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SCORE SETS IN ORIENTED GRAPHS

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The score of a vertex v in an oriented graph D is $a_v = n-1+d_v^+ - d_v^-$, where d_v^+ and d_v^- are the outdegree and indegree respectively of v and n is the number of vertices in D. The set of distinct scores of the vertices in an oriented graph D is called its score set. If $a > 0$ and $d > 1$ are positive integers, we show there exists an oriented graph with score set $\{a, ad, ad^2, \ldots, ad^n\}$ except for $a = 1$, $d = 2$, $n > 0$, and for $a = 1$, $d = 3$, $n > 0$. It is also shown that there exists no oriented graph with score set $\{a, ad, ad^2, \ldots, ad^n\}$, $n > 0$ when either $a = 1$, $d = 2$, or $a = 1$, $d = 3$. Also we prove for the non-negative integers $a_1, a_2, \ldots, a_n$ with $a_1 < a_2 < \cdots < a_n$, there is always an oriented graph with $a_n + 1$ vertices with score set $\{a'_1, a'_2, \ldots, a'_n\}$, where

$$a'_i = \begin{cases} a_{i-1} + a_i + 1, & \text{for } i > 1, \\ a_i, & \text{for } i = 1. \end{cases}$$

1. INTRODUCTION

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Let D be an oriented graph with set $V = \{v_1, v_2, \ldots, v_n\}$, and let d_v^+ and d_v^- respectively be the outdegree and indegree of vertex v_i. Define a_{v_i} (or simply a_i)$= n - 1 + d_v^+ - d_v^-$, as the score of v_i. Clearly, $0 \le a_{v_i} \le 2n - 2$. The sequence $A = [a_1, a_2, \ldots, a_n]$ in non-decreasing order is the score sequence of an oriented graph D.

For any two distinct vertices u and v in an oriented graph D, we have one of the following possibilities. (i). An arc directed from u to v denoted by $u \to v$. (ii). An arc directed from v to u denoted by $u \leftarrow v$. (iii). There is no arc from u to v and there is no arc from v to u. This is denoted by $u \sim v$.

Let D be an oriented graph with vertex set V and let $X, Y \subseteq V$. If there is an arc from each vertex of X to every vertex of Y, then it is denoted by $X \to Y$.

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If d_v^* is the number of those vertices u in D which have $v \sim u$, then $d_v^+ + d_v^- + d_v^* = n - 1$. Therefore $a_v = 2d_v^+ + d_v^*$. This implies that each vertex u with $v \to u$ contributes two to the score of v and each vertex u with $v \sim u$ contributes one to the score of v. Since the number of arcs and non-arcs in an oriented graph of order n is $\binom{n}{2}$ and each $v \sim u$ contributes two (one each at u and v) to scores, the sum total of all the scores is $2\binom{n}{2} = n(n-1)$.

The following result [**1**, Theorem 2.1] characterizes score sequences of all oriented graphs.

Theorem 1.1. *A non-decreasing sequence of non-negative integers $A = [a_1, a_2, \ldots, a_n]$ is the score sequence of an oriented graph if and only if*

$$\sum_{i=1}^{k} a_i \geq k(k-1) \tag{1.1}$$

for $1 \leq k \leq n$ with equality when $k = n$.

A tournament is an orientation of a complete simple graph. The score s_v of a vertex v in a tournament T is the outdegree of v. The score sequence of a tournament is formed by listing the vertex scores in non-decreasing order. The set S of distinct scores of the vertices in a tournament T is called its score set. Reid [**5**] conjectured that every finite set S of non-negative integers is a score set of some tournament and verified this conjecture for $|S| = 1, 2, 3$ or S is in arithmetic or geometric progression. Hager [**2**] verified Reid's conjecture for the cases $|S| = 4, 5$. In 1986 Yao proved Reid's conjecture by pure arithmetical analysis which appeared in Chinese [**6**] in 1986 and in English [**7**] in 1989. Recently Pirzada and Naikoo [**3**] proved by construction that the set of non-negative integers $S = \{s_1, \sum_{i=1}^{2} s_i, \ldots, \sum_{i=1}^{n} s_i\}$, with $s_1 < s_2 < \cdots < s_n$, is a score set of some tournament. In [**4**] it has been proved that every set of n non-negative integers, except $\{0\}$ and $\{0, 1\}$, is a score set of some 3-partite tournament. Also it is shown that every set of n non-negative integers is a score set of some k-partite tournament for every $n \geq k \geq 2$.

2. SCORE SETS IN ORIENTED GRAPHS

We start with the following observation:

Lemma 2.1. *The number of vertices in an oriented graph with at least two distinct scores does not exceed its largest score.*

Proof. Clearly an oriented graph with at least two distinct scores has more than one vertex. Let D be an oriented graph with $n > 1$ vertices, say $v_1, v_2, \ldots, v_n$ with their respective scores $a_{v_1}, a_{v_2}, \ldots, a_{v_n}$ such that $a_{v_1} \leq a_{v_2} \leq \cdots \leq a_{v_n}$. We assume without loss of generality that the scores a_{v_i} and a_{v_n} are distinct so that $a_{v_i} < a_{v_n}$ for some i, where $1 \leq i \leq n-1$.

Therefore for all j, where $1 \leq j \leq i$, we have $a_{v_j} < a_{v_n}$, which gives $a_{v_n} \geq a_{v_j} + 1$, and for all k, where $i+1 \leq k \leq n-1$, we have $a_{v_k} \leq a_{v_n}$.

Claim $n \leq a_{v_n}$.

Assume to the contrary that $n > a_{v_n}$. Then for all j, where $1 \leq j \leq i$, we have $n > a_{v_j} + 1$, which gives $n-1 \geq a_{v_j} + 1$, and for all k, where $i+1 \leq k \leq n-1$, we have $n > a_{v_k}$, which gives $n-1 \geq a_{v_k}$. Also, $n-1 \geq a_{v_n}$.

Thus

$$n-1 \geq a_{v_1} + 1, \ldots, n-1 \geq a_{v_i} + 1, n-1 \geq a_{v_{i+1}}, \ldots, n-1 \geq a_{v_n}.$$

Adding these inequalities we have

$$n(n-1) \geq \sum_{r=1}^{n} a_{v_r} + i.$$

Since $[a_{v_1}, a_{v_2}, \ldots, a_{v_n}]$ is the score sequence of D, by Theorem 1.1 we have

$$\sum_{r=1}^{n} a_{v_r} = n(n-1).$$

Thus $n(n-1) \geq n(n-1) + i$ so that $i \leq 0$, which is a contradiction since $1 \leq i \leq n-1$. This establishes the claim. □

Now we obtain the following result:

Theorem 2.2. *Let $A = \{a, ad, ad^2, \ldots, ad^n\}$, where a and d are positive integers with $a > 0$ and $d > 1$. Then there exists an oriented graph with score set A except for $a = 1$, $d = 2$, $n > 0$ and for $a = 1$, $d = 3$, $n > 0$.*

Proof. We use induction on n. Let $n = 0$. As $a > 0$, $a + 1 > 0$. Let D be an oriented graph on $a + 1$ vertices with no arcs (that is the complement of K_{a+1}). Then each vertex of D has score $a + 1 - 1 + 0 - 0 = a$. Therefore the score set of D is $A = \{a\}$, proving the result for $n = 0$.

If $n = 1$, then three cases arise. (i) $a = 1$, $d > 3$, (ii) $a > 1$, $d = 2$ and (iii) $a > 1$, $d > 2$.

Case (i). $a = 1$, $d > 3$. Then $a + 1 > 0$ and $ad - 2a - 1 = a(d-2) - 1 = d - 3 > 0$. Construct an oriented graph D with vertex set $V = X \cup Y$, where $X \cap Y = \emptyset$, $|X| = a + 1$, $|Y| = ad - 2a - 1$ and $Y \to X$. Therefore D has $|V| = |X| + |Y| = a + 1 + ad - 2a - 1 = ad - a$ vertices and the scores of the vertices are

$$a_x = |V| - 1 + 0 - |Y| = ad - a - 1 - (ad - 2a - 1) = a$$

for all $x \in X$ and

$$a_y = |V| - 1 + |X| - 0 = ad - a - 1 + a + 1 = ad$$

for all $y \in Y$.

Therefore the score set of D is $A = \{a, ad\}$.

Case (ii). $a > 1$, $d = 2$. Firstly take $a = 2$, $d = 2$, so that $ad = 4$. Consider an oriented graph D with vertex set $V = \{v_1, v_2, v_3, v_4\}$ in which $v_1 \to v_3$ and $v_2 \to v_4$. Then D has $ad = 4$ vertices and the scores of the vertices are $a_{v_1} = a_{v_2} = 4 - 1 + 1 - 0 = 4 = ad$ and $a_{v_3} = a_{v_4} = 4 - 1 + 0 - 1 = 2 = a$.

Therefore the score set of D is $A = \{a, ad\}$.

Now take $a > 2$, $d = 2$. Then $a > 0$ and $a - 2 > 0$. Consider an oriented graph D with vertex set $V = X \cup Y \cup Z$, where $X \cap Y = \emptyset$, $Y \cap Z = \emptyset$, $Z \cap X = \emptyset$, $|X| = 2$, $|Y| = a - 2$, $|Z| = a$. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, \ldots, y_{a-2}\}$ and $Z = \{z_1, z_2, \ldots, z_a\}$. Let $y_i \to x_1$, $y_i \to x_2$ for all i, where $1 \le i \le a - 2$; $z_1 \to x_1$; $z_2 \to x_2$; $z_{i+2} \to y_i$ for all i, where $1 \le i \le a - 2$.

Then D has $|V| = |X| + |Y| + |Z| = 2 + a - 2 + a = 2a = ad$ vertices and the score of the vertices are

$$a_{x_1} = a_{x_2} = |V| - 1 + 0 - (|Y| + 1) = ad - 1 - (a - 2 + 1) = ad - a = 2a - a = a,$$
$$a_{y_i} = |V| - 1 + 2 - 1 = ad \text{ for all } i, \text{ where } 1 \le i \le a - 2$$

and

$$a_{z_i} = |V| - 1 + 1 - 0 = ad \text{ for all } i, \text{ where } 1 \le i \le a.$$

Therefore the score set of D is $A = \{a, ad\}$.

Case (iii). $a > 1$, $d > 2$. Then $a + 1 > 0$ and $ad - 2a - 1 = a(d - 2) - 1 > 0$, and the result follows from case (i).

Hence in all these cases we obtain an oriented graph D with score set $A = \{a, ad\}$. This proves the result for $n = 1$.

Assume that the result is true for $n = 0, 1, 2, 3, \ldots, p$ for some integer $p \ge 1$. We show that the result is true for $p + 1$.

Let a and d be positive integers with $a > 0$ and $d > 1$ and for $a = 1$, $d \ne 2, 3$. Therefore by the inductive hypothesis there exists an oriented graph D with score set $\{a, ad, ad^2, \ldots, ad^p\}$. That is, $a, ad, ad^2, \ldots, ad^p$ are the distinct score of the vertices of D. Let V be the vertex set of D.

Once again we have either (i). $a = 1$, $d > 3$, (ii). $a > 1$, $d = 2$ or (iii). $a > 1$, $d > 2$. Obviously for $d > 1$ in all the above cases we have $ad^{p+1} \ge 2ad^p$. Also the score set of D, namely $\{a, ad, ad^2, \ldots, ad^p\}$, has at least two distinct scores for $p \ge 1$. Therefore by Lemma 2.1 we have $|V| \le ad^p$. Hence $ad^{p+1} \ge 2|V|$ so that $ad^{p+1} - 2|V| + 1 > 0$.

Consider now a new oriented graph D_1 with vertex set $V_1 = V \cup X$, where $V \cap X = \emptyset$, $|X| = ad^{p+1} - 2|V| + 1$ and arc set containing all the arcs of D together with $X \to V$. Then D_1 has $|V_1| = |V| + |X| = |V| + ad^{p+1} - 2|V| + 1 = ad^{p+1} - |V| + 1$ vertices and $a + |X| - |X| = a$, $ad + |X| - |X| = ad$, $ad^2 + |X| - |X| = ad^2, \ldots, ad^p + |X| - |X| = ad^p$ are the distinct scores of the vertices of V and

$$a_x = |V_1| - 1 + |V| - 0 = ad^{p+1} - |V| + 1 - 1 + |V| = ad^{p+1} \text{ for all } x \in X.$$

Therefore the score set of D_1 is $A = \{a, ad, ad^2, \ldots, ad^p, ad^{p+1}\}$ which proves the result for $p + 1$. Hence the result follows. □

For any two integers m and n with $m \neq 0$ we denote by m/n to mean that m divides n.

As noted in Theorem 2.2, there exists no oriented graph when either $a = 1$, $d = 2$, $n > 0$ or $a = 1$, $d = 3$, $n > 0$, which is now proved in the next Theorem.

Theorem 2.3. *There exists no oriented graph with score set* $A = \{a, ad, ad^2, \dots, ad^n\}$, $n > 0$, *when either* (i). $a = 1$, $d = 2$ *or* (ii). $a = 1$, $d = 3$.

Proof. Case(i). Assume $A = \{1, 2, 2^2, \dots, 2^n\}$ is a score set of some oriented graph D for $n > 0$. Then there exist positive integers, say $x_1, x_2, x_3, \dots, x_{n+1}$ such that

$$A_1 = [\underbrace{1, 1, \dots, 1}_{x_1}, \underbrace{2, 2, \dots, 2}_{x_2}, \underbrace{2^2, 2^2, \dots, 2^2}_{x_3}, \dots, \underbrace{2^n, 2^n, \dots, 2^n}_{x_{n+1}}]$$

is the score sequence of D. Therefore by Theorem 1.1 we have

$$x_1 + 2x_2 + 2^2x_3 + \cdots + 2^n x_{n+1} = \left(\sum_{i=1}^{n+1} x_i\right)\left(\left(\sum_{i=1}^{n+1} x_i\right) - 1\right)$$

which implies that x_1 is even. However, x_1 is a positive integer, therefore $x_1 \geq 2$. Let $a_1 = 1$, $a_2 = 1$ and $a_3 \geq 1$. By equation(1.1) $a_1 + a_2 + a_3 \geq 3(3-1)$, $2 + a_3 \geq 6$, or $a_3 \geq 4$. This implies that $x_2 = 0$, a contradiction.

Case(ii). Assume $A = \{1, 3, 3^2, \dots, 3^n\}$ is a score set of some oriented graph D for $n > 0$. Then there exist positive integers, say $y_1, y_2, y_3, \dots, y_{n+1}$ such that

$$A_2 = [\underbrace{1, 1, \dots, 1}_{y_1}, \underbrace{3, 3, \dots, 3}_{y_2}, \underbrace{3^2, 3^2, \dots, 3^2}_{y_3}, \dots, \underbrace{3^n, 3^n, \dots, 3^n}_{y_{n+1}}]$$

is the score sequence of D. Therefore by Theorem 1.1 we have

$$y_1 + 3y_2 + 3^2y_3 + \cdots + 3^n y_{n+1} = \left(\sum_{i=1}^{n+1} y_i\right)\left(\left(\sum_{i=1}^{n+1} y_i\right) - 1\right)$$

so that

$$y_1 + 3y_2 + 3^2y_3 + \cdots + 3^n y_{n+1} = q(q-1),$$

where $q = \sum_{i=1}^{n+1} y_i > 1$ as $n > 0$ and $y_1, y_2, y_3, \dots, y_{n+1}$ are positive integers.

Firstly suppose $y_1 > 1$ or $y_1 \geq 2$. Let $a_1 = 1$, $a_2 = 1$ and $a_3 \geq 1$. By equation (1.1) $a_1 + a_2 + a_3 \geq 3(3-1)$, $2 + a_3 \geq 6$ or $a_3 \geq 4$. This implies that $y_2 = 0$, a contradiction.

Now suppose $y_1 = 1$. Therefore $1 + 3y_2 + 3^2y_3 + \cdots + 3^n y_{n+1} = q(q-1)$, where $q > 1$, that is, $3(y_2 + 3y_3 + \cdots + 3^{n-1}y_{n+1}) = q^2 - q - 1$. Since the expression in the lefthand side is a multiple of 3, therefore $3/q^2 - q - 1$. For $q = 2$, we have $3/2^2 - 2 - 1$, or $3/1$, which is absurd. Now suppose $q > 2$. Since every positive integer $q > 2$ is one of the forms $3r$, $3r+1$, $3r+2$, where $r > 0$, therefore, $3/(3r)^2 - 3r - 1$, or $3/(3r+1)^2 - (3r+1) - 1$, or $3/(3r+2)^2 - (3r+2) - 1$, respectively, which gives $3/9r^2 - 3r - 1$, or $3/9r^2 + 3r - 1$, or $3/9r^2 + 3r + 1$, respectively.

Since $3/9r^2 - 3r$, $3/9r^2 + 3r$ and $3/9r^2 + 9r$, $3/-1$, or $3/1$, which is absurd. □

Theorem 2.4. *If $a_1, a_2, \ldots, a_n$ are non-negative integers with $a_1 < a_2 < \cdots < a_n$, then there exists an oriented graph with $a_n + 1$ vertices and with score set $A = \{a'_1, a'_2, \ldots, a'_n\}$, where*

$$a'_i = \begin{cases} a_{i-1} + a_i + 1 & \text{for } i > 1, \\ a_i & \text{for } i = 1. \end{cases}$$

Proof. We use induction on n. For $n = 1$ let D be an oriented graph on $a_1 + 1$ vertices with no arcs (that is the complement of K_{a_1+1}). Then each vertex of D has score $a_1 + 1 - 1 + 0 - 0 = a_1 = d_1$. Therefore the score set of D is $A = \{a'_1\}$. This verifies the result for $n = 1$.

If $n = 2$, then there are two non-negative integers a_1 and a_2 with $a_1 < a_2$. Clearly $a_1 + 1 > 0$ and $a_2 - a_1 > 0$. Consider an oriented graph D with vertex set $V = X \cup Y$, where $X \cap Y = \emptyset$, $|X| = a_1 + 1$, $|Y| = a_2 - a_1$ and $Y \to X$. Therefore D has $|V| = |X| + |Y| = a_1 + 1 + a_2 - a_1 = a_2 + 1$ vertices, and the score of the vertices are $a_x = |V| - 1 + 0 - |Y| = a_2 + 1 - 1 - (a_2 - a_1) = a_1 = a'_1$ for all $x \in X$ and $a_y = |V| - 1 + |X| - 0 = a_2 + 1 - 1 + a_1 + 1 = a_1 + a_2 + 1 = a'_2$ for all $y \in Y$.

Therefore the score set of D is $A = \{a'_1, a'_2\}$ which proves the result for $n = 2$.

Assume that the result is true for $n = 1, 2, 3, \cdots, p$, for some integer $p \geq 2$. We show that the result is true for $p + 1$.

Let $a_1, a_2, \ldots, a_{p+1}$ be non-negative integers with $a_1 < a_2 < \cdots < a_{p+1}$. Since $a_1 < a_2 < \cdots < a_p$, by the inductive hypothesis there exists an oriented graph D on $a_p + 1$ vertices with score set $\{a'_1, a'_2, \ldots, a'_p\}$ where

$$a'_i = \begin{cases} a_{i-1} + a_i + 1 & \text{for } i > 1, \\ a_i & \text{for } i = 1. \end{cases}$$

That is, score set of D is $\{a_1, a_1 + a_2 + 1, a_2 + a_3 + 1, \cdots, a_{p-1} + a_p + 1\}$. So $a_1, a_1 + a_2 + 1, a_2 + a_3 + 1, \ldots, a_{p-1} + a_p + 1$ are the distinct scores of the vertices of D. Let V be the vertex set of D so that $|V| = a_p + 1$.

Since $a_{p+1} > a_p$, $a_{p+1} - a_p > 0$. Consider now a new oriented graph D_1 with vertex set $V_1 = V \cup X$, where $V \cap X = \emptyset$, $|X| = a_{p+1} - a_p$, and arc set containing all the arcs of D together with $X \to V$. Then D_1 has $|V_1| = |V| + |X| = a_p + 1 + a_{p+1} - a_p = a_{p+1} + 1$ vertices, $a_1 + |X| - |X| = a_1 = a'_1$, $a_1 + a_2 + 1 + |X| - |X| = a_1 + a_2 + 1 = a'_2$, $a_2 + a_3 + 1 + |X| - |X| = a_2 + a_3 + 1 = a'_3, \ldots, a_{p-1} + a_p + 1 + |X| - |X| = a_{p-1} + a_p + 1 = a'_p$ are the distinct scores of the vertices of V and

$$a_x = |V_1| - 1 + |V| - 0 = a_{p+1} + 1 - 1 + a_p + 1 = a_p + a_{p+1} + 1 = a'_{p+1}$$

for all $x \in X$.

Therefore the score set of D_1 is $A = \{a'_1, a'_2, \ldots, a'_p, a'_{p+1}\}$ which proves the result for $p + 1$. Hence by induction, the result follows. □

From Theorem 2.4 it follows that every singleton set of non-negative integers is a score set of some oriented graph.

As we have shown in Theorem 2.3 that the sets $\{1, 2, 2^2, \ldots, 2^n\}$ and $\{1, 3, 3^2, \ldots, 3^n\}$ cannot be the score sets of an oriented graph, for $n > 0$. It therefore follows that the above results cannot be generalized to say that any set of non-negative integers forms the score set of some oriented graph. However, there can be other special classes of non-negative integers which can form the score set of an oriented graph, and the problem needs further investigation.

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