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NEW SUFFICIENT CONDITIONS FOR INTEGRAL COMPLETE 3-PARTITE GRAPHS

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A graph is integral if all the eigenvalues of its adjacency matrix are integers. In this paper we give sufficient conditions for complete 3-partite graphs to be integral, from which we construct infinitely many new classes of such integral graphs.

1. INTRODUCTION

The notion of integral graphs was introduced by F. Harary and A. J. Schwenk in 1974 (see [**5**]). A graph G is called integral if all the zeros of the characteristic polynomial $P(G,x)$ are integers. In general, the problem of characterizing integral graphs seems to be very difficult. Thus, it makes sense to restrict our investigations to some families of graphs, for instance cubic graphs, trees, etc. The results on integral graphs can be found in [**1, 2, 3, 4, 6, 10**]. The first class of integral complete 3-partite graphs were found in [**9**]. More classes of integral complete 3-partite graphs were found in [**11, 12**]. Moreover, some necessary and sufficient conditions for integral complete r-partite graphs were given in [**12**], as well as several open problems for integral complete r-partite graphs. Finally, the classes of integral complete 4-partite graphs were constructed in [**8**] and [**11**]. For all other facts on terminology of graph spectra see [**3, 4**].

We shall consider only simple undirected graphs. For a graph G, let $V(G)$ denote the vertex set and $E(G)$ the edge set. The characteristic polynomial $|xI-A|$ of the adjacency matrix A of G is called the characteristic polynomial of G and is denoted by $P(G,x)$. The spectrum of A is also called the spectrum of G.

A complete 3-partite graph $K_{m,n,p}$ is a graph with a vertex set $V = V_1 \bigcup V_2 \bigcup V_3$, where $|V_1| = m$, $|V_2| = n$, $|V_3| = p$, such that two vertices in V are adjacent if and only if they belong to different V_i's.

In this paper we give sufficient conditions for complete $3-$partite graphs to be integral, from which we construct infinitely many new classes of such integral graphs.

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2. RESULTS

From the theory of divisors and co-divisors of a graph follows that the divisor of the graph $K_{m,n,p}$ has characteristic polynomial

$$P(D,x) = x^3 - (mn + mp + np)x - 2mnp.$$

Moreover, the characteristic polynomial of the co-divisor is

$$P(C,x) = x^{m+n+p-3}.$$

More details about the theory of divisors and co-divisors can be found in [**3**, **6**, **7**]. Thus, the following theorem holds:

Theorem 1. *A graph $K_{m,n,p}$ is integral if and only if the characteristic polynomial of its divisor has only integer zeros, which means that the zeros x_1, x_2, x_3 have to satisfy the following equations*:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0; \quad x_2 < 0;\ x_3 < 0, \\ x_1x_2 + x_1x_3 + x_2x_3 &= -mn - mp - np, \\ x_1x_2x_3 &= 2mnp. \end{aligned} \tag{1}$$

In addition, in [**12**] it is proved that if $m < n < p$ and x_1, x_2, x_3 are the zeros of $P(D,x)$, then

$$-p < x_3 < -n < x_2 < -m < 0 < x_1. \tag{2}$$

The following theorem was given in [**12**].

Theorem 2. *For any positive integer q, the complete 3-partite graph $K_{m,n,p}$ is integral if and only if the complete 3-partite graph $K_{mq,nq,pq}$ is integral.*

The above theorem shows that it is reasonable to study only complete 3-partite graphs where $(m,n,p) = 1$. Let us call such a vector primitive.

The following theorem was given in [**9**].

Theorem 3. *Let $m = 4u^2(u^2+v^2)^3, n = 3u^2v^2(u^2+6uv+v^2)(-u^2+6uv-v^2)$, $p = 4v^2(u^2+v^2)^3$ such that $(3-\sqrt{8})v < u < v$, and let $x_1 = 24u^2v^2(u^2+v^2)^2$, $x_2 = -2uv(u^2+v^2)^2(-u^2+6uv-v^2), x_3 = -2uv(u^2+v^2)^2(u^2+6uv+v^2)$, such that u, v are positive integers. Then $K_{m,n,p}$ is integral.*

The above theorem gives infinitely many integral complete 3-partite graphs $K_{m,n,p}$.

Using a computer we have found 137 primitive integral complete 3-partite graphs $K_{m,n,p}$, where $m < n < p$, $1 \le m \le 1000$, $m+1 \le n \le 1000$, $n+1 \le p \le 1000$. Some of them are in Table 1.

No.	m	n	p	x_1	x_2	x_3	No.	m	n	p	x_1	x_2	x_3
1	3	17	65	39	–5	–34	13	112	288	585	576	–156	–420
2	4	13	48	32	–6	–26	14	144	441	980	882	–210	–672
3	5	53	207	115	–9	–106	15	200	225	252	450	–210	–240
4	7	13	45	35	–9	–26	16	125	357	500	600	–175	–425
5	7	109	429	231	–13	–218	17	5	12	77	40	–7	–33
6	8	65	252	144	–14	–130	18	25	297	675	495	–45	–450
7	13	24	28	42	–16	–26	19	33	98	833	357	–49	–308
8	17	33	35	55	–21	–34	20	45	136	396	306	–66	–240
9	29	36	80	90	–32	–58	21	49	220	441	392	–77	–315
10	29	39	77	91	–33	–58	22	216	385	684	798	–270	–528
11	5	8	12	16	–6	–10	23	297	437	585	858	–345	–513
12	44	200	525	400	–70	-330	24	693	760	828	1515	–720	–798

Table 1.

Analyzing these 137 primitive integral complete 3-partite graphs, one can see that 62 of them have the property $x_3 = -2n$ (see for example graphs No. 1-6 in Table 1), 62 of them have the property $x_3 = -2m$ (see for example graphs No. 7-11 in Table 1).

Based on similar exploration, but on a sample of only 34 complete 3-partite integral graphs, the authors of [**12**] proved the following theorem using (1) and the assumption $x_3 = -2m$ or $x_3 = -2n$.

Theorem 4. *Let* $m = \dfrac{a^2+b^2}{d}$, $n = \dfrac{2a(2a+b)}{d}$, $p = \dfrac{2b(2b-a)}{d}$, $x_1 = \dfrac{2b(b+2a)}{d}$, $x_2 = -\dfrac{2a(2b-a)}{d}$, $x_3 = -\dfrac{2(a^2+b^2)}{d}$, *where* $a > 0, b > 0, d = (2b(2b-a), 2a(2a+b), a^2+b^2)$. *Then* $K_{m,n,p}$ *is integral, where* $d \in \{1, 2, 5, 10\}$.

Remark. From 137 primitive integral complete 3-partite graphs found by computer, all 124 graphs $K_{m,n,p}$, where $x_3 = -2m$ or $x_3 = -2n$ can be obtained using Theorem 4 for suitable a and b. Moreover, the graph No. 16 from Table 1 can be obtained by Theorem 3.

Using (1) and (2) it is clear that none of the following cases can happen: $x_3 = -2p, x_2 = -2m, x_2 = -2n, x_2 = -2p, x_1 = 2m, x_1 = 2p$. From Table 1 we can see that there exist integral complete 3-partite graphs where $x_1 = 2n$. In the following part of the paper we investigate these graphs and give sufficient conditions for these graphs to be integral.

Using the substitution $x_1 = v, -x_2 = w, -x_3 = u$ in (1) we get

$$\begin{aligned} u + w &= v; \quad u > 0;\ v > 0;\ w > 0, \\ mn + mp + np &= vw + vu - wu, \\ vwu &= 2mnp. \end{aligned} \tag{3}$$

Let $x_1 = v = 2n$. From (3) we get

$$uw = pm, \qquad u + w = 2n. \tag{4}$$

Equations (4) have integral solutions for u, w if and only if there exists integer x such that

$$n^2 - mp = x^2. \tag{5}$$

Then

$$u, w = n \pm x. \tag{6}$$

From equations (3) and (4) it follows

$$4n^2 = 2pm + n(p + m). \tag{7}$$

Using (5) to (7) we have

$$2n^2 - n(p + m) + 2x^2 = 0. \tag{8}$$

Equation (8) have an integral solution for n if and only if there exists integer y such that

$$(p + m)^2 - 16x^2 = y^2. \tag{9}$$

Using the substitution $q = p + m$ in (9) we have $(q - y)(q + y) = 4x \cdot 4x$ and after simplification

$$\frac{q + y}{4x} = \frac{4x}{q - y} = \frac{a}{b}\,, \text{ where } a > b, (a, b) = 1. \tag{10}$$

From (10) after routine simplification we have

$$2aby = 4(a^2 - b^2)x, \qquad 2abq = 4(a^2 + b^2)x,$$

from which we get

$$y : x : q = 2(a^2 - b^2) : ab : 2(a^2 + b^2)$$

.

If $d = (ab, 2(a^2 - b^2),\, 2(a^2 + b^2))$, then

$$y = \frac{2(a^2 - b^2)}{d}\,, \quad x = \frac{ab}{d}\,, \quad q = \frac{2(a^2 + b^2)}{d}. \tag{11}$$

It is easy to verify that if both a and b are odd then $d = 1$, otherwise $d = 2$. From (8) we have $n = \frac{p + m \pm y}{4} = \frac{q \pm y}{4}$. Using (11) we get $n = \frac{a^2}{d}$ or $n = \frac{b^2}{d}$. If we combine (5), (11) and $n = \frac{a^2}{d}$, we have $uw = pm = n^2 - x^2 = \frac{a^2(a^2 - b^2)}{d}$. As pm is a positive integer, the case $n = \frac{b^2}{d}$ does not suit. Now, let us use $n = \frac{a^2}{d}$ and $x = \frac{ab}{d}$ in (6). We have

$$u, w = n \pm x = \frac{a^2 \pm ab}{d}\,, \tag{12}$$

where $(a, b) = 1$, $a > b$ and if ab is odd, then $d = 1$, otherwise $d = 2$. Moreover, the following formula holds.

$$(p - m)^2 = (p + m)^2 - 4mp = q^2 - 4uw = \frac{4b^2(3a^2 + b^2)}{d^2} \tag{13}$$

It means that there exists an integer c, for which $3a^2 + b^2 = c^2$. After simplification we have $3a^2 = (c + b)(c - b)$, from which we get

$$\frac{3a}{c + b} = \frac{c - b}{a} = \frac{s}{t}\,, \tag{14}$$

where $(s, t) = 1, s > t$. After routine simplification of equations (14) we get

$$a(3t^2 - s^2) = 2bst, \qquad a(3t^2 + s^2) = 2cst. \tag{15}$$

Then $a : b : c = 2st : 3t^2 - s^2 : 3t^2 + s^2$. Let us denote $k = (2st, 3t^2 - s^2, 3t^2 + s^2)$. Then we have

$$a = \frac{2st}{k}\,, \quad b = \frac{3t^2 - s^2}{k}\,, \quad c = \frac{3t^2 + s^2}{k}\,. \tag{16}$$

It is easy to prove that $k \in \{1, 2, 3, 6\}$. We can get the following cases:

1. $k = 6$ if $3|s$ and both s and t are odd,
2. $k = 3$ if $3|s$ and st is even,
3. $k = 2$ if $3 \nmid s$ and both s and t are odd,
4. $k = 1$ if $3 \nmid s$ and st is even.

Moreover, if st is even, then $2|a$ and so $d = 2$.

If st is odd, then $s = 2r + 1, t = 2\ell + 1$ and so $4 \nmid 2st$ and $2 \nmid a$. Then $3t^2 - s^2 = 4(3\ell^2 + 3\ell - r^2 - r) + 2$, so $4 \nmid (3t^2 - s^2)$, $2 \nmid b$ and $d = 1$.

For k and d the following table holds.

		k	d	dk^2
$3\|s$	s and t are odd	6	1	36
$3\|s$	$s \cdot t$ is even	3	2	18
$3 \nmid s$	s and t are odd	2	1	4
$3 \nmid s$	$s \cdot t$ is even	1	2	2

Table 2.

The following conditions hold for parameters a, b, s, t:

1. If $b > 0$, then $3t^2 > s^2$, so $\sqrt{3}t > s$.

2. If $a > b$, then $2st > 3t^2 - s^2$ and after simplification we have $(s + 3t)(s - t) > 0$, from which $s > t$.

Using (12) we have $u, w = \dfrac{a^2 \pm ab}{d}$ and $v = \dfrac{2a^2}{d}$, where $d = 1$ if $a \cdot b$ is odd and $d = 2$ if $a \cdot b$ is even. Using (13) we have $|p - m| = \dfrac{2bc}{d}$ and by (11) $p + m = q = \dfrac{2(a^2 + b^2)}{d}$. From the previous equations we get $n = \dfrac{a^2}{d}$ $m = \dfrac{a^2 + b^2 - bc}{d}$, $p = \dfrac{a^2 + b^2 + bc}{d}$ and after substitution for a, b, c we have the following equations.

$$m = \frac{2s^2(s^2 - t^2)}{dk^2}, \quad n = \frac{4t^2 s^2}{dk^2}, \quad p = \frac{2t^2(9t^2 - s^2)}{dk^2}, \tag{17}$$

$$u = \frac{2st(3t - s)(s + t)}{dk^2}, \quad v = \frac{8t^2 s^2}{dk^2}, \quad w = \frac{2st(3t + s)(s - t)}{dk^2}. \tag{18}$$

Using previous equations we can formulate Theorem 5, which gives new infinite classes of integral complete 3-partite graphs for suitable values of parameters s and t.

Theorem 5. *Let* $m = \dfrac{2s^2(s^2 - t^2)}{dk^2}$, $n = \dfrac{4t^2 s^2}{dk^2}$, $p = \dfrac{2t^2(9t^2 - s^2)}{dk^2}$, *where* $t < s < \sqrt{3}t$, $dk^2 = (2s^2(s^2-t^2), 4t^2s^2, 2t^2(9t^2-s^2))$. *Then the graph* $K_{m,n,p}$ *is integral and zeros of its divisor are* $x_1 = \dfrac{8t^2 s^2}{dk^2}$, $x_2 = \dfrac{2ts(s - 3t)(t + s)}{dk^2}$, $x_3 = \dfrac{2ts(s + 3t)(t - s)}{dk^2}$.

For the values $(s, t) \in \{(2, 3), (3, 4), (3, 5), (5, 6), (7, 9)\}$ we get graphs No. 11-15 in Table 1. Next graphs for $t < 9$ can be found in Table 3. Let us remark that for graph No. 11 in Table 1 it holds both $x_1 = 2n$ and $x_3 = -2m$.

No.	t	s	mdk^2	ndk^2	pdk^2	dk^2	m	n	p	x_1	x_2	x_3
1	2	3	90	144	216	18	5	8	12	16	–10	–6
2	3	4	224	576	1170	2	112	288	585	576	–420	–156
3	3	5	800	900	1008	4	200	225	252	450	–240	–210
4	4	5	450	1600	3808	2	225	800	1904	1600	–1260	–340
5	5	6	792	3600	9450	18	44	200	525	400	–330	–70
6	5	7	2352	4900	8800	4	588	1225	2200	2450	–1680	–770
7	5	8	4992	6400	8050	2	2496	3200	4025	6400	–3640	–2760
8	6	7	1274	7056	19800	2	637	3528	9900	7056	–6006	–1050
9	7	8	1920	12544	36946	2	960	6272	18473	12544	–10920	–1624
10	7	9	5184	15876	35280	36	144	441	980	882	–672	–210
11	7	10	10200	19600	33418	2	5100	9800	16709	19600	–13090	–6510
12	7	11	17424	23716	31360	4	4356	5929	7840	11858	–6930	–4928
13	7	12	27360	28224	29106	18	1520	1568	1617	3136	–1596	–1540
14	8	9	2754	20736	63360	18	153	1152	3520	2304	–2040	–264
15	8	11	13794	30976	58240	2	6897	15488	29120	30976	–21736	–9240
16	8	13	35490	43264	52096	2	17745	21632	26048	43264	–24024	–19240

Table 3.

Using Theorem 2 in Theorem 5 we get the following result.

Corollary 6. *For every m, n, p, where* $m = \dfrac{2s^2(s^2-t^2)}{dk^2}$, $n = \dfrac{4t^2s^2}{dk^2}$, $p = \dfrac{2t^2(9t^2-s^2)}{dk^2}$, $t < s < \sqrt{3}t$, $dk^2 = (2s^2(s^2-t^2), 4t^2s^2, \ 2t^2(9t^2-s^2))$ *and for arbitrary $q \in N$ the graph $K_{mq,nq,pq}$ is integral complete 3-partite graph.*

In the next part of the paper we give different sufficient conditions for graph $K_{m,n,p}$ to be integral. The main difference between these conditions for m, n, p and conditions in Theorems 4 and 5 is that these conditions depend on just one parameter.

Let substitute $x_1 = mp_2$, $x_2 = -p_1$, $x_3 = -2n$, where $p = p_1.p_2, (m,n,p) = 1, m < n < p$. After simplification we get

$$m = \frac{p_1(2+p_2)}{2p_2-1}, \quad n = \frac{p_1(p_2^2+1)}{2(2p_2-1)}, p = p_1 \cdot p_2. \tag{19}$$

Let us consider two cases.

1. Let p_2 be even, which means $p_2 = 2t$. Then

$$m = \frac{p_1(2+2t)}{4t-1}, \quad n = \frac{p_1(4t^2+1)}{2(4t-1)}, \quad p = 2p_1t. \tag{20}$$

Clearly, $(4t^2+1, 4t-1) = 1$ if $t \in N$ and $t \neq 4+5s$, otherwise $(4t^2+1, 4t-1) = 5$. As m, n, p are integers, $p_1 = 2k(4t-1)$. Then $m = 4+4t, n = 4t^2+1, p = 4t(4t-1)$.

2. Let p_2 be odd, which means $p_2 = 2t+1$. Then

$$m = \frac{p_1(3+2t)}{4t+1}, \quad n = \frac{p_1(2t^2+2t+1)}{(4t+1)}, \quad p = p_1(2t+1). \tag{21}$$

Clearly, $(2t+3, 4t+1) = 1$ if $t \in \mathbb{N}$ and $t \neq 1+5s$, otherwise $(2t+3, 4t+1) = 5$. As m, n, p are integers, $p_1 = k(4t+1)$. Then $m = 2t+3, n = 2t^2+2t+1, p = (2t+1)(4t+1)$.

We get the following theorem.

Theorem 6.

1. *Let* $m = \dfrac{4+4t}{d}$, $n = \dfrac{4t^2+1}{d}$, $p = \dfrac{4t(4t-1)}{d}$, $t \in \mathbb{N}, d = 5$ *if* $t = 4+5s$, *otherwise* $d = 1$. *Then $K_{m,n,p}$ is integral.*
2. *Let* $m = \dfrac{3+2t}{d}$, $n = \dfrac{2t^2+2t+1}{d}$, $p = \dfrac{(2t+1)(4t+1)}{d}$, $t \in \mathbb{N}$, $d = 5$ *if* $t = 1+5s$, *otherwise* $d = 1$. *Then $K_{m,n,p}$ is integral.*

The integral graphs from Theorem 6 for parameter $t \in \{1, 2, \ldots, 10\}$ are in Table 4.

t	m	n	p	d	t	m	n	p	d
1	5	5	15	5	1	8	5	12	1
2	7	13	45	1	2	12	17	56	1
3	9	25	91	1	3	16	37	132	1
4	11	41	153	1	4	20	65	240	5
5	13	61	231	1	5	24	101	380	1
6	15	85	325	5	6	28	145	552	1
7	17	113	435	1	7	32	197	756	1
8	19	145	561	1	8	36	257	992	1
9	21	181	703	1	9	40	325	1260	5
10	23	221	861	1	10	44	401	1560	1

Table 4.

REMARK. Integral graphs obtained by Theorem 6 can be obtained also by Theorem 4. However, while in Theorem 4 m, n, p depend on two parameters, in Theorem 6 they depend on just one parameter.

3. CONCLUSION

Theorems 3-6 give sufficient conditions by which we can construct infinite classes of integral complete 3-partite graphs. However, by these conditions we are not able to generate integral graphs $K_{m,n,p}$, where

$$(m,n,p) \in \{(5,12,77),\ (25,297,675),\ (33,98,833),\ (45,136,396),\\ (49,220,441),\ (216,385,684), (297,437,585), (693,760,828)\}.$$

These integral graphs were obtained using computers and are in Table 1, No. 17-24. Is it possible to find sufficient conditions by which we can generate these graphs? Characterization of integral complete 3-partite graphs remains open.

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