u^n &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{n+j} \binom{n+j}{j} \\ &\quad \times \left(-\binom{2}{1} t \right)^n \left(-\binom{2}{2} t(1-t) \right)^j u^{n+2j}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^n B(n, k; 2) t^k u^n &= \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-j} \binom{n-j}{j} (-2t)^{n-2j} \\ &\quad \times (-t(1-t))^j u^n. \end{aligned}$$

Equating the coefficients of u^n on both sides of the above equation, we obtain the following result:

Corollary 17. Let $n \in \mathbb{N}_0$. Then we have

$$(29) \quad \sum_{k=0}^n B(n, k; 2) t^k = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} 2^{n-2j} t^{n-j} (1-t)^j.$$

Combining (29) with (2), we also have the following result:

$$B_n(t; 2) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} 2^{n-2j} t^{n-j} (1-t)^j.$$

3. SOME COMPUTATIONAL ILLUSTRATIONS OF CARLITZ NUMBERS AND CARLITZ POLYNOMIALS

In this section, we present some computational illustrations of the Carlitz numbers and polynomials. Wolfram Language implementations of these numbers and polynomials are provided via the Wolfram Cloud platform [23]. Using these implementations, some special values of these numbers and polynomials are computed and presented in tables. Furthermore, graphical representations are included to illustrate the behavior of these polynomials.

Before giving these applications, we note that many calculation algorithms have been developed in different programming languages for special numbers and polynomials in the literature, along with their applications (for details, see [5, 8, 11, 18, 20, 21]). Among these, the Wolfram Language, implemented in Mathematica and available via the Wolfram Cloud platform, is frequently used for such computations. In this paper, we use the Eqs. (1) and (2) to demonstrate these applications. Using these equations, the Carlitz numbers and polynomials are computed and their graphs are plotted. In order to calculate of the Carlitz numbers, we implement Eq. (1) in the Wolfram Language as follows.

Implementation 1: Using (1), the following Wolfram Language code is presented for the Carlitz numbers.

```

1  Unprotect[Power];
2  Power[0, 0] = 1;
3  Protect[Power];
4  CarlitzNumbers[0, 0, s] := 1;
5  CarlitzNumbers[n, 0, s] /; n > 0 := 0;
6  CarlitzNumbers[n, k, s] /; n < 0 || k < 0 || k > n := 0;
7  CarlitzNumbers[n_, k_, s_] := Sum[(-1)^r * Binomial[n + 1, r] * Binomial[s * (k - r), n], {r, 0, k}]
8  TableForm[FullSimplify[Table[CarlitzNumbers[n, k, s], {n, 0, 5}, {k, 0, 4}]], TableHeadings ->
  {Range[0, 5], Range[0, 4]}]

```

Based on the Implementation 1, numerical values of the Carlitz numbers for $n \in \{0, 1, 2, 3, 4, 5\}$ and $k \in \{0, 1, 2, 3, 4\}$ are given in Table 1, which illustrates how these values vary with n and k .

Table 1: The first few values of the numbers $B(n, k; s)$.

$n \backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	0	s	0	0	0
2	0	$\frac{1}{2}(s-1)s$	$\frac{1}{2}(s+1)s$	0	0
3	0	$\frac{1}{6}(s-2)(s-1)s$	$\frac{2}{3}(s^2-1)s$	$\frac{1}{6}(s+2)(s+1)s$	0
4	0	$\frac{1}{24}(s-3)(s-2)(s-1)s$	$\frac{1}{24}(s-1)s(s+1)(11s-18)$	$\frac{1}{24}(s-1)s(s+1)(11s+18)$	$\frac{1}{24}s(s+1)(s+2)(s+3)$
5	0	$\frac{1}{120}(s-4)(s-3)(s-2)(s-1)s$	$\frac{1}{60}(s-2)(s-1)s(s+1)(13s-24)$	$\frac{1}{20}s(11s^4-35s^2+24)$	$\frac{1}{60}(s-1)s(s+1)(s+2)(13s+24)$

Implementation 2: Some calculation of the Carlitz numbers in Wolfram Language.

```

1 TableForm[Table[CarlitzNumbers[n,k,7],{n,0,9},{k,0,6}], TableHeadings -> {Range[0,9],
Range[0,6]}}]
2 TableForm[Table[CarlitzNumbers[n,k,10],{n,0,9},{k,0,6}], TableHeadings -> {Range[0,9],
Range[0,6]}}]

```

Using the Implementation 2, for $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $k \in \{0, 1, 2, 3, 4, 5, 6\}$, some numerical values of the Carlitz numbers are listed in Table 2 and Table 3.

Table 2: Some special values of the numbers $B(n, k; 7)$

$n \backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	7	0	0	0	0	0
2	0	21	28	0	0	0	0
3	0	35	224	84	0	0	0
4	0	35	826	1330	210	0	0
5	0	21	1876	8652	5796	462	0
6	0	7	2954	33390	59710	20664	924
7	0	1	3424	88852	349840	315918	63792
8	0	0	3003	176463	1384803	2636263	1384803
9	0	0	2002	273910	4057690	14525070	15776530

Table 3: Some special values of the numbers $B(n, k; 10)$

$n \backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	10	0	0	0	0	0
2	0	45	55	0	0	0	0
3	0	120	660	220	0	0	0
4	0	210	3795	5280	715	0	0
5	0	252	13992	53262	30492	2002	0
6	0	210	37290	326865	488565	142065	5005
7	0	120	76560	1419000	4521000	3405600	566280
8	0	45	125565	4720815	28759500	44865975	19502505
9	0	10	167860	12628000	137924380	394713550	351966340

To compute the Carlitz polynomials, we use Eq. (2) and Implementation 1 in the Wolfram Language as follows.

Implementation 3: Using (2), the following Wolfram Language code is presented for the Carlitz polynomials $B_n(t; s)$.

```

1 CarlitzPolynomials [n_, t_, s_] := Sum[CarlitzNumbers[n, k, s] * t^k, {k, 0, n}]
2 TableForm[Table[TraditionalForm[Subscript[B, n][t; s] == FullSimplify[ CarlitzPolynomials [n
, t, s ]]], { n, 0, 5}]]

```

By using Implementation 3, for $n \in \{0, 1, 2, 3, 4, 5\}$, some values of the Carlitz polynomials are given as follows:

$$\begin{aligned}
B_0(t; s) &= 1, \\
B_1(t; s) &= st, \\
B_2(t; s) &= \frac{1}{2} st (st + s + t - 1), \\
B_3(t; s) &= \frac{1}{6} st (4(s^2 - 1)t + s^2 + (s + 1)(s + 2)t^2 - 3s + 2), \\
B_4(t; s) &= \frac{1}{24} st (s^3(t + 1)(t(t + 10) + 1) + 6s^2(t - 1)(t(t + 4) + 1)) \\
&\quad + \frac{1}{24} st (11s(t - 1)^2(t + 1) + 6(t - 1)^3), \\
B_5(t; s) &= \frac{1}{120} st (s^4(t(t(t(t + 26) + 66) + 26) + 1)) \\
&\quad + \frac{1}{120} st (10s^3(t - 1)(t + 1)(t(t + 10) + 1) + 35s^2(t - 1)^2 t(t + 4) + 1) \\
&\quad + \frac{1}{120} st (50s(t - 1)^3(t + 1) + 24(t - 1)^4).
\end{aligned}$$

Implementation 4: Some calculation of the Carlitz polynomials $B_n(t; s)$ in Wolfram Language.

```

1 TableForm[Table[TraditionalForm[Subscript[B, n][t; 2] == CarlitzPolynomials[n, t, 2]], { n
, 0, 10}]]
2 TableForm[Table[TraditionalForm[Subscript[B, n][t; 4] == CarlitzPolynomials[n, t, 4]], { n
, 0, 10}]]

```

By using Implementation 4, for $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, some values of the

polynomials $B_n(t; 2)$ are given as follows:

$$\begin{aligned}
B_0(t; 2) &= 1, \\
B_1(t; 2) &= 2t, \\
B_2(t; 2) &= 3t^2 + t, \\
B_3(t; 2) &= 4t^3 + 4t^2, \\
B_4(t; 2) &= 5t^4 + 10t^3 + t^2, \\
B_5(t; 2) &= 6t^5 + 20t^4 + 6t^3, \\
B_6(t; 2) &= 7t^6 + 35t^5 + 21t^4 + t^3, \\
B_7(t; 2) &= 8t^7 + 56t^6 + 56t^5 + 8t^4, \\
B_8(t; 2) &= 9t^8 + 84t^7 + 126t^6 + 36t^5 + t^4, \\
B_9(t; 2) &= 10t^9 + 120t^8 + 252t^7 + 120t^6 + 10t^5, \\
B_{10}(t; 2) &= 11t^{10} + 165t^9 + 462t^8 + 330t^7 + 55t^6 + t^5.
\end{aligned}$$

By using Implementation 4, for $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, some values of the polynomials $B_n(t; 4)$ are given as follows:

$$\begin{aligned}
B_0(t; 4) &= 1, \\
B_1(t; 4) &= 4t, \\
B_2(t; 4) &= 10t^2 + 6t, \\
B_3(t; 4) &= 20t^3 + 40t^2 + 4t, \\
B_4(t; 4) &= 35t^4 + 155t^3 + 65t^2 + t, \\
B_5(t; 4) &= 56t^5 + 456t^4 + 456t^3 + 56t^2, \\
B_6(t; 4) &= 84t^6 + 1128t^5 + 2128t^4 + 728t^3 + 28t^2, \\
B_7(t; 4) &= 120t^7 + 2472t^6 + 7728t^5 + 5328t^4 + 728t^3 + 8t^2, \\
B_8(t; 4) &= 165t^8 + 4950t^7 + 23607t^6 + 27876t^5 + 8451t^4 + 486t^3 + t^2, \\
B_9(t; 4) &= 220t^9 + 9240t^8 + 63460t^7 + 116304t^6 + 63460t^5 + 9240t^4 + 220t^3, \\
B_{10}(t; 4) &= 286t^{10} + 16302t^9 + 154518t^8 + 411334t^7 + 358490t^6 \\
&\quad + 100298t^5 + 7282t^4 + 66t^3.
\end{aligned}$$

For instance, using the recurrence relation in Theorem 13 with $s = 2$ and $n = 2$, we obtain

$$B_2(t; 2) = \binom{2}{1}tB_1(t; 2) + \binom{2}{2}t(1-t)B_0(t; 2) = 3t^2 + t.$$

On the other hand, Corollary 17 gives

$$B_2(t; 2) = \binom{2}{0}2^2t^2 + \binom{1}{1}2^0t(1-t) = 3t^2 + t.$$

Thus, both formulas agree with the value obtained in Implementation 4.

To graphical representations of the Carlitz polynomials, the following Wolfram Language implementation is used.

Implementation 5: Some plots of the polynomials $B_n(t; s)$ in Wolfram Language.

```

1 Plot[Evaluate@Table[CarlitzPolynomials[2, t, s], {s, 0, 5}], {t, -10, 10}, AxesLabel -> {Style[
TraditionalForm[t], 12], Style[TraditionalForm[Subscript[B, 2][t; s]], 12]}, PlotLegends -> {
"s=0", "s=1", "s=2", "s=3", "s=4", "s=5"}]
2 Plot[Evaluate@Table[CarlitzPolynomials[5, t, s], {s, 0, 5}], {t, -10, 10}, AxesLabel -> {Style[
TraditionalForm[t], 12], Style[TraditionalForm[Subscript[B, 5][t; s]], 12]}, PlotLegends -> {
"s=0", "s=1", "s=2", "s=3", "s=4", "s=5"}]
3 Plot[Evaluate@Table[CarlitzPolynomials[8, t, s], {s, 0, 5}], {t, -10, 10}, AxesLabel -> {Style[
TraditionalForm[t], 12], Style[TraditionalForm[Subscript[B, 8][t; s]], 12]}, PlotLegends -> {
"s=0", "s=1", "s=2", "s=3", "s=4", "s=5"}]
4 Plot[Evaluate@Table[CarlitzPolynomials[15, t, s], {s, 0, 5}], {t, -10, 10}, AxesLabel -> {Style[
TraditionalForm[t], 12], Style[TraditionalForm[Subscript[B, 15][t; s]], 12]}, PlotLegends -> {
"s=0", "s=1", "s=2", "s=3", "s=4", "s=5"}]

```

Using Implementation 5, for $n = 2, 5, 8, 15$, $s \in \{0, 1, 2, 3, 4, 5\}$ and $t \in [-10, 10]$, some plots of the polynomials $B_n(t; s)$ are presented in Figures 1-4:

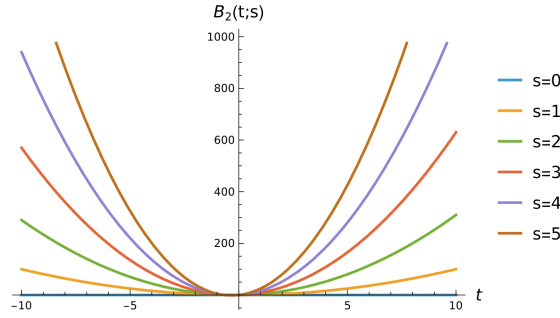


Figure 1: Plots of the polynomials $B_2(t; s)$ in state that $s \in \{0, 1, 2, 3, 4, 5\}$ and $t \in [-10, 10]$.

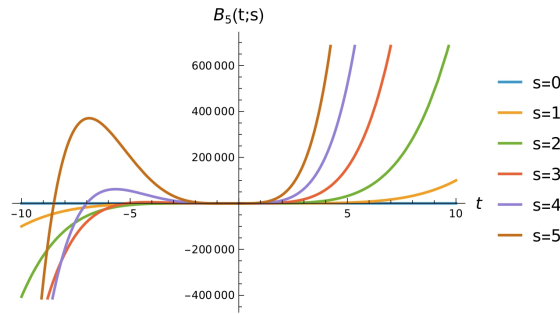


Figure 2: Plots of the polynomials $B_5(t; s)$ in state that $s \in \{0, 1, 2, 3, 4, 5\}$ and $t \in [-10, 10]$.

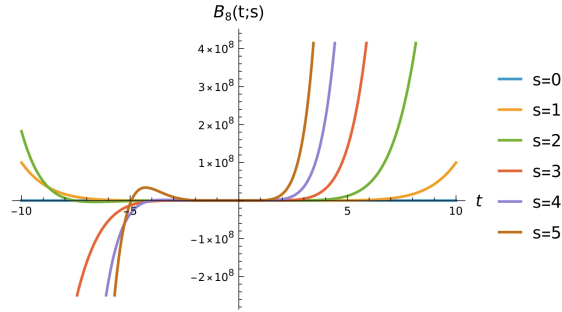


Figure 3: Plots of the polynomials $B_8(t; s)$ in state that $s \in \{0, 1, 2, 3, 4, 5\}$ and $t \in [-10, 10]$.

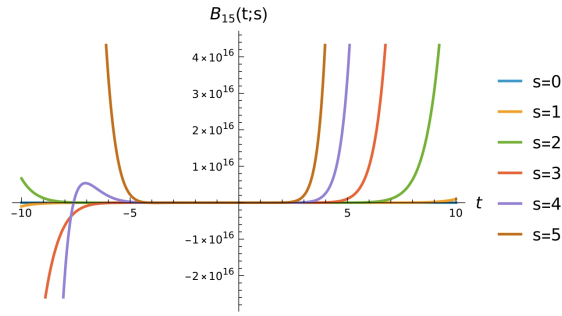


Figure 4: Plots of the polynomials $B_{15}(t; s)$ in state that $s \in \{0, 1, 2, 3, 4, 5\}$ and $t \in [-10, 10]$.

To surface plots of the Carlitz polynomials, the following Wolfram Language implementation is used.

Implementation 6: Some surface plots of the polynomials $B_n(t; s)$ in Wolfram Language.

```

1  Plot3D[CarlitzPolynomials[2, t, s], {t, -20, 20}, {s, 3, 5}, AxesLabel -> {Style[TraditionalForm[t], 12], Style[TraditionalForm[s], 12], Style[TraditionalForm[Subscript[B, 2][t; s]], 12]},
   PlotRange -> All]
2  Plot3D[CarlitzPolynomials[5, t, s], {t, -20, 20}, {s, 3, 5}, AxesLabel -> {Style[TraditionalForm[t], 12], Style[TraditionalForm[s], 12], Style[TraditionalForm[Subscript[B, 5][t; s]], 12]},
   PlotRange -> All]

```

Using Implementation 6, for $n = 2, 5$, $t \in [-20, 20]$ and $s \in [3, 5]$, some surface plots of the polynomials $B_n(t; s)$ are presented in Figures 5-6.

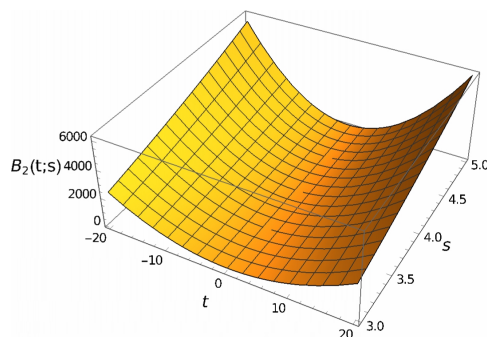


Figure 5: Surface plots of the polynomials $B_2(t; s)$ in state that $s \in [3, 5]$ and $t \in [-20, 20]$.

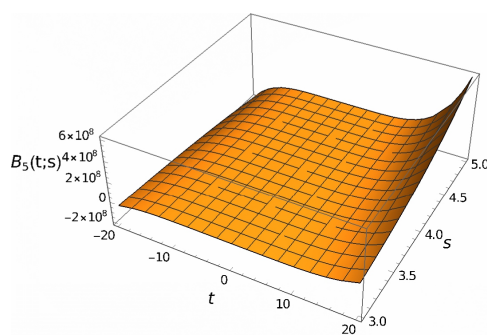


Figure 6: Surface plots of the polynomials $B_5(t; s)$ in state that $s \in [3, 5]$ and $t \in [-20, 20]$.

4. CONCLUSION

In this paper, the Carlitz numbers and polynomials are investigated. These numbers are defined by means of generalized binomial coefficients. By employing generating function techniques, binomial coefficients, and differential operators, various identities and recurrence relations involving the Carlitz numbers and polynomials are derived. Furthermore, with the aid of the generating function for the Carlitz numbers, several relations and explicit formulas connecting the Carlitz numbers and polynomials with multivariable polynomials are established.

In addition, many relations are obtained linking these numbers and polynomials with the Stirling numbers of the first and second kinds, the Apostol-type Peters

numbers and polynomials, the Peters-type Simsek numbers of the second kind, the generalized Fubini numbers, the Fubini numbers, and the Apostol–Euler numbers. The results presented in this paper may have potential applications in discrete mathematics and related areas of mathematics. Thus, this work contributes to the existing literature and opens new directions for future research in related areas.

Acknowledgments. The author would like to thank the referees for their valuable comments.

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(Received 09.03.2026.)

(Revised 14.10.2026.)