Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 137–146. doi:10.2298/AADM0901137L

CLIQUE IRREDUCIBILITY AND CLIQUE VERTEX IRREDUCIBILITY OF GRAPHS

Aparna Lakshmanan S., A. Vijayakumar

A graphs G is clique irreducible if every clique in G of size at least two, has an edge which does not lie in any other clique of G and is clique reducible if it is not clique irreducible. A graph G is clique vertex irreducible if every clique in G has a vertex which does not lie in any other clique of G and clique vertex reducible if it is not clique vertex irreducible. The clique vertex irreducibility and clique irreducibility of graphs which are non-complete extended p-sums (NEPS) of two graphs are studied. We prove that if G^c has at least two non-trivial components then G is clique vertex reducible and if it has at least three non-trivial components then G is clique reducible. The cographs and the distance hereditary graphs which are clique vertex irreducible and clique irreducible are also recursively characterized.

1. INTRODUCTION

We consider only finite, simple graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$.

A clique of a graph G is a maximal complete subgraph of G. The vast literature on cliques of a graph is very well summarized in [**11**]. A graph G is clique irreducible if every clique in G of size at least two, has an edge which does not lie in any other clique of G and is clique reducible if it is not clique irreducible [**10**]. A graph G is clique vertex irreducible if every clique in G has a vertex which does not lie in any other clique of G and is clique vertex reducible if it is not clique vertex irreducible [**1**]. The clique vertex irreducibility implies clique irreducibility, whereas the converse is not true.

In [**10**], it is proved that the interval graphs are clique irreducible. Wallis and Zhang [**12**] generalized this result and attempted to characterize clique irreducible graphs. In [**1**], the line graphs and its iterations, the Gallai graphs, the

2000 Mathematics Subject Classification. 05C99
Keywords and Phrases. Clique vertex irreducible graphs, clique irreducible graphs, non-complete extended p-sum (NEPS), cographs, distance hereditary graphs.

anti-Gallai graphs and its iterations which are clique irreducible and which are clique vertex irreducible are obtained.

The complement of a graph G is denoted by G^c. A graph whose edge set is empty is called a trivial graph. A vertex of degree one is called a pendant vertex. The graph induced by the vertices in $N(v)$ is denoted by $\langle N(v) \rangle$. The join (sum) of two graphs G and H, denoted by $G \vee H$ is defined as the graph with

$$V(G \vee H) = V(G) \cup V(H)$$

and

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv, \text{ where } u \in V(G) \text{ and } v \in V(H)\}$$

. A graph G is H-free, if G does not contain H as an induced subgraph. For all graph theoretic notations and preliminaries, we follow [**2**]

In this paper, the clique vertex irreducibility and clique irreducibility of graphs which are non-complete extended p-sums (NEPS) of two graphs are studied. We prove that if G^c has at least two non-trivial components then G is clique vertex reducible and if it has at least three non-trivial components then G is clique reducible. The cographs and the distance hereditary graphs which are clique vertex irreducible and clique irreducible are also recursively characterized.

2. NEPS OF TWO GRAPHS

The non-complete extended p-sum of graphs (NEPS) were first introduced in [**6**] in the context of studying eigen values of graphs. Let $\mathcal{B}$ be a non-empty subset of the collection of all binary n-tuples which does not include $(0, 0, \ldots, 0)$. The non-complete extended p-sum of graphs $G_1, G_2, \ldots, G_p$ with basis $\mathcal{B}$ denoted by NEPS$(G_1, G_2, \ldots, G_p; \mathcal{B})$, is the graph with vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_p)$, in which two vertices $(u_1, u_2, \ldots, u_p)$ and $(v_1, v_2, \ldots, v_p)$ are adjacent if and only if there exists $(\beta_1, \beta_2, \ldots, \beta_p) \in \mathcal{B}$ such that u_i is adjacent to v_i in G_i whenever $\beta_i = 1$ and $u_i = v_i$ whenever $\beta_i = 0$. The graphs $G_1, G_2, \ldots, G_p$ are called the factors of the NEPS [**6**]. Most of the well known graph products are special cases of the NEPS.

There are seven possible ways of choosing the basis $\mathcal{B}$ when $p = 2$.
$\mathcal{B}_1 = \{(0, 1)\}$
$\mathcal{B}_2 = \{(1, 0)\}$
$\mathcal{B}_3 = \{(1, 1)\}$
$\mathcal{B}_4 = \{(0, 1), (1, 0)\}$
$\mathcal{B}_5 = \{(0, 1), (1, 1)\}$
$\mathcal{B}_6 = \{(1, 0), (1, 1)\}$
$\mathcal{B}_7 = \{(0, 1), (1, 0), (1, 1)\}$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs with $|V_i| = n_i$ and $|E_i| = m_i$ for $i = 1, 2$.

The NEPS$(G_1, G_2; \mathcal{B}_1)$ is G_2 repeated n_1 times and NEPS$(G_1, G_2; \mathcal{B}_2)$ = NEPS$(G_2, G_1; \mathcal{B}_1)$.

In the NEPS$(G_1, G_2; \mathcal{B}_j)$ two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if

(1) $j = 3$: u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2. This is same as the tensor product [**8**] of G_1 and G_2.

(2) $j = 4$: $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$. This is same as the cartesian product [**8**] of G_1 and G_2.

(3) $j = 5$: Either $u_1 = u_2$ or u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2.

(4) $j = 6$: This is same as NEPS$(G_2, G_1; \mathcal{B}_5)$.

(5) $j = 7$: Either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$ or u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2. This is same as the strong product [**8**] of G_1 and G_2.

Theorem 2.1. *The* NEPS$(G_1, G_2; \mathcal{B}_3)$ *is clique vertex reducible except for* $G_1 = K_{1,n}$ *and* $G_2 = K_2$.

Proof.

Case 1. Both G_1 and G_2 contains K_3 as a subgraph.

Let $\langle u_1, u_2, \ldots, u_k \rangle$ and $\langle v_1, v_2, \ldots, v_l \rangle$ be cliques in G_1 and G_2 respectively, where k and ℓ are both greater than or equal to three. Then $\langle (u_1, v_1), (u_2, v_2), \ldots, (u_i, v_i) \rangle$, where $i = min\{k, \ell\}$ is a clique in NEPS$(G_1, G_2; \mathcal{B}_3)$, all of whose vertices are present in at least one of the cliques $\langle (u_1, v_1), (u_2, v_3), (u_3, v_2), (u_4, v_4), \ldots, (u_i, v_i) \rangle$, $\langle (u_1, v_2), (u_2, v_1), (u_3, v_3), (u_4, v_4), \ldots, (u_i, v_i) \rangle$ and $\langle (u_1, v_3), (u_2, v_2), (u_3, v_1), (u_4, v_4), \ldots, (u_i, v_i) \rangle$.

Case 2. At least one among G_1 and G_2 does not contain K_3 as a subgraph.

In this case, NEPS$(G_1, G_2; \mathcal{B}_3)$ also does not contain K_3 as a subgraph. The K_3-free graphs which are clique vertex irreducible are of the form $\bigcup K_{1,n}$. If both G_1 and G_2 has P_3 as an induced subgraph, then NEPS$(G_1, G_2; \mathcal{B}_3)$ contains $K_{1,4} \cup C_4$ as an induced subgraph which is a contradiction. Therefore, one of them, say G_2, must be K_2. But, then NEPS$(G_1, G_2; \mathcal{B}_3)$ is $\bigcup K_{1,n}$ only if $G_1 = K_{1,n}$. □

Theorem 2.2. *The* NEPS$(G_1, G_2; \mathcal{B}_3)$ *is clique irreducible if and only if one of the following holds.*

(1) G_1 *or* G_2 *is* K_3*-free.*

(2) G_1 *and* G_2 *are clique irreducible* K_4*-free graphs.*

Proof.

Case 1. G_1 or G_2 is K_3-free.

In this case, NEPS$(G_1, G_2; \mathcal{B}_3)$ is also K_3-free and hence is clique irreducible.

Case 2. G_1 and G_2 contains K_4 as an induced subgraph.

Let $\langle u_1, u_2, \ldots, u_k\rangle$ and $\langle v_1, v_2, \ldots, v_l\rangle$ be cliques of size greater than or equal to four in G_1 and G_2 respectively. All the edges of the clique $\langle (u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4), (u_i, v_i)$, for $i = 5, 6, \ldots, min\{k, l\}\rangle$ in NEPS$(G_1, G_2; \mathcal{B}_3)$ will be present in at least one of the cliques

$\langle (u_1, v_2), (u_2, v_1), (u_3, v_3), (u_4, v_4), (u_i, v_i)$, for $i = 5, 6, \ldots, min\{k, \ell\}\rangle$,
$\langle (u_1, v_3), (u_2, v_2), (u_3, v_1), (u_4, v_4), (u_i, v_i)$, for $i = 5, 6, \ldots, \min\{k, l\}\rangle$,
$\langle (u_1, v_4), (u_2, v_2), (u_3, v_3), (u_4, v_1), (u_i, v_i)$, for $i = 5, 6, \ldots, min\{k, \ell\}\rangle$,
$\langle (u_1, v_1), (u_2, v_3), (u_3, v_2), (u_4, v_4), (u_i, v_i)$, for $i = 5, 6, \ldots, \min\{k, \ell\}\rangle$,
$\langle (u_1, v_1), (u_2, v_4), (u_3, v_3), (u_4, v_2), (u_i, v_i)$, for $i = 5, 6, \ldots, min\{k, \ell\}\rangle$

and

$\langle (u_1, v_1), (u_2, v_2), (u_3, v_4), (u_4, v_3), (u_i, v_i)$, for $i = 5, 6, \ldots, \min\{k, \ell\}\rangle$

in NEPS$(G_1, G_2; \mathcal{B}_3)$. Therefore, NEPS$(G_1, G_2; \mathcal{B}_3)$ is clique reducible.

Case 3. G_1 contains K_3 but not K_4 and G_2 contains K_4 as a subgraph.

Let $\langle u_1, u_2, u_3\rangle$ be a clique in G_1 and let $\langle v_1, v_2, \ldots, v_\ell\rangle$ be a clique in G_2, where $\ell \geq 4$. Then all the edges of the clique $\langle (u_1, v_1), (u_2, v_2), (u_3, v_3)\rangle$ will be present in at least one of the cliques $\langle (u_1, v_4), (u_2, v_2), (u_3, v_3)\rangle$, $\langle (u_1, v_1), (u_2, v_4), (u_3, v_3)\rangle$ and $\langle (u_1, v_1), (u_2, v_2), (u_3, v_4)\rangle$ in NEPS$(G_1, G_2; \mathcal{B}_3)$.

Therefore, NEPS$(G_1, G_2; \mathcal{B}_3)$ is clique reducible.

Case 4. Both G_1 and G_2 contains a K_3, but are K_4-free.

Since G_1 and G_2 are K_4-free, NEPS$(G_1, G_2; \mathcal{B}_3)$ is also K_4-free. The cliques of size two always has an edge which does not lie in any other clique. Let $C = \langle (u_1, v_1), (u_2, v_2), (u_3, v_3)\rangle$ be a clique in NEPS$(G_1, G_2; \mathcal{B}_3)$. Therefore, $\langle u_1, u_2, u_3\rangle$ and $\langle v_1, v_2, v_3\rangle$ are cliques in G_1 and G_2 respectively. If both G_1 and G_2 are clique irreducible then there exist edges u_1u_2 and v_1v_2 in G_1 and G_2 respectively, which are not present in any other cliques of G_1 and G_2. Therefore, $(u_1, v_1)(u_2, v_2)$ is an edge in C which is not present in any other clique of NEPS$(G_1, G_2; \mathcal{B}_3)$. Therefore, NEPS$(G_1, G_2; \mathcal{B}_3)$ is clique irreducible. If G_1 or G_2 is not clique irreducible, using a similar argument, we can prove that there exists a clique C in NEPS$(G_1, G_2; \mathcal{B}_3)$ all of whose edges are present in some other clique of NEPS$(G_1, G_2; \mathcal{B}_3)$. □

Theorem 2.3. *The* NEPS$(G_1, G_2; \mathcal{B}_4)$ *is*

(1) *Clique vertex irreducible if and only if G_1 is trivial and G_2 is clique vertex irreducible or viceversa.*

(2) *Clique irreducible if and only if both G_1 and G_2 are clique irreducible.*

Proof. The cliques of NEPS$(G_1, G_2; \mathcal{B}_4)$ are of the form (H_1, v_2) or (v_1, H_2) where $v_i \in V(G_i)$ and H_i is a clique in G_i for $i = 1, 2$. Therefore, every vertex (v_1, v_2) in NEPS$(G_1, G_2; \mathcal{B}_4)$, where v_1 and v_2 are not isolated in G_1 and G_2 respectively, will be present in at least two cliques.

Also, the cliques of NEPS$(G_1, G_2; \mathcal{B}_4)$ has an edge of its own if and only if each H_i has an edge of its own. Hence, the theorem. □

Theorem 2.4. *The* $\text{NEPS}(G_1, G_2; \mathcal{B}_5)$ *is*
(1) *Clique vertex irreducible if and only if* G_1 *is trivial and* G_2 *is clique vertex irreducible.*
(2) *Clique irreducible if and only if either* G_1 *is trivial and* G_2 *is clique irreducible or* G_2 *is* K_3*-free.*

Proof. Let $u_1 \in V(G_1)$ be such that $u_2 \in N(u_1)$. Let $C = \langle v_1, v_2, \ldots, v_l \rangle$ be a clique in G_2. Every vertex of the clique $\langle (u_1, v_j) : j = 1, 2, \ldots, \ell \rangle$ in $\text{NEPS}(G_1, G_2; \mathcal{B}_5)$ will be present in at least one of the cliques $\langle (u_2, v_1), (u_1, v_j)$, for $j = 2, 3, \ldots, \ell \rangle$ and $\langle (u_1, v_j)$, for $j = 1, 2, \ldots \ell - 1, (u_2, v_l) \rangle$. Therefore, $\text{NEPS}(G_1, G_2; \mathcal{B}_5)$ is clique vertex reducible. If G_1 is trivial, then $\text{NEPS}(G_1, G_2; \mathcal{B}_5)$ is n_1 copies of G_2 which is clique vertex irreducible if and only if G_2 is clique vertex irreducible.

If $\ell \geq 3$, then again $\langle (u_1, v_j) : j = 1, 2, \ldots, \ell \rangle$ is a clique in $\text{NEPS}(G_1, G_2; \mathcal{B}_5)$ all of whose edges are present in at least one of the cliques $\langle (u_2, v_1), (u_1, v_j), \forall v_j \in C, j \neq 1 \rangle$, $\langle (u_2, v_2), (u_1, v_j), \forall v_j \in C, j \neq 2 \rangle$ and $\langle (u_2, v_3), (u_1, v_j), \forall v_j \in C, j \neq 3 \rangle$. Therefore, $\text{NEPS}(G_1, G_2; \mathcal{B}_5)$ is clique reducible. Conversely, if G_2 is K_3-free, then $\text{NEPS}(G_1, G_2; \mathcal{B}_5)$ is also K_3-free and hence is clique irreducible. Again, if G_1 is trivial, then $\text{NEPS}(G_1, G_2; \mathcal{B}_5)$ is n_1 copies of G_2 which is clique irreducible if and only if G_2 is clique irreducible. □

Theorem 2.5. *The* $\text{NEPS}(G_1, G_2; \mathcal{B}_7)$ *is clique vertex irreducible* (*clique irreducible*) *if and only if both* G_1 *and* G_2 *are clique vertex irreducible* (*clique irreducible*).

3. COMPLEMENT OF CLIQUE IRREDUCIBLE AND CLIQUE VERTEX IRREDUCIBLE GRAPHS

Theorem 3.1. *If* G^c *has at least three non-trivial components then* G *is clique reducible.*

Proof, Let G be a graph such that G^c has at least three non trivial components. Let $H_1, H_2, \ldots, H_p$ be the components of G^c. Let $G_i = H_i^c$ for $i = 1, 2, \ldots, p$. Then, $G = G_1 \vee G_2 \vee \cdots \vee G_p$. Also, any clique of G is the join of the cliques of G_i s for $i = 1, 2, \ldots, p$. At least three of the H_i s are non-trivial and hence at least three of the G_i s have more than one clique. Let C_{ij} for $j = 1, 2$ be any two of the cliques of G_i for $i = 1, 2, 3$. Let S_i be a clique of G_i for $i = 4, 5, \ldots, p$. Consider the clique $C_{11} \vee C_{21} \vee C_{31} \vee S_4 \vee \cdots \vee S_p$. Every edge of this clique is present in at least one of the cliques $C_{11} \vee C_{21} \vee C_{32} \vee S_4 \vee \cdots \vee S_p$, $C_{11} \vee C_{22} \vee C_{31} \vee S_4 \vee \cdots \vee S_p$, $C_{12} \vee C_{21} \vee C_{31} \vee S_4 \vee \cdots \vee S_p$. Therefore, G is clique reducible. □

Theorem 3.2. *If* G^c *has at least two non-trivial components then* G *is clique vertex reducible.*

Proof. Let G be a graph whose complement has at least two non trivial components. Let H_i, G_i, C_{ij} for $i = 1, 2, \ldots p$ and $j = 1, 2$ and S_i for $i = 3, 4, \ldots, p$ be defined as in the proof of Theorem 3.1 and consider the clique $C_{11} \vee C_{21} \vee S_3 \vee \cdots \vee S_p$. Every vertex of this clique is present in at least one of the cliques

$C_{11} \vee C_{22} \vee S_3 \vee \cdots \vee S_p$, $C_{12} \vee C_{21} \vee S_3 \vee \cdots \vee S_p$. Therefore, G is clique vertex reducible. □

Note. If G is clique irreducible then G^c is either connected or has exactly two non trivial components and if G is clique vertex irreducible then G^c is either connected or has exactly one non-trivial component.

4. COGRAPHS AND DISTANCE HEREDITARY GRAPHS

A graph G is complement reducible (cograph) if it can be reduced to edge less graphs by taking complements with in components [**5**]. Cographs can also be recursively defined as,

(1) K_1 is a cograph

(2) If G is a cograph, so is G^c and

(3) If G and H are cographs, so is $G \vee H$.

It is also known [**9**] that, G is a cograph if and only if G does not contain P_4-the path on four vertices, as an induced subgraph.

A graph G is distance hereditary, if $d_G(u,v) = d_H(u,v)$ for every connected induced subgraph H of G, where $u, v \in V(H)$ [**7**]. The distance hereditary graphs can also be obtained from K_1 by recursively,

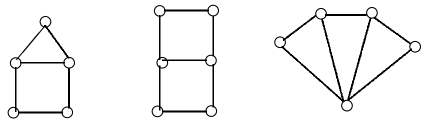
(1) Attaching pendant vertices

(2) Attaching true twins and

(3) Attaching false twins

where a true twin of a vertex u is a vertex v which is adjacent only to u and all its neighbors and a false twin of a vertex u' is a vertex v' which is adjacent only to the neighbors of u' [**3**].

The forbidden subgraphs of a distance hereditary graph are house, hole, domino and gem; where hole is an odd cycle of length greater than or equal to 5 and others are as given in the following figure [**3**].



House Domino Gem

As indicated from the forbidden subgraph characterizations, the cographs form a subclass of the distance hereditary graphs.

Lemma 4.1. *The clique veterx reducible graphs and the clique reducible graphs are closed for the operations of union and join.*

Proof. If G_1 and G_2 are clique vertex reducible (clique reducible) graphs, then their union is also clique vertex reducible (clique reducible), since the cliques of $G_1 \cup G_2$ are precisely the cliques of G_1 and the cliques of G_2.

If every vertex (edge) of the clique H_1 in G_1 is present in some other cliques of G_1, then every vertex (edge) of the clique $H_1 \vee H_2$ will be present in some other clique of G, where H_2 is a clique in G_2. Hence, the lemma. □

Theorem 4.1. *A cograph G is clique vertex irreducible if and only if it can be reduced to a trivial graph by recursively deleting vertices of full degree in each of the components.*

Proof. The proof is by induction on $|V| = n$. For $n = 1$ the theorem is trivially true. Assume that the theorem is true for any cograph with less than n vertices.

Let G be a graph with n vertices. A disconnected graph is clique vertex irreducible if and only if each of its components is clique vertex irreducible. Therefore, we may assume that, G is a connected cograph with n vertices. Then $G = G_1 \vee G_2$. If both G_is are not complete, then G^c will have at least two non trivial components which by Theorem 3.2 is a contradiction. Therefore, let G_1 be complete. Every vertex of G_1 is a universal vertex of G. Deleting these vertices we get a cograph G_2 with less than n vertices. Any clique C of G_2 corresponds to a clique $G_1 \vee C$ of G and hence has a vertex which does not lie in any other clique of G_2. Therefore, G_2 is a clique irreducible cograph with less than n vertices and hence by the induction hypothesis G_2 can be reduced to trivial graph by deleting universal vertices. Hence, the theorem. □

Theorem 4.2. *A connected non-trivial cograph G is clique irreducible if and only if $G = G_1 \vee G_2 \vee K_p$ where G_1 and G_2 are clique vertex irreducible cographs such that G_i^c is connected for $i = 1, 2$ and $p \geq 0$.*

Proof. Let $G = G_1 \vee G_2 \vee K_p$ where G_1 and G_2 are connected clique vertex irreducible cographs and $p \geq 0$. Any clique of G is of the form $H = H_1 \vee H_2 \vee K_p$, where H_1 and H_2 are cliques of G_1 and G_2 respectively. Since, G_1 and G_2 are clique vertex irreducible, there exist vertices $v_1 \in H_1$ and $v_2 \in H_2$ such that they do not lie in any other clique of G. Therefore, the edge v_1v_2 of H does not lie in any other clique of G and hence G is clique irreducible.

Conversely, assume that G is clique irreducible. Since G is a cograph G^c must be disconnected. Therefore by Theorem 3.1, G^c has exactly two non trivial components. So, $G = G_1 \vee G_2 \vee K_p$, where G_1^c and G_2^c are both connected. Let H_{11} and H_{12} be any two cliques of G_1 and H_{21} and H_{22} be any two cliques of G_2. $H = H_{11} \vee H_{21} \vee K_p$ is a clique of G. Every edge in H_{11}, every edge which joins H_{11} to a vertex of K_p and every edge in K_p will be present in the clique $H_{11} \vee H_{22} \vee K_p$. Again, every edge in H_{21}, every edge which joins H_{21} to a vertex of K_p and every edge in K_p will be present in the clique $H_{12} \vee H_{21} \vee K_p$. But, H has an edge which does not lie in any other clique of G. Therefore, that edge must be an edge which joins a vertex of H_{11} to a vertex of H_{21}. Let that edge be u_1u_2. But, then u_1 and u_2 cannot be present in any other clique of G_1 and G_2 respectively. Therefore, G_1 and G_2 are clique vertex irreducible. □

Lemma 4.2. *The clique vertex reducible* (*clique reducible*) *graphs are closed under the operations of attaching a pendant vertex, a true twin and a false twin.*

Proof. Let G be a clique vertex reducible (clique reducible) graph and C be a clique in G, all of whose vertices (edges) are present in some other clique in G.

The cliques of the graph obtained by attaching a pendant vertex u to a vertex v of G are the cliques of G together with the clique uv. Therefore C is a clique in this new graph and all of its vertices (edges) are present in some other clique.

The cliques of the graph obtained by attaching a true twin u to the vertex v of G are the cliques of G which does not contain the vertex v and the cliques of G which contains v together with the vertex u. If $v \notin C$, then C is a clique in the new graph and all its vertices (edges) are present in some other clique. If $v \in C$, then all the vertices (edges) in C other than u (the edges with one end vertex u) are already present in some other clique. Since v is (the edges with one end vertex v are) present in some other clique, u (the edges with one end vertex u) also must be present in some other clique.

The cliques of the graph obtained by attaching a false twin u to the vertex v of G are the cliques of G and the cliques of the form $(S \cup \{u\}) - \{v\}$, where S is a clique in G which contains the vertex v. Therefore, C is a clique in this new graph and all of its vertices (edges) are present in some other clique. □

Notation. Let $\mathcal{G}_1$ be the class of graphs recursively obtained as follows.
(1) $K_1 \in \mathcal{G}_1$.
(2) $\mathcal{G}_1$ is closed for attaching pendant vertices to a vertex v if either $N(v)$ is not complete or there exists $w \in N(v)$ such that $N(w) = N(v)$.
(3) $\mathcal{G}_1$ is closed for attaching true twins.
(4) $\mathcal{G}_1$ is closed for attaching false twins to a vertex v if $< \langle (v) \rangle$ is complete.

Theorem 4.3. A distance hereditary graph G is clique vertex irreducible if and only if $G \in \mathcal{G}_1$.

Proof. The graph K_1 is clique vertex irreducible. Let G be a clique vertex irreducible graph. Let G' be the graph obtained by attaching a pendant vertex u to a vertex $v \in V(G)$, such that $G' \in \mathcal{G}_1$. The cliques of G' are precisely, the cliques of G and the edge uv. The clique uv contains the vertex u which does not belong to any other clique of G'. Every clique of G' which does not contain v also has a vertex which does not lie in any other clique of G', since G is clique vertex irreducible. Let C be a clique of G which contains the vertex v. If $N(v)$ is not complete then C contains a vertex $v' \neq v$ which is not present in any other clique of G and hence of G'. If $N(v)$ is complete, then C contains a vertex which does not belong to any other clique of G' if and only if there exist a vertex $w \in V(C)$ which does not belong to any other clique of G. i.e; if and only if $N(w) = N(v)$.

Let G be a clique vertex irreducible graph. Let G' be the graph obtained by attaching a true twin u to a vertex v of G. The cliques of G' are precisely, the cliques of G which does not contain v and the cliques of G which contains v together with the vertex u. Each such clique contains a vertex which does not lie in any other clique of G', since G is clique vertex irreducible and hence G' is also clique vertex irreducible.

Let G' be the graph obtained by attaching a false twin u to a vertex v of G. The cliques of G' are the cliques of G together with the cliques of the form $(C \cup \{u\}) - \{v\}$ where C is a clique of G which contains v. The cliques of G' which does not contain v will continue to have a vertex which does not lie in any other clique. Let C be a clique of G which contains the vertex v. Every vertex of the clique C other than v will be present in the clique $(C \cup \{u\}) - \{v\}$ also. Therefore, C contains a vertex which does not lie in any other clique of G' if and only if v does not belong to any other clique of G, which happens if and only if $\langle N(v) \rangle$ is complete.

Therefore, if $G \in \mathcal{G}_1$ then G is clique vertex irreducible.

Since, any distance hereditary graph G can be obtained from K_1 by the operations of attaching a pendant vertex, introducing true twin and introducing false twin and by Lemma 4.2, G is clique vertex irreducible if and only if $G \in \mathcal{G}_1$.□

Notation. Let $\mathcal{G}_2$ be the class of graphs recursively obtained as follows.
(1) $K_2 \in \mathcal{G}_2$.
(2) $\mathcal{G}_2$ is closed for attaching pendant vertices.
(3) $\mathcal{G}_2$ is closed for attaching true twins.
(4) $\mathcal{G}_2$ is closed for attaching false twins to a vertex v if $\langle N(v) \rangle$ is clique vertex irreducible.

Theorem 4.4. *A distance hereditary graph G is clique irreducible if and only if $G \in \mathcal{G}_2$.*

Proof. K_2 is clique irreducible. Let G be a clique irreducible graph. Let G' be the graph obtained by attaching a pendant vertex u to a vertex v of G. The cliques of G' are precisely, the cliques of G and the edge uv. Every clique contains an edge which does not lie in any other clique of G' and hence G' is clique irreducible.

Let G be a clique irreducible graph. Let G' be the graph obtained by attaching a true twin u to a vertex v of G. The cliques of G' are precisely, the cliques of G which does not contain v and the cliques of G which contains v together with the vertex u. Every such clique contains an edge which does not lie in any other clique, since G is clique irreducible and hence G' is also clique irreducible.

Let G' be the graph obtained by attaching a false twin u to a vertex v of G. The cliques of G' are the cliques of G together with the cliques of the form $(C \cup \{u\}) - \{v\}$ where C is a clique of G which contains v. The cliques of G' which does not contain v will continue to have an edge which does not lie in any other clique. Let C be a clique of G which contains the vertex v. Every edge of C which does not contain v will be present in the clique $(C \cup \{u\}) - \{v\}$ also. Therefore, C contains an edge which does not lie in any other clique of G' if and only if there exists an edge vv' which does not lie in any other clique of G. Therefore, the vertex v' is not present in any clique of $\rangle(v)\langle$ other than $C - \{v\}$. So, $\langle \{v\} \rangle$ is clique vertex irreducible.

Therefore, if $G \in \mathcal{G}_2$ then G is clique irreducible.

Since, any distance hereditary graph G other than the trivial graphs can be obtained from K_2 by the operations of attaching a pendant vertex, introducing true twin and introducing false twin and by Lemma 4.2, G is clique irreducible if and only if $G \in \mathcal{G}_2$. □

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Department of Mathematics,
Cochin University of Science and Technology,
Cochin-682 022,
India
E–mails: aparnaren@gmail.com
vijay@cusat.ac.in

(Received May 25, 2008)
(Revised December 11, 2008)