

SOME PROPERTIES OF A LINEAR OPERATOR DEFINED THROUGH THE SCHUR-SZEGŐ COMPOSITION THEOREM AND BERNSTEIN TYPE INEQUALITIES

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In this manuscript, we investigate a linear operator $\mathcal{T}_g$ acting on complex polynomials, defined through the Schur-Szegő composition theorem. For this operator, we establish sharp Bernstein-type inequalities for polynomials on the unit disk. Our results not only recover several known theorems as special cases but also provide new estimates for wider classes of B_n -Operators. This work strengthens the link between operator theory and polynomial inequalities, with applications to geometric function theory and approximation theory.

1. INTRODUCTION AND PRELIMINARIES

Polynomial inequalities form a cornerstone of modern analysis, particularly within geometric function theory and approximation theory, where they provide deep insights into the behavior of analytic and algebraic functions. Among these, Bernstein-type inequalities have emerged as a significant aspect of geometric function theory, offering powerful tools for studying the extremal properties of polynomials and related functions. The growing attention they have received in recent years within geometric function theory is largely attributed to their extensive range

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of applications across various domains of science, engineering, and discrete dynamical systems. Indeed, such inequalities offer a simple yet efficient framework for expressing and analyzing the intricate relationships among the variables of a system, thereby deepening our understanding of both theoretical and applied aspects of mathematical modeling.

Motivated by these developments, considerable attention has been devoted to identifying and characterizing classes of linear operators that preserve or refine Bernstein-type inequalities for polynomials. Such operators not only provide a unifying framework for studying extremal properties of analytic and algebraic functions but also facilitate the extension of classical results to more general functional settings. We denote by $\mathcal{P}_n$ the class of all complex polynomials $p(z) := \sum_{j=0}^n a_j z^j$ of degree n , where $z \in \Omega \subset \mathbb{C}$, then $p'(z) = \sum_{j=1}^n j a_j z^{j-1}$. Here, we study a new operator $\mathcal{T}_g : \mathcal{P}_n \rightarrow \mathcal{P}_n$, induced by a polynomial g depending on several parameters, which generalizes and extends some of the well-known B_n -Operators. The study of this operator enables a deeper understanding of the structural properties of polynomial spaces and yields new insights into inequalities of Bernstein type, enriching their interplay with operator theory and geometric function theory.

For a given analytic function f defined on some subset Ω of the complex plane, observing the growth of f with respect to some norm has been a fundamental problem in geometric function theory. This leads to the foundation of many Banach spaces of analytic functions on Ω , for example Hardy spaces, Bergman spaces, Fock spaces, and so on (see [25]). For $p \in \mathcal{P}_n$, it is natural to ask how fast the derivative p' in the uniform norm grows with respect to the growth of p . This question was answered by Sergei Bernstein [4] in the following way.

Theorem 1. *If $p \in \mathcal{P}_n$, then*

$$(1.1) \quad \max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$

The above inequality cannot be improved and holds with equality if and only if $p(z) = \gamma z^n$ for some $\gamma \in \mathbb{R}$. The above inequality is known as Bernstein's inequality. A related result derived from the Maximum Modulus Principle for $\rho > 1$ is the following:

Theorem 2. *If $p \in \mathcal{P}_n$, then*

$$(1.2) \quad \max_{|z|=1} |p(\rho z)| \leq \rho^n \max_{|z|=1} |p(z)|.$$

This inequality is also sharp and holds with equality if and only if $p(z) = \zeta z^n$, where $\zeta \neq 0$. The inequalities (1.1) and (1.2) can be improved if we restrict ourselves to the class of polynomials that have no zeros in $|z| < 1$. In fact, if $p \in \mathcal{P}_n$ and does not vanish in $|z| < 1$, then

$$(1.3) \quad \max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|$$

and for $\rho > 1$,

$$(1.4) \quad \max_{|z|=1} |p(\rho z)| \leq \frac{\rho^n + 1}{2} \max_{|z|=1} |p(z)|.$$

The inequality (1.3) was conjectured by Erdős and proved by Lax [11], while (1.4) was established by Ankeny and Rivlin [1]. Further improvements were obtained by Aziz and Dawood [2], who proved:

If $p \in \mathcal{P}_n$ has no zeros in $|z| < 1$, then

$$(1.5) \quad \max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left[\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right]$$

and for $\rho > 1$,

$$(1.6) \quad \max_{|z|=1} |p(\rho z)| \leq \frac{\rho^n + 1}{2} \max_{|z|=1} |p(z)| - \frac{\rho^n - 1}{2} \min_{|z|=1} |p(z)|.$$

Over the past several decades, extensive research has been devoted to refining and generalizing these inequalities, leading to numerous significant developments (see, e.g., [2, 5, 8, 12, 14–16, 18–21, 23]). In this work, we establish new Bernstein-type inequalities for a class of linear operators known as B_n -Operators, focusing on their relationship with the maximum norm of polynomials on the unit disk.

Definition 1 (B_n -Operator). Let $T : \mathcal{P}_n \rightarrow \mathcal{P}_n$ be a linear operator. We shall say that T is a B_n -Operator if, for every polynomial $f \in \mathcal{P}_n$ having all zeros inside the closed unit disk, $T[f]$ has all its zeros inside the closed unit disk.

In this paper, we introduce a new class of B_n -Operators. Before defining this operator, we first consider the operator T_g acting on $\mathcal{P}_n$, induced by a polynomial $g(z) = \sum_{\nu=0}^n \binom{n}{\nu} \lambda_\nu z^\nu$, whose zeros lie inside the open unit disk. For a polynomial $f(z) = \sum_{\nu=0}^n \binom{n}{\nu} a_\nu z^\nu$, the operator T_g is defined by

$$(1.7) \quad T_g[f(z)] := \sum_{\nu=0}^n \binom{n}{\nu} a_\nu \lambda_\nu z^\nu.$$

One can easily check that T_g is linear. Now we are ready to define the operator $\mathcal{T}_g : \mathcal{P}_n \rightarrow \mathcal{P}_n$ induced by the polynomial g that depends on several parameters and extends the T_g -Operator.

Definition 2. Assume that $f \in \mathcal{P}_n$ and $\rho > \tau \geq k$, $k > 0$, while the complex numbers α and β are such that $|\alpha| \leq 1$ and $|\beta| \leq 1$, and the operator T_g is given by (1.7). We define the operator $\mathcal{T}_g : \mathcal{P}_n \rightarrow \mathcal{P}_n$ by

$$(1.8) \quad \mathcal{T}_g[f](z) \equiv \mathcal{T}_g[f(\rho, \tau, \alpha, \beta, k)](z) := T_g[f(\rho z)] - \psi(\rho, \tau, \alpha, \beta, k) T_g[f(\tau z)],$$

where

$$(1.9) \quad \psi(\rho, \tau, \alpha, \beta, k) := \alpha - \beta \left[\left(\frac{\rho + k}{\tau + k} \right)^n - |\alpha| \right].$$

In this paper, we investigate the operator $\mathcal{T}_g$ and establish generalizations of the inequalities (1.1)–(1.6). The obtained results subsume several classical inequalities as particular cases, thereby illustrating the wide applicability of the proposed extensions. For related applications in approximation theory and numerical approximation, refer to [24]. This paper is organized as follows. Section 2 is devoted to preliminary results needed for the proofs of the main theorems. The main results are stated in Section 3, while their proofs are presented in Section 4. Concluding remarks are provided in Section 5.

2. AUXILIARY RESULTS

To prove our main theorems, we first present the following two lemmas. The first result is an immediate consequence of the Schur-Szegő composition theorem [13, p. 66].

Lemma 3. *Let f and g be two polynomials of degree n . If all the zeros of f lie in $|z| \leq \tau$ and all the zeros of g lie in $|z| \leq s$, then all the zeros of $T_g[f]$ lie in $|z| \leq \tau s$.*

The next lemma is due to Aziz and Zargar [3].

Lemma 4. *If $p \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k > 0$, then for every $\rho > \tau \geq k$,*

$$|p(\rho z)| \geq \left(\frac{\rho + k}{\tau + k} \right)^n |p(\tau z)| \quad \text{for } |z| = 1.$$

3. MAIN RESULTS

In this section, we present our main results and their proofs are given in the subsequent section. Unless stated otherwise, all parameters ρ, τ, k, α , and β are assumed to satisfy the conditions specified in Definition 2. The following theorem offers concise generalizations of (1.1) and (1.2).

Theorem 5. *If $p, q \in \mathcal{P}_n$ are such that*

- (i) $|p(z)| \leq |q(z)| \quad \text{for } |z| = k,$
- (ii) $q(z)$ has all zeros in $|z| \leq k$, $k > 0$,

then, for $|z| \geq 1$,

$$(3.10) \quad |\mathcal{T}_g[p](z)| \leq |\mathcal{T}_g[q](z)|.$$

Equality in (3.10) holds for $p(z) = e^{i\gamma} q(z)$, $\gamma \in \mathbb{R}$.

A variety of interesting results can be deduced from Theorem 5 as special cases, and here, we mention a few of them. Let $q(z) = Mz^n/k^n$, where $M := \max_{|z|=k} |p(z)|$, so that $|p(z)| \leq |q(z)|$ for $|z| = k$. By applying Theorem 5 together with the definition of $\mathcal{T}_g$, we obtain the following corollary:

Corollary 1. *If $p \in \mathcal{P}_n$, then for $|z| \geq 1$, we have*

$$(3.11) \quad |\mathcal{T}_g[p](z)| \leq \frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k)\tau^n| |T_g[z^n]| \max_{|z|=k} |p(z)|,$$

where $\psi(\rho, \tau, \alpha, \beta, k)$ is defined in (1.9). Equality in (3.11) holds for $p(z) = az^n$, $a \neq 0$.

Remark 1. If we take $\beta = 0$ in Corollary 1, we obtain the following inequality.

$$|T_g[p(\rho z)] - \alpha T_g[p(\tau z)]| \leq \frac{1}{k^n} |\rho^n - \alpha\tau^n| |T_g[z^n]| \max_{|z|=k} |p(z)|.$$

For $k = 1$ the result was independently proved by Qasim and Mir in [22].

For $g(z) = \sum_{j=0}^n \binom{n}{j} z^j$, we obtain the following result from Corollary 1.

Corollary 2. *If $p \in \mathcal{P}_n$, then for $|z| \geq 1$, we have*

$$(3.12) \quad |p(\rho z) - \psi(\rho, \tau, \alpha, \beta, k)p(\tau z)| \leq \frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k)\tau^n| |z|^n \max_{|z|=k} |p(z)|.$$

Equality in (3.12) holds for $p(z) = az^n$, $a \neq 0$.

Remark 2. If in inequality (3.12) we take $\alpha = \tau = k = 1$, $\beta = 0$, divide both sides of it by $\rho - 1$ and make $\rho \rightarrow 1$, we obtain

$$|zp'(z)| \leq n|z|^n \max_{|z|=1} |p(z)| \quad \text{for } |z| \geq 1,$$

which in particular yields inequality (1.1). Inequality (1.2) follows from inequality (3.12) for $k = 1$ and $\alpha = \beta = 0$.

Theorem 5 also yields the following comparison result:

Corollary 3. *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in $|z| < k$, $k > 0$, then*

$$(3.13) \quad |\mathcal{T}_g[p](z)| \leq \frac{1}{k^n} |\mathcal{T}_g[Q](z)|, \quad \text{for } |z| \geq 1,$$

where $Q(z) := z^n \overline{p(k^2/\bar{z})}$.

Next, we prove the following result for polynomials having all zeros in $|z| \leq k$, $k > 0$. The result gives an analog of Corollary 1 for minimum modulus of a polynomial $p(z)$ on $|z| = k$.

Theorem 6. If $p \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k > 0$, then

$$(3.14) \quad |\mathcal{T}_g[p](z)| \geq \frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n| |T_g[z^n]| m_k \quad \text{for } |z| \geq 1,$$

where $m_k = \min_{|z|=k} |p(z)|$. Equality in (3.14) holds for $p(z) = az^n$, $a \neq 0$.

Remark 3. If in (3.14), we take $k = 1$ and $\beta = 0$, it follows that if $p \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then for every $|\alpha| \leq 1$ and $\rho > \tau \geq 1$, we have

$$|T_g[p(\rho z)] - \alpha T_g[p(\tau z)]| \geq |\rho^n - \alpha \tau^n| |T_g[z^n]| m_1 \quad \text{for } |z| \geq 1,$$

We also present the following result with the convention $\lambda_0 = g(0)$ used throughout.

Theorem 7. If $p \in \mathcal{P}_n$, then for $|z| \geq 1$, we have

$$(3.15) \quad |\mathcal{T}_g[p](z)| + \frac{1}{k^n} |\mathcal{T}_g[Q](z)| \leq \left[\frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n| |T_g[z^n]| + |1 - \psi(\rho, \tau, \alpha, \beta, k)| |\lambda_0| \right] M,$$

where $M := \max_{|z|=k} |p(z)|$ and $Q(z) := z^n \overline{p(k^2/\bar{z})}$. Equality in (3.15) holds for $p(z) = az^n$, $a \neq 0$.

Remark 4. For $k = 1$ the theorem reduced to a result previously proved Mir and Qasim in [19].

For $k = 1$ and $\alpha = \beta = 0$, Theorem 7 reduces to the following result due to Gulzar and Rather [7].

Corollary 4. If $p \in \mathcal{P}_n$, then for $\rho > 1$ and $|z| \geq 1$,

$$|T_g[p](z)| + |T_g[Q](z)| \leq \left[\rho^n |T_g[z^n]| + |\lambda_0| \right] M,$$

where $M := \max_{|z|=1} |p(z)|$ and $Q(z) := z^n \overline{p(1/\bar{z})}$.

Next, we have the following result for polynomials which does not vanish in $|z| < k$, $k > 0$.

Theorem 8. If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in $|z| < k$, $k > 0$, then for $|z| \geq 1$,

$$\begin{aligned} |\mathcal{T}_g[p](z)| \leq & \frac{1}{2} \left[\left(\frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n| |T_g[z^n]| + |1 - \psi(\rho, \tau, \alpha, \beta, k)| |\lambda_0| \right) M \right. \\ & \left. - \left(\frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n| |T_g[z^n]| - |1 - \psi(\rho, \tau, \alpha, \beta, k)| |\lambda_0| \right) m_k \right], \end{aligned}$$

where $M := \max_{|z|=k} |p(z)|$ and $m_k := \min_{|z|=k} |p(z)|$.

For $\alpha = \beta = 0$ in the previous theorem, we have the following corollary which gives refinement of a result due to Gulzar and Rather [7].

Corollary 5. *If $p \in \mathcal{P}_n$ has no zero in $|z| < k$, $k > 0$, then for $\rho > k$ and $|z| \geq 1$,*

$$|T_g[p(\rho z)]| \leq \frac{1}{2} \left[\left(\frac{1}{k^n} \rho^n |T_g[z^n]| + |\lambda_0| \right) \max_{|z|=k} |p(z)| - \left(\frac{1}{k^n} \rho^n |T_g[z^n]| - |\lambda_0| \right) \min_{|z|=k} |p(z)| \right].$$

Remark 5. For $g(z) = \sum_{j=0}^n \binom{n}{j} j z^j$, $k = 1$, as $\rho \rightarrow 1$ and $|z| = 1$, Corollary 5 reduces to inequality (1.5). Similarly, if we take $g(z) = \sum_{j=0}^n \binom{n}{j} z^j$ with $|z| = 1$ in Corollary 5, we obtain inequality (1.6).

By considering $g(z) = z^n + z^t$ with $0 \leq t \leq n-1$, taking $|z| = 1$, and letting $\rho \rightarrow 1$ in Corollary 5, we obtain the following result.

Corollary 6. *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n which does not vanish in $|z| < k$, $0 < k \leq 1$, then,*

$$|a_n| + \frac{|a_t|}{\binom{n}{t}} \leq \frac{1}{2} \left[\Lambda \max_{|z|=k} |p(z)| - \zeta \min_{|z|=k} |p(z)| \right], \quad 0 \leq t \leq n-1,$$

where

$$\Lambda = \begin{cases} \frac{1}{k^n}, & \text{if } 1 \leq t \leq n-1, \\ \frac{1+k^n}{k^n}, & \text{if } t = 0, \end{cases}$$

and

$$\zeta = \begin{cases} \frac{1}{k^n}, & \text{if } 1 \leq t \leq n-1, \\ \frac{1-k^n}{k^n}, & \text{if } t = 0. \end{cases}$$

The result is sharp.

Theorem 9. *If $p \in \mathcal{P}_n$ and $p(z) = (\zeta/k^n)Q(z)$, where $Q(z) := z^n \overline{p(k^2/\bar{z})}$, then for $|z| \geq 1$, we have*

$$(3.16) \quad |\mathcal{T}_g[p](z)| \leq \frac{1}{2} \left[\frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n| |T_g[z^n]| + |1 - \psi(\rho, \tau, \alpha, \beta, k)| |\lambda_0| \right] \max_{|z|=k} |p(z)|.$$

Remark 6. For $\alpha = \beta = 0$ and $k = 1$ in (3.16), we recover a result of Gulzar and Rather [7]. For $g(z) = \sum_{j=0}^n \binom{n}{j} z^j$, we obtain the following result, which includes, as special cases, many well-known polynomial inequalities for self-inversive polynomials.

Corollary 7. If $p \in \mathcal{P}_n$ and $p(z) = (\zeta/k^n)Q(z)$, where $Q(z) := z^n \overline{p(k^2/\bar{z})}$, then for $|z| \geq 1$, we have

$$|p(\rho z) - \psi(\rho, \tau, \alpha, \beta, k)p(\tau z)| \leq \frac{1}{2} \left[\frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k)\tau^n| |z^n| + |1 - \psi(\rho, \tau, \alpha, \beta, k)| \right] M,$$

where $M = \max_{|z|=k} |p(z)|$.

Remark 7. A variety of interesting results can be readily deduced from Theorem 9, just as we have done with Theorem 8. Taking $k = 1$ in Theorem 9 and Corollary 7, we obtain the corresponding estimates for self-inversive polynomials.

4. PROOFS OF THE MAIN RESULTS

Now, we give proofs of our main results.

Proof of Theorem 5. Since $q(z)$ has all zeros in $|z| \leq k$ and $|p(z)| \leq |q(z)|$ for $|z| = k$, it follows by Rouché's theorem that for every complex number λ with $|\lambda| > 1$, the polynomial $f(z) = p(z) - \lambda q(z)$ has all its zeros in $|z| \leq k$. Applying Lemma 4 to $f(z)$, we obtain for $\rho > \tau \geq k$,

$$(4.17) \quad |f(\rho z)| \geq \left(\frac{\rho + k}{\tau + k} \right)^n |f(\tau z)| \quad \text{for } |z| = 1.$$

Since $f(\rho z) \neq 0$ and $(k + \rho)/(k + \tau) > 1$, for every $\rho > \tau \geq k$, $k > 0$ it follows from (4.17) that for $|z| = 1$,

$$|f(\rho z)| > |f(\tau z)|.$$

All zeros of $f(\rho z)$ lie in $|z| < k/\rho < 1$. Using Rouché's theorem we conclude that the polynomial $f(\rho z) - \alpha f(\tau z)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Also, for $|z| = 1$

$$(4.18) \quad |f(\rho z) - \alpha f(\tau z)| \geq |f(\rho z)| - |\alpha| |f(\tau z)| \geq \left\{ \left(\frac{\rho + k}{\tau + k} \right)^n - |\alpha| \right\} |f(\tau z)|.$$

Therefore, it follows from inequality (4.18) by direct application of Rouché's theorem that the polynomial

$$\Psi(z) = f(\rho z) - \alpha f(\tau z) + \beta \left\{ \left(\frac{\rho + k}{\tau + k} \right)^n - |\alpha| \right\} f(\tau z)$$

has all its zeros in $|z| < 1$, for any complex number β with $|\beta| < 1$ and $\rho > \tau \geq k$. Applying Lemma 3, it follows that:

$$\begin{aligned} \phi(z) &:= T_g[\Psi(z)] = T_g[f(\rho z)] - \alpha T_g[f(\tau z)] + \beta \left\{ \left(\frac{\rho + k}{\tau + k} \right)^n - |\alpha| \right\} T_g[f(\tau z)] \\ &= \mathcal{T}_g[f](z) \end{aligned}$$

has all its zeros in $|z| < 1$. Replacing $f(z) = p(z) - \lambda q(z)$, we conclude that all the zeros of the polynomial

$$(4.19) \quad \phi(z) := \mathcal{T}_g[p](z) - \lambda \mathcal{T}_g[q](z)$$

lie in $|z| < 1$. We claim that for $|z| \geq 1$,

$$(4.20) \quad |\mathcal{T}_g[p](z)| \leq |\mathcal{T}_g[q](z)|.$$

For if inequality (4.20) is not true, then there exists $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$ such that

$$(4.21) \quad |\mathcal{T}_g[p](z_0)| > |\mathcal{T}_g[q](z_0)|.$$

Taking

$$\lambda = \frac{\mathcal{T}_g[p](z_0)}{\mathcal{T}_g[q](z_0)}$$

so that by (4.21), $|\lambda| > 1$. For this value of λ it follows from equation (4.19) that $\phi(z_0) = 0$, which is a contradiction to the fact that all the zeros of $\phi(z)$ lie in $|z| < 1$. For β with $|\beta| = 1$, inequality (4.20) follows by continuity and this completes the proof of the theorem. $\square$

Proof of Corollary 3. Since $p(z)$ has all zeros in $|z| \geq k$, $Q(z)$ has all zeros in $|z| \leq k$. Also

$$|Q(z)| = k^n |p(z)| \quad \text{for } |z| = k.$$

Therefore using Theorem 5, the required result follows. $\square$

Proof of Theorem 6. Let $m_k = \min_{|z|=k} |p(z)|$, then

$$m_k |z|^n \leq k^n |p(z)| \quad \text{for } |z| = k.$$

If $p(z)$ has zero on $|z| = k$, then $m_k = 0$. Assume that $p(z)$ has all its zeros in $|z| < k$. Applying Theorem 5 with $p(z)$ replaced by $m_k z^n$ and $q(z)$ replaced by $k^n p(z)$, we obtain for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $\rho > \tau \geq k$,

$$|\mathcal{T}_g[p](z)| \geq \frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n| |T_g[z^n]| m_k \quad \text{for } |z| \geq 1.$$

This gives the required assertion and the proof complete. $\square$

Proof of Theorem 7. We have

$$|p(z)| \leq M \quad \text{for } |z| = k.$$

By Rouché's theorem $F(z) = p(z) - \mu M$ does not vanish in $|z| < k$ for some $\mu \in \mathbb{C}$ with $|\mu| > 1$. By Corollary 3, it follows that for $\rho > \tau \geq k$, $|\alpha| \leq 1$, $|\beta| \leq 1$, and $|z| \geq 1$,

$$|\mathcal{T}_g[F](z)| \leq \frac{1}{k^n} |\mathcal{T}_g[H](z)|,$$

where $H(z) = z^n \overline{F(k^2/\bar{z})} = Q(z) - \bar{\mu} z^n M$. On substituting the values of $F(z)$ and $H(z)$, we obtain for $|z| \geq 1$,

$$\begin{aligned} & |\mathcal{T}_g[p](z) - \mu [1 - \psi(\rho, \tau, \alpha, \beta, k)] \lambda_0 M| \\ & \leq \frac{1}{k^n} |\mathcal{T}_g[Q](z) - \bar{\mu} [\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n] T_g[z^n] M|. \end{aligned}$$

Now choosing the argument of μ on the right side, we obtain for $\rho > \tau \geq k$, $|\alpha| \leq 1$, $|\beta| \leq 1$ and $|z| \geq 1$,

$$\begin{aligned} (4.22) \quad & |\mathcal{T}_g[p](z)| + \frac{1}{k^n} |\mathcal{T}_g[Q](z)| \\ & \leq |\mu| \left[\frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n| |T_g[z^n]| + |1 - \psi(\rho, \tau, \alpha, \beta, k)| |\lambda_0| \right] M. \end{aligned}$$

On making $|\mu| \rightarrow 1$ in (4.22) gives (3.15) and hence the proof is complete. $\square$

Proof of Theorem 8. If $p(z)$ has a zero on $|z| = k$, then $m_k = 0$, so we suppose that all the zeros of $p(z)$ lie in $|z| > k$, so that $m_k > 0$ and

$$m_k \leq |p(z)| \quad \text{for } |z| = k.$$

Therefore for every complex number λ with $|\lambda| < 1$, it follows by Rouché's theorem that all the zeros of $F(z) = p(z) - m_k \lambda$ lie in $|z| > k$. We note that $F(z)$ has no zeros on $|z| = k$, because if for some $z = z_0$ with $|z_0| = k$, we have

$$F(z_0) = p(z_0) - m_k \lambda = 0,$$

then

$$|p(z_0)| = m_k |\lambda| < m_k,$$

which is a contradiction. Now if

$$\begin{aligned} G(z) &= z^n \overline{F(k^2/\bar{z})} \\ &= z^n \overline{p(k^2/\bar{z})} - \bar{\lambda} m_k z^n \\ &= Q(z) - \bar{\lambda} m_k z^n, \end{aligned}$$

then, all the zeros of $G(z)$ lie in $|z| < k$ and

$$|G(z)| = k^n |F(z)| \quad \text{for } |z| = k.$$

Therefore, by Theorem 5, we obtain for $\rho > \tau \geq k$, $|\alpha| \leq 1$, $|\beta| \leq 1$ and $|z| \geq 1$,

$$|k^n \mathcal{T}_g[F](z)| \leq |\mathcal{T}_g[G](z)|.$$

On substituting back the value of $F(z)$ and $G(z)$, the above inequality simplifies to:

$$\begin{aligned} & \left| k^n \mathcal{T}_g[p](z) - m_k k^n \lambda [1 - \psi(\rho, \tau, \alpha, \beta, k)] \lambda_0 \right| \\ & \leq \left| \mathcal{T}_g[Q](z) - \bar{\lambda} [\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n] T_g[z^n] m_k \right|. \end{aligned}$$

By choosing the argument of λ on the right-hand side and applying the triangle inequality to the left-hand side of the above inequality, then letting $|\lambda| \rightarrow 1$, we obtain:

$$(4.23) \quad |\mathcal{T}_g[p](z)| \leq \frac{1}{k^n} |\mathcal{T}_g[Q](z)| \\ - \left[\frac{1}{k^n} |\rho^n - \psi(\rho, \tau, \alpha, \beta, k) \tau^n| |T_g[z^n]| - |1 - \psi(\rho, \tau, \alpha, \beta, k)| |\lambda_0| \right] m_k.$$

Adding $|\mathcal{T}_g[p](z)|$ on both sides of (4.23) and using Theorem 7, yields the required result. $\square$

Proof of Theorem 9. By the given condition it easily follows that for $\rho > \tau \geq k$, $|\alpha| \leq 1$, $|\beta| \leq 1$ and for all z , we have

$$|\mathcal{T}_g[p](z)| = \frac{1}{k^n} |\mathcal{T}_g[Q](z)|.$$

This, when combined with Theorem 7, yields the required result. $\square$

5. CONCLUSION

In this work, we have introduced the operator $\mathcal{T}_g[f]$ and demonstrated how several classical polynomial inequalities can be extended within this new framework. While not every known inequality admits a meaningful generalization, the operator $\mathcal{T}_g[f]$ provides a flexible and powerful tool for extending a wide class of results in approximation theory. This approach opens avenues for further exploration, and a rich collection of related inequalities that may serve as a basis for future generalizations can be found in [23].

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