Applicable Analysis and Discrete Mathematics, **2** (2008), 56–66.
Available electronically at **http://pefmath.etf.bg.ac.yu**

SOME MORE IDENTITIES INVOLVING RATIONAL SUMS

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We establish integral identities for sums involving binomial coefficients and Harmonic numbers. Using the identities we recapture some established closed form representations of binomial sums and announce some new results.

1. INTRODUCTION

The representation of sums in closed form can in some cases be achieved through a variety of different methods, including transform techniques, W–Z methods, Riordan arrays and integral representations. The interested reader is referred to the works of Egorychev [**2**], Gould [**3**], Merlini, Sprugnoli and Verri [**4**], Petkovšek, Wilf and Zeilberger [**5**] and Sofo [**6**], [**7**] and [**8**]. Recently Diaz-Barrero et. al. [**1**] gave a general procedure to represent sums in the form of:

$$\sum_{n=1}^{p}(-1)^n\binom{p}{n}Q^{(r)}(j)=f(j,p)$$

where $Q(j)=\dfrac{1}{\binom{n+j}{j}}$ is the reciprocal binomial coefficient and $Q^{(r)}(j)=\dfrac{\mathrm{d}^r}{\mathrm{d}j^r}\big(Q(j)\big)$ is the r^{th} derivative of the reciprocal binomial coefficient.

Binomial coefficients play an important role in many areas of mathematics, including number theory, statistics and probability. The binomial coefficient is defined as

$$\binom{z}{r}=\begin{cases}\dfrac{1}{r!}\prod\limits_{i=1}^{r}(z+1-i), & \text{if } r>0\\ 1, & \text{if } r=0\\ 0, & \text{if } r<0\end{cases}\qquad \text{for } (r,z)\in\mathbb{Z}.$$

2000 Mathematics Subject Classification. 11B65, 05A19, 05A10.
Keywords and Phrases. Binomial coefficients, combinatorial identities, integral representations, finite sums, identities

In particular Diaz-Barrero [**1**] used combinatorial properties in developing a procedure to generalize the result

$$\sum_{n=1}^{p}(-1)^n\binom{p}{n}\sum_{1\le r\le s\le n}\frac{1}{rs}=-\frac{1}{p^2}.$$

In this paper we use integral representations of series to recapture Diaz-Barrero's [**1**] results. Moreover we use our integral representation methods to highlight a number of new results in representing combinatorial sums in closed form.

2. THE MAIN RESULTS

The following Lemma deals with the derivatives of the reciprocal of a binomial coefficient and will be useful in the proof of the main result of this paper.

Lemma 1. *Let a be a positive integer, $j\ge 0$, $n>0$ and* $Q(j)=\dfrac{1}{\binom{an+j}{j}}$. *Then,*

$$Q'(j)=\frac{\mathrm{d}Q}{\mathrm{d}j}=\begin{cases}-\dfrac{1}{\binom{an+j}{j}}\displaystyle\sum_{r=1}^{an}\frac{1}{r+j}, & \text{for } j>0\\ -H_n^{(1)}, & \text{for } j=0 \text{ and } a=1\end{cases}\tag{2.1}$$

and

$$Q''(j)=\frac{\mathrm{d}^2Q}{\mathrm{d}j^2}=\begin{cases}\dfrac{1}{\binom{an+j}{j}}\left(\left(\displaystyle\sum_{r=1}^{an}\frac{1}{r+j}\right)^2+\displaystyle\sum_{r=1}^{an}\frac{1}{(r+j)^2}\right), & \text{for } j>0\\ \dfrac{1}{\binom{an+j}{j}}\begin{pmatrix}\left(\psi(an+j+1)-\psi(j+1)\right)^2\\ +\psi'(j+1)-\psi'(an+j+1)\end{pmatrix} & \\ \left(H_n^{(1)}\right)^2+H_n^{(2)}, & \text{for } j=0 \text{ and } a=1\end{cases}\tag{2.2}$$

where $H_n^{(\alpha)}=\displaystyle\sum_{r=1}^{n}\frac{1}{r^\alpha}$ *is the generalized Harmonic number in power* α. *The polygamma functions* $\psi^{(k)}(z)$, $k\in\mathbb{N}$ *are defined by*

$$\begin{aligned}\psi^{(k)}(z)\ :&=\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}\log\Gamma(z)=\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right),\quad k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\\ &=-\int_{t=0}^{1}\frac{\left(\log(t)\right)^k\, t^{z-1}}{1-t}\,\mathrm{d}t\end{aligned}$$

and $\psi^{(0)}(z) = \psi(z)$, denotes the Psi, or digamma function, defined by

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t.$$

Proof. Let

$$Q(j) = \frac{1}{\binom{an+j}{j}} = \frac{\Gamma(j+1)\Gamma(an+1)}{\Gamma(an+j+1)} = \frac{\Gamma(an+1)}{\prod_{r=1}^{an}(r+j)}, \tag{2.3}$$

taking logs of both sides and differentiating with respect to j we obtain the result (2.1). Differentiating again with respect to j we obtain the result (2.2).

For the specific case of $a = 1$ we can also claim that

$$\begin{aligned}\left(\sum_{r=1}^{n} \frac{1}{r+j}\right)^2 + \sum_{r=1}^{n} \frac{1}{(r+j)^2} &= \sum_{r=1}^{n}\sum_{s=1}^{r} \frac{2}{(r+j)(s+j)} \\ &= \big(\psi(n+j+1) - \psi(j+1)\big)^2 + \psi'(j+1) - \psi'(n+j+1),\end{aligned} \tag{2.4}$$

and for $j = 0$

$$\begin{aligned}\left(\sum_{r=1}^{n} \frac{1}{r}\right)^2 + \sum_{r=1}^{n} \frac{1}{r^2} = \left(H_n^{(1)}\right)^2 + H_n^{(2)} &= 2\sum_{r=1}^{n}\sum_{s=1}^{r} \frac{1}{rs} \\ &= \big(\psi(n+1) - \psi(1)\big)^2 + \psi'(1) - \psi'(n+1) \\ &= 2\sum_{r=1}^{n} \frac{(-1)^{r+1}\binom{n}{r}}{r^2}\end{aligned} \tag{2.5}$$

□

Now we can state the following theorem.

Theorem 1. *Let a be a positive integer, $t \in \mathbb{R}$, $p > 0$ and $j \geq 0$. Then*

$$\begin{aligned}S(a,j,p,t) &= \sum_{n=0}^{p} \frac{t^n \binom{p}{n}}{\binom{an+j}{j}} \sum_{r=1}^{an} \frac{1}{r+j} \\ &= -\int_0^1 (1-x)^{j-1}(1+tx^a)^p\,\mathrm{d}x \\ &\qquad - j\int_0^1 (1-x)^{j-1}(1+tx^a)^p \log(1-x)\,\mathrm{d}x.\end{aligned} \tag{2.6}$$

Proof. We have

$$\sum_{n=0}^{p}\frac{t^n\binom{p}{n}}{\binom{an+j}{j}}=\sum_{n=0}^{p}t^n\binom{p}{n}j\,\frac{\Gamma(j)\Gamma(an+1)}{\Gamma(an+j+1)}=j\sum_{n=0}^{p}t^n\binom{p}{n}B(an+1,j)$$
$$=j\sum_{n=0}^{p}t^n\binom{p}{n}\int_0^1(1-x)^{j-1}x^{an}\,\mathrm{d}x,$$

where $B(\alpha,\beta)=\int_0^1(1-y)^{\alpha-1}y^{\beta-1}\,\mathrm{d}y=\int_0^1(1-y)^{\beta-1}y^{\alpha-1}\,\mathrm{d}y$ for $\alpha>0$ and $\beta>0$ is the classical Beta function and $\Gamma(\cdot)$ is the Gamma function.

By an allowable interchange of sum and integral, we have

$$\sum_{n=0}^{p}\frac{t^n\binom{p}{n}}{\binom{an+j}{j}}=j\int_0^1(1-x)^{j-1}\sum_{n=0}^{p}\binom{p}{n}(tx^a)^n\,\mathrm{d}x \tag{2.7}$$
$$=j\int_0^1(1-x)^{j-1}(1+tx^a)^p\,\mathrm{d}x.$$

Now we differentiate, with respect to j, both sides of (2.7) and utilize (2.1) in Lemma 1, so that

$$\sum_{n=0}^{p}\frac{t^n\binom{p}{n}}{\binom{an+j}{j}}\sum_{r=1}^{an}\frac{1}{r+j}=-\frac{\partial}{\partial j}\left(j\int_0^1(1-x)^{j-1}(1+tx^a)^p\,\mathrm{d}x\right)$$
$$=-\int_0^1(1-x)^{j-1}(1+tx^a)^p\,\mathrm{d}x$$
$$-j\int_0^1(1-x)^{j-1}(1+tx^a)^p\log(1-x)\,\mathrm{d}x. \qquad \square$$

The following Corollary can now be stated.

Corollary 1. *Choosing the values $t=-1$ and $a=1$ we obtain*

$$\sum_{n=0}^{p}\frac{(-1)^n\binom{p}{n}}{\binom{n+j}{j}}\sum_{r=1}^{n}\frac{1}{r+j}=-\frac{p}{(p+j)^2}$$

and for $j=0$

$$\sum_{n=0}^{p}(-1)^n\binom{p}{n}H_n^{(1)}=-\frac{1}{p},$$

by definition we take $H_0^{(\alpha)}=0$.

It is clear that for other values of a and t a multitude of different particular identities are possible, however these are intrinsically more involved and hence will be reported in another paper.

Theorem 2. *Let a be a positive integer, $t \in \mathbb{R}$, $p > 0$ and $j \geq 0$, then*

$$(2.8)\qquad T(a,j,p,t) = \sum_{n=0}^{p} \frac{t^n \binom{p}{n}}{\binom{an+j}{j}} \left(\left(\sum_{r=1}^{an} \frac{1}{r+j} \right)^2 + \sum_{r=1}^{an} \frac{1}{(r+j)^2} \right)$$
$$= 2\int_0^1 (1-x)^{j-1}(1+tx^a)^p \log(1-x)\,\mathrm{d}x$$
$$+ j\int_0^1 (1-x)^{j-1}(1+tx^a)^p \big(\log(1-x)\big)^2\,\mathrm{d}x.$$

Proof. The proof follows by utilizing Lemma (1) and differentiating the right hand side of (2.7) twice with respect to j. $\square$

Corollary 2. *Choosing the values $t = -1$ and $a = 1$ we obtain*

$$\sum_{n=0}^{p} \frac{(-1)^n \binom{p}{n}}{\binom{n+j}{j}} \left(\left(\sum_{r=1}^{n} \frac{1}{r+j} \right)^2 + \sum_{r=1}^{n} \frac{1}{(r+j)^2} \right) = -\frac{2p}{(p+j)^3}$$
$$= \sum_{n=1}^{p} \frac{(-1)^n \binom{p}{n}}{\binom{n+j}{j}} \sum_{1\leq r\leq s\leq n} \frac{2}{(r+j)(s+j)}$$

$$\sum_{n=0}^{p} \frac{(-1)^n \binom{p}{n}}{\binom{n+j}{j}} \begin{pmatrix} \big(\psi(n+j+1)-\psi(j+1)\big)^2 \\ +\psi'(j+1)-\psi'(n+j+1) \end{pmatrix} = -\frac{2p}{(p+j)^3}\,,$$

and for $j = 0$

$$\sum_{n=0}^{p} (-1)^n \binom{p}{n} \left(\big(H_n^{(1)}\big)^2 + H_n^{(2)} \right) = -\frac{2}{p^2}$$
$$= -2\sum_{n=1}^{p} (-1)^n \binom{p}{n} \sum_{r=1}^{n} \frac{(-1)^{r+1}\binom{n}{r}}{r^2}\,.$$

Diaz-Barrero et. al. [**1**] obtained the same results using combinatorial relationships.

REMARK 1. To obtain the result of Theorem 3 as given by DIAZ-BARRERO et. al. [**1**], namely

$$\sum_{n=1}^{p} \frac{(-1)^n \binom{p}{n}}{\binom{n+j}{j}} \left(\left(\sum_{r=1}^{n} \frac{1}{r+j} \right)^3 + 3 \left(\sum_{r=1}^{n} \frac{1}{r+j} \right) \left(\sum_{r=1}^{n} \frac{1}{(r+j)^2} \right) + 2 \sum_{r=1}^{n} \frac{1}{(r+j)^3} \right) = -\frac{6p}{(p+j)^4}$$

and for $j = 0$

$$\sum_{n=1}^{p} (-1)^n \binom{p}{n} \left(\left(H_n^{(1)}\right)^3 + 3H_n^{(1)} H_n^{(2)} + 2H_n^{(3)} \right) = -\frac{6}{p^3}$$

we simply differentiate both sides (2.7) three times and evaluate the resulting integral. Again we can continue this process of differentiation many times to obtain a multitude of identities.

The method of integral identities is useful for obtaining closed form representations of sums and we can extend the results of the previous section as follows.

3. EXTENSION OF RESULTS

The previous results can be extended in various ways; we give one such extension.

Theorem 3. *Let a be a positive integer, $t \in \mathbb{R}$, $p > 0$ and $j \geq 0$. Then*

$$\sum_{n=0}^{p} \frac{t^n n \binom{p}{n}}{\binom{an+j}{j}} \sum_{r=1}^{an} \frac{1}{r+j} = -pt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} \, \mathrm{d}x - jpt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} \log(1-x) \, \mathrm{d}x, \tag{3.1}$$

and

$$\sum_{n=0}^{p} \frac{t^n \, n \binom{p}{n}}{\binom{an+j}{j}} \left(\left(\sum_{r=1}^{an} \frac{1}{r+j} \right)^2 + \sum_{r=1}^{an} \frac{1}{(r+j)^2} \right) = 2pt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} \log(1-x) \, \mathrm{d}x + jpt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} \big(\log(1-x)\big)^2 \, \mathrm{d}x. \tag{3.2}$$

Proof. From (2.7) in Theorem 1 we have

$$\sum_{n=0}^{p} \frac{t^n \binom{p}{n}}{\binom{an+j}{j}} = j \int_0^1 (1-x)^{j-1}(1+tx^a)^p \, \mathrm{d}x$$

and applying the operator $x\frac{\mathrm{d}}{\mathrm{d}x}\left((1+tx^a)^p\right)$ we may write

$$\sum_{n=0}^{p} \frac{t^n \, n\binom{p}{n}}{\binom{an+j}{j}} = jpt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} \, \mathrm{d}x.$$

Differentiating with respect to j and applying Lemma 1 we obtain the result

$$\begin{aligned}\sum_{n=0}^{p} \frac{t^n n\binom{p}{n}}{\binom{an+j}{j}} \sum_{r=1}^{an} \frac{1}{r+j} &= -\frac{\partial}{\partial j}\left(jpt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} \, \mathrm{d}x\right) \\ &= -pt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} \, \mathrm{d}x \\ &\quad -jpt \int_0^1 (1-x)^{j-1} x^a (1+tx^a)^{p-1} \log(1-x) \, \mathrm{d}x,\end{aligned}$$

which is the result (3.1).

Again differentiating with respect to j and applying Lemma 1 we obtain the result (3.2). □

Corollary 3. *Choosing the values $t=-1$ and $a=1$ we obtain*

$$\sum_{n=1}^{p} \frac{(-1)^n \, n\binom{p}{n}}{\binom{n+j}{j}} \sum_{r=1}^{n} \frac{1}{r+j} = \frac{p(p^2-p-j^2)}{(p+j)^2(p+j-1)^2}$$

and

$$\sum_{n=1}^{p} \frac{(-1)^n \, n\binom{p}{n}}{\binom{n+j}{j}} \left(\left(\sum_{r=1}^{n} \frac{1}{r+j}\right)^2 + \sum_{r=1}^{n} \frac{1}{(r+j)^2}\right) = \frac{2p\left(2p^3+3p^2(j-1)-p(3j-1)-j^3\right)}{(p+j)^3(p+j-1)^3}$$

and for $j=0$

$$\sum_{n=1}^{p}(-1)^n\, n\binom{p}{n}H_n^{(1)}=\frac{1}{p-1}$$

and

$$\sum_{n=1}^{p}(-1)^n\, n\binom{p}{n}\left(\left(H_n^{(1)}\right)^2+H_n^{(2)}\right)=\frac{2\,(2p-1)}{p(p-1)^2},\quad p>1.$$

Remark 2. Differentiating (3.2), once more we have

$$\sum_{n=1}^{p}\frac{(-1)^n\, n\binom{p}{n}}{\binom{n+j}{j}}\left(\left(\sum_{r=1}^{n}\frac{1}{r+j}\right)^3+3\left(\sum_{r=1}^{n}\frac{1}{r+j}\right)\left(\sum_{r=1}^{n}\frac{1}{(r+j)^2}\right)+2\sum_{r=1}^{n}\frac{1}{(r+j)^3}\right)$$
$$=\frac{6p(3p^4+2p^3(4j-3)+2p^2(3j^2-6j+2)-p(6j^2-4j+1)-j^4)}{(p+j)^4(p+j-1)^4}$$

and for $j=0$

$$\sum_{n=1}^{p}(-1)^n\, n\binom{p}{n}\left(\left(H_n^{(1)}\right)^3+3H_n^{(1)}H_n^{(2)}+2H_n^{(3)}\right)=\frac{6(3p^2-3p+1)}{p^2(p-1)^3},\quad p>1.$$

These results can also be extended in a variety of different ways.

4. CONCLUSION

We have applied the method of integral representation for binomial sums and recaptured in a natural way the results of Diaz-Barrero et. al. [**1**]. We have also extended the range of identities of finite sums with Harmonic numbers and detailed a procedure that allows for many other identities depending on the parameter values chosen. The technique of integral representation of binomial sums can be extended in various other directions, these extensions will be reported in another forum.

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(Received November 30, 2007)
(Revised February 11, 2008)