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SOME SOLVABILITY THEOREMS FOR NONLINEAR EQUATIONS WITH APPLICATIONS TO PROJECTED DYNAMICAL SYSTEMS

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Dedicated to the Memory of Professor D. S. Mitrinović (1908–1995)

We present in this paper some solvability results with applications to the study of existence of periodic orbits for projected dynamical systems.

1. INTRODUCTION

The study of solvability of nonlinear equations defined in topological vector spaces is a fundamental problem considered in Nonlinear Analysis. This problem is so important because it is related to the study of solvability of differential or integral equations.

The literature related to this subject is huge. See [**1**]–[**4**], [**7**], [**10**], [**11**], [**17**], [**23**]–[**25**] among the other papers and books published until now. The solvability of nonlinear equations has been studied by many authors using several kinds of mathematical tools and related to this problem several chapters of Nonlinear Analysis have been created as for example, the fixed point theory, the theory of monotone operators, the theory of accretive operators, the theory of normal solvable operator, etc. Several solvability theorems have been obtained using as assumptions the monotonicity [**17**], [**19**], [**24**], [25] or some geometrical assumptions, [**1**], or topological degrees.

We present in this paper some simple solvability theorems applicable to the study of some problems considered in the theory of projected dynamical systems [**5**], [**6**], [**8**], [**16**], [**21**], [**22**]. The theory of projected dynamical systems is a new chapter in the theory of dynamical systems. In our solvability results some inequalities are essential. The inequalities are so important in mathematics [**20**].

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We note that, we will give an answer to an open subject related to projected dynamical systems [**6**]. This open subject motivated us to write this paper. To the end we will add some comments.

2. PRELIMINARIES

The notions and the notations given in this section are necessary for this paper. We denote by $(E, \|\cdot\|)$ a BANACH space and by $(E, \langle\cdot\,,\cdot\rangle)$ a HILBERT space. Let $(E, \|\cdot\|)$ be a BANACH space. The open ball of radius $r > 0$ centered at the origin defined by the norm $\|\cdot\|$ is the set

$$B_r = \{x \in E \big| \|x\| < r\}.$$

The closure of B_r is $\overline{B_r} = \{x \in E \big| \|x\| \leqslant r\}$ and the sphere of radius r centered at the origin is $S_r = \{x \in E \big| \|x\| = r\}$.

A semi-inner-product on E is a mapping satisfying the following properties:

(s_1) $[x + y, z]_\ell = [x, z]_\ell + [y, z]_\ell\,,$ *for all* $x, y, z \in E,$

(s_2) $[\lambda x, y]_\ell = \lambda [x, y]_\ell\,,$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y \in E,$

(s_3) $[x, x]_\ell > 0\,,$ *for all* $x \in E, x \neq 0,$

(s_4) $|[x, y]_\ell| \leqslant [x, x]_\ell \cdot [y, y]_\ell\,,$ *for all* $x, y \in E.$

It is known [**18**] that the mapping $x \to [x, x]^{1/2}$ for any $x \in E$ is a norm on E. We denote this norm by $\|\cdot\|_s$. If $\|x\|_s^2 = \|x\|^2$ for any $x \in E$, we say that the semi-inner-product is subordinated to the norm $\|\cdot\|$ given on E. The notion of *semi-inner-product* is due to G. LUMER [**18**].

It is known that on any BANACH space it is possible to define a semi-inner-product. For more information about semi-inner-products the reader is referred to [**9**] and [**18**]. We note that on an arbitrary BANACH space $(E, \|\cdot\|)$, it is possible to define another kind of semi-inner-product which has no all the properties of the semi-inner-product in LUMER's sense, but which is useful. This is the semi-inner-product in DEIMLING's sense [**7**], that is,

$$[x, y]_d = \|y\| \cdot \lim_{t \to 0_+} \frac{\|y + tx\| - \|y\|}{t}\,, \ \textit{for all } x, y \in E.$$

About this semi-inner-product we note the following two properties (consequences of the definition):

(d_1) $[\lambda x, y]_d = \lambda [x, y]_d$ *, for any* $x, y \in E$ *and* $\lambda \in \mathbb{R}, \lambda > 0,$

(d_2) $[x, x]_d > 0,$ *for any* $x \in E,$ $x \neq 0.$

Let $(E, \|\cdot\|)$ be a BANACH space and $X \subset E$ a non-empty subset. We say that a mapping $f : X \to E$ is *compact* if it is continuous and $f(X)$ is relatively compact in E, (i.e.,$\overline{f(X)}$ is compact, where $\overline{f(X)}$ is the topological closure of $f(x)$).

We say that a mapping $f : E \to E$ is completely continuous if f is continuous and for any bounded subset $D \subset E$, we have that $\overline{f(D)}$ is compact.

Let $(E, \langle \cdot\,, \cdot \rangle)$ be a HILBERT space and $\Omega \subset E$ a non-empty closed convex set. Recall that for each $x \in \Omega$ the set $T_\Omega = \overline{\bigcup_{\lambda>0} \frac{1}{\lambda}(\Omega - x)}$ is the *tangent cone* to Ω at the point x.

Also we recall that for any $x \in E$, there exists a unique element in Ω, denoted by $P_\Omega(x)$ such that $\|P_\Omega(x) - y\| = \inf_{y\in\Omega} \|x - y\|$. This defines a mapping $P_\Omega : E \to \Omega$ given by $x \to P_\Omega(x)$, called *the projection operator* of the space E onto the subset Ω. The properties of the projection operator on HILBERT spaces are well known. It is known [**5**], [**6**], [**21**] that for any $x \in \Omega$ and any element $v \in E$ the limit

$$\Pi_\Omega(x; v) := \lim_{\delta\to 0_+} \frac{P_\Omega(x + \delta v) - x}{\delta}$$

exists and $\Pi_\Omega(x; v) := P_{T_\Omega(x)}(v)$. The mapping $\Pi_\Omega(x; v) := \Omega \times E \to E$, generally is discontinuous of the boundary of Ω [**21**], [**6**].

3. SOME SOLVABILITY THEOREMS

First, we need to recall the following classical result:

Theorem 1 (SCHAUDER). *Let D be a convex (not necessarily closed) subset of a normed vector space $(E, \|\cdot\|)$. Then each compact mapping $f : D \to D$ has at least one fixed point.*

The original proof of $Theorem 1$ is in [**21**], but the reader is referred also to [**10**].

Let $(E, \|\cdot\|)$ be an arbitrary BANACH space, $r > 0$ and $f : E \to E$ a completely continuous mapping. We consider in $\overline{B_r}$ the equation

$$f(x) = 0. \tag{1}$$

Definition 1. [**13**] *We say that equation* (1) *is almost solvable in $\overline{B_r}$ if $0 \in \overline{f(\overline{B_r})}$*

Obviously, in the n-dimensional Euclidean space $(\mathbb{R}^n, \langle\cdot, \cdot\rangle)$ the almost solvability implies the solvability, because the set $\overline{B_r}$ is a compact set. Our goal in this section is to study the almost solvability and also the solvability of equation (1).

Suppose given a mapping $G : E \times E \to \mathbb{R}$ satisfying for some $r > 0$ the following conditions:

(g_1) $G(x, x) \geq 0$ *, for all $x \in S_r$,*

(g_2) $G(\lambda x, x) \geq \lambda G(x, x)$ *, for all $x \in S_r$ and $\lambda > 0, \lambda \in \mathbb{R}$.*

EXAMPLES

1. If $(E, \langle \cdot\,, \cdot \rangle)$ is a HILBERT space, then in this case we can take $G(\cdot\,, \cdot) = \langle \cdot\,, \cdot \rangle$ where $\langle \cdot\,, \cdot \rangle$ is the inner-product defined on E.

2. If $(E, \|\cdot\|)$ is an arbitrary BANACH space, we can take $G(\cdot\,, \cdot) = [\cdot\,, \cdot]_\ell$, where $[\cdot\,, \cdot]_\ell$ is a semi-inner-product in LUMER's sense defined on E, or $G(\cdot\,, \cdot) = [\cdot\,, \cdot]_d$, where $[\cdot\,, \cdot]_d$ is a semi-inner-product in DEIMLING's sense.

3. Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space and $E = C([0,1], H)$ the vector space of continuous functions from [0, 1] into H. Consider on $E = C([0,1], H)$ the norm $\|x\| = \sup_{t\in[0,1]} \|x(t)\|_H$, where $\|\cdot\|_H$ is the norm of the space H. Then in this case we may take:

$$G(x,y) = \sup_{t\in[0,1]} \langle x(t), y(t)\rangle\,, \quad \text{or} \quad G(x,y) = \int_0^1 \langle x(t), y(t)\rangle \,\mathrm{d}t.$$

The following result is an almost solvability test.

Theorem 2. *Let $(E, \|\cdot\|)$ be a Banach space and $f : E \to E$ a completely continuous mapping. Suppose given a mapping $G : E \times E \to \mathbb{R}$. If there exists $r > 0$ such that the following assumptions are satisfied*:

(1) *G satisfies properties (g_1) and (g_1) with respect to S_r,*

(2) *$G(f(x), x) < G(x, x)$ for any $x \in S_r$,*

(3) *$\|f(x)\| \geq r$, for any $x \in S_r$,*

then the equation $f(x) = 0$ is almost solvable in $\overline{B_r}$.

Proof. If there is an element $x_0 \in \overline{B_r}$ such that $f(x) = 0$, then the equation $f(x) = 0$ is solvable and consequently it is almost solvable. Suppose that $0 \notin f(\overline{B_r})$. In this case we will show that $0 \in \overline{f(\overline{B_r})}$. Indeed, we suppose that $0 \notin \overline{f(\overline{B_r})}$. In this case we consider the mapping

$$F(x) = \frac{r}{\|f(x)\|} f(x), \quad x \in \overline{B_r}.$$

The mapping $F : \overline{B_r} \to E$ is well defined and continuous. Obviously, we have the inclusions $F(\overline{B_r}) \subset S_r \subset \overline{B_r}$. Now we show that $F(\overline{B_r})$ is a relatively compact set. Indeed, let $\{u_n\}_{n\in N}$ be a sequence in $F(\overline{B_r})$. Then, for any $n \in \mathbb{N}$ we have that $u_n = F(x_n)$, with $x_n \in \overline{B_r}$, that is we have

$$u_n = \frac{r}{\|f(x_n)\|} f(x_n), \text{ for all n } \in \mathbb{N}.$$

Since $f(\overline{B_r})$ is relatively compact, we have that $f(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $\{f(x_{n_k})\}_{k\in\mathbb{N}}$. We denote $z = \lim_{k\to\infty} f(x_{n_k})$ and we have that $z \in \overline{f(\overline{B_r})}$. To show that $\{u_{n_k}\}_{k\in\mathbb{N}}$ is convergent, we must show that $z \neq 0$, which is true, since we suppose that $0 \notin \overline{f(\overline{B_r})}$.

Therefore, applying the Schauder Fixed Point Theorem, we obtain the existence of a point $x_* \in \overline{B_r}$ such that $F(x_*) = x_*$. Moreover, we have $x_* \in S_r$. Considering our assumptions we have

$$G(x_*, x_*) > G(f(x_*), x_*) = G\left(\frac{\|f(x_*)\|}{r} x_*, x_*\right) \geq \frac{f(x_*)}{r} G(x_*, x_*) \geq G(x_*, x_*)$$

which is a contradition.

Hence, we must have $0 \in \overline{f(\overline{B_r})}$, and the proof is complete. □

Corollary 1. *Let $(E, \|\cdot\|)$ be a Banach space and $f : E \to E$ a completely continuous mapping. Suppose given a mapping $G : E \times E \to \mathbb{R}$. If there exist $r > 0$ and $c \geq 0$ such that:*

(1) *G satisfies properties (g_1) and (g_2) with respect to S_r,*

(2) *$G(x, x) \geq c$, for any $x \in S_r$,*

(3) *$G(f(x), x) < c$, for any $x \in S_r$,*

(4) *$\|f(x)\| \geq r$, for any $x \in S_r$,*

then the equation $f(x) = 0$ is almost solvable in $\overline{B_r}$.

Corollary 2. *Let $(E, \|\cdot\|)$ be a Banach space and $f : E \to E$ a completely continuous mapping. If there exist $r > 0$ and a mapping $G : E \times E \to \mathbb{R}$ such that:*

(1) *G satisfies properties (g_1) and (g_2) with respect to S_r,*

(2) *$G(f(x), x) < 0$, for any $x \in S_r$,*

then the equation $f(x) = 0$ is almost solvable in $\overline{B_r}$.

Proof. We remark that in this case assumption (3) of Theorem 2 is not necessary. □

Corollary 3. *Let $(E, \|\cdot\|)$ be a Banach space and $h : E \to E$ a completely continuous mapping. Suppose given a mapping $G : E \times E \to \mathbb{R}$. If there exists $r > 0$ such that:*

(1) *G satisfies property (g_1) for all $x \in S_r$,*

(2) *$G(\lambda x, x) = \lambda G(x, x)$, for all $x \in S_r$, and all $\lambda \in \mathbb{R}$,*

(3) *$G(h(x), x) > 0$, for all $x \in S_r$,*

then the equation $h(x) = 0$ is almost solvable in $\overline{B_r}$.

Proof. We apply Corollary 2 taking $f(x) = -h(x)$ for any $x \in E$ and we observe that $0 \notin \overline{(-h)(\overline{B_r})}$ implies $0 \notin \overline{h(\overline{B_r})}$. □

The next result is a solvability test.

Let $(E, \|\cdot\|)$ be a BANACH space, E^* the topological dual of E and let $\langle E, E^* \rangle$ be a duality (pairing) between E and E^*. We recall some notions, well known in Nonlinear Analysis, [**3**] and [**4**].

We say that $f : E \to E^*$ satisfies condition $(S)_0$ if any sequence $\{x_n\}_{n \in \mathbb{N}} \subset E$, weakly convergent to an element $x^* \in E$ and such that $\{f(x_n)\}_{n \in N}$ is weakly-($*$)-convergent to an element $u \in E^*$ with $\lim_{n \to \infty} \langle x_n, f(x_n) \rangle = \langle x_*, u \rangle$, has a subsequence $\{x_{n_k}\}$ convergent in norm to x_*.

We introduced in [**15**] the following condition.

We say that $f : E \to E^*$ satisfies condition $(S)^1_+$ if any sequence $\{x_n\}_{n \in N} \subset E$, weakly convergent to an element $x_* \in E$ and such that $\{f(x_n)\}_{n \in \mathbb{N}}$ is weakly-($*$)-convergent to an element $u \in E^*$ and $\lim_{n \to \infty} \langle x_n, f(x_n) \rangle \leq \langle x_*, u \rangle$ has a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ convergent in norm to x_*.

Obviously, if f satisfies condition $(S)_+^1$ then f satisfies condition $(S)_0$. We note that condition $(S)_+$ defined in [**3**] and [**4**] implies condition $(S)_+^1$ and hence condition $(S)_0$. Examples of mappings satisfying conditions $(S)_0$, $(S)_+$ and $(S)_+^1$ are given in [**2**]–[**4**], [**15**].

Now, we consider a Hilbert space $(H, \langle \cdot\,, \cdot\rangle)$. In this case we take $G(\cdot\,,\cdot) = \langle \cdot\,, \cdot \rangle$.

Theorem 3. *Let $(H, \langle \cdot\,, \cdot\rangle)$ be a separable Hilbert space and $f : H \to H$ a continuous mapping. Let $b \in H$ be an arbitrary element. If there exists $r > 0$ such that the following assumptions are satisfied*:

(1) *f is bounded*,

(2) *f satisfies condition $(S)_0$*,

(3) *$\langle F(x), x\rangle > \langle b, x\rangle$ for any $x \in H$ with $\|x\| = r$*,

then there exists x_* such that $f(x_*) = b$ and $\|x_*\| \le r$.

Proof. Because H is separable, there exists a sequence $\{v_1, v_2, \ldots, v_m, \ldots\}$ of elements in H such that the linear subspace generated by this sequence is dense in H. We can suppose that $\{v_1, v_2, \ldots, v_m, \ldots\}$ is linearly independent. Let H_m denote the linear subspace generated by $\{v_1, v_2, \ldots, v_m\}$, for each $m \geqslant 1$. We denote by $j_m : \mathbb{R}^m \to H_m$ the linear mapping defined by the representation with respect to this basis.

Denote by $\langle \cdot\,, \cdot\rangle$ the inner-product on $\mathbb{R}^m$ defined by

$$\langle x, y\rangle = \langle j_m(x), j_m(y)\rangle$$

We denote by H^* the topological dual of H (which is obviously isomorph to H) and consider f as a mapping from H into H^*. Because $H_m \subset H$, consider $j_m : \mathbb{R}^m \to H_m \subset H$ and denote by j_m^* the adjoint mapping of j_m. We have $j_m^* : H^* \to \mathbb{R}^m$. Fix $m \geqslant 1$ and consider the problem:

$$\text{(2)} \qquad \begin{cases} \text{find } u_m \in H_m \text{ such that} \\ \langle f(u_m), v_k\rangle = \langle b, v_k\rangle \\ 1 \leqslant k \leqslant m. \end{cases}$$

We have that an element $u_m \in H_m$ is a solution to problem (2) if $u_m = j_m(\widehat{u}_m)$, where $\widehat{u}_m \in \mathbb{R}^m$, is such that

$$\text{(3)} \qquad F(\widehat{u}_m) \equiv (j_m^* \circ f \circ j_m)(\widehat{u}_m) - j_m^*(b) = 0,$$

Indeed, if $v_k = j_m(\widehat{v}_k)$, then we have

$$\langle (j_m^* \circ f \circ j_m)(\widehat{u}_m), \widehat{v}_k\rangle = \langle j_m^*(b), \widehat{v}_k\rangle$$

which implies

$$\langle f(u_m), v_k\rangle = \langle b, v_k\rangle\,; k = 1, 2, \cdots, m,$$

that is u_m solves (2).

We must show that for any $m \geqslant 1$ the mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ has a zero $\widehat{u}_m$. For $m \geqslant 1$ fixed, the mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and hence completely continuous and $\langle F(u), u\rangle > 0$ for any $u \in \mathbb{R}^m$ with $\|u\| = r$, because $\|j_m(u)\| = \|u\| = r$ and

$$\langle F(u), u\rangle = \langle j_m^* \circ f \circ j_m(u) - j_m^*(b), u\rangle = \langle f(j_m(u)), j_m(u)\rangle - \langle b, j_m(u)\rangle > 0.$$

Applying Corollary 3 of Theorem 2 we have that the equation $F(u) = 0$ is solvable, that is there exists $\widehat{u}_m \in \mathbb{R}^m$ such that $F(\widehat{u}_m) = 0$ and $\|\widehat{u}_m\| = \|j_m(\widehat{u}_m)\| \leqslant r$. Then we have $\langle F(\widehat{u}_m), \widehat{u}_m\rangle = 0$, which implies $\langle j_m^* \circ f \circ j_m(u) - j_m^*(b), \widehat{u}_m\rangle = 0$, that is $\langle f(u_m), u_m\rangle = \langle b, u_m\rangle$, where $u_m = j_m(\widehat{u}_m)$ and $\|u_m\| \leqslant r$. Hence for any $m \geqslant 1$ there exists $u_m \in H$ with $\|u_m\| \leqslant r$ and

$$\langle f(u_m), u_m\rangle = \langle b, u_m\rangle\,, \tag{4}$$

The sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded, then there exists a subsequence of $\{u_m\}_{m\in\mathbb{N}}$, denoted again by $\{u_m\}_{m\in\mathbb{N}}$ which is weakly convergent to an element x_*. Because the mapping f is bounded we have that the sequence $\{f(u_m)\}_{m\in\mathbb{N}}$ has a subsequence, denoted again by $\{f(u_m)\}_{m\in\mathbb{N}}$, which is weakly convergent to an element $z \in H$.

Considering relation (2) we have that for any $m \geqslant k$ $\langle f(u_m), v_k\rangle = \langle b, v_k\rangle$. Computing the limit when $m \to +\infty$ we obtain $\langle z, v_k\rangle = \langle b, v_k\rangle$, for any k, and because H is separable we deduce that $z = b$ and consequently $\{f(u_m)\}_{m\in\mathbb{N}}$ is weakly convergent to b. From (4) we have that

$$\lim_{m\to\infty} \langle f(u_m), u_m\rangle = \lim_{m\to\infty} \langle b, u_m\rangle = \langle b, x_*\rangle$$

Applying condition $(S)_0$ we have that $\{u_m\}_{m\in\mathbb{N}}$ has a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ convergent in norm to x_*, and because f is continuous we have that $\lim_{i\to\infty} f(u_{n_i}) = b$, that is $f(x_*) = b$, and the proof is complete. □

4. APPLICATION TO PROJECTED DYNAMICAL SYSTEMS

In this section we present some applications to the study of existence of periodic orbits for projected dynamical systems. Our applications are related to the results presented in [**6**].

Let $(H, \langle\cdot\,,\cdot\rangle)$ be a separate HILBERT space and $\Omega \subset H$ a non-empty closed convex subset. Let $AC([0,+\infty],\Omega)$ denote the class of absolutely continuous functions from $[0,+\infty]$ into Ω. Let $F : \Omega \to H$ be a mapping and $x_0 \in H$. It is known [**6**] that the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \Pi_\Omega(x(t) - F(x)), \\ x(0) = x_0 \end{cases} \tag{5}$$

has a unique solution in $AC([0,+\infty],\Omega)$ if F is a Lipschitzian continuous mapping.

We recall that a *projected dynamical system* (PDS) is given by a mapping $\phi:\mathbb{R}_+\times\Omega\to\Omega$ which solves the initial value problem:

$$\begin{cases}\phi'(t,x)=\Pi_\Omega(\phi(t,x)-F(\phi(t,x))),\\ \phi(0,x)=x\in\Omega.\end{cases}\tag{6}$$

If we denote by $x(t):=\phi(t,x)$ we define a mapping from Ω into $AC([0,+\infty],\Omega)$ by $x\to x(t)=\phi(t,x)$. If $T\in[0,+\infty]$ is given, we define the mapping $Q^T:\Omega\to H$ by

$$x\to Q^T(x)=x(T)=x+\int_0^T\Pi_\Omega(x(s)-F(x(s)))\,\mathrm{d}s.$$

The problem of finding a periodic orbit of the *PDS* is equivalent to the problem of finding a fixed point of Q^T, or of a zero of the mapping $x\to\int_0^T\Pi_\Omega(x(s)-F(x(s)))\,\mathrm{d}s$. We recall the following result proved by R. I. Kachurovskii in 1968 [**17**].

Theorem (Kachurovskii**).** *Let $(E,\|\cdot\|)$ be a reflexive Banach space and $D\subset E$ closed convex set, such that $0\in int(D)$. If there exists $r>0$ such that $D\subset B_r$ and $f:B_r\to E^*$ is a monotone operator, such that the following assumptions are satisfied:*

(1) *f is hemicontinuous (i.e. for any $u,v\in B_r$ the real function $t\to\langle f(u+tv),v\rangle$ is continuous),*

(2) *$\langle x,f(x)\rangle\geqslant 0$ for any $x\in\partial D$*

then there exists at least one element $x_0\in D$ such that $f(x_0)=0$.

Using Kachurovskii's theorem we obtain the following more flexible variant of Theorem 3.4 proved in [**6**].

Theorem 4. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\Omega\subset H$ a non-empty closed convex set and $f:\Omega\to H$ a monotone Lipschitz continuous mapping. Let $r>0$ be such that $B_r\subset\Omega$. Let D be an arbitrary closed convex set such that $0\in int(D)$ and $D\subset B_r$. If there exists, $T>0$ such that $\big\langle(Id_\Omega-Q^T)(x),x\big\rangle\geqslant 0$ for any $x\in\partial D$, then there exist $x_*\in D$ such that $x_*(T)=x_*$, that is, the PDS has a periodic orbit.*

Proof. Because we know [**6**] that under the assumptions of Theorem 4 the mapping $Id_\Omega-Q^T$ is continuous and monotone, we apply Kachurovskii's Theorem and the conclusion of theorem follows. □

In the n-dimensional Euclidean space $(R^n,\langle\cdot,\cdot\rangle)$ it is not necessary to have that the operator $Id_\Omega-Q^T$ is monotone. Indeed, we have the following result.

Theorem 5. *Let $(R^n,\langle\cdot,\cdot\rangle)$ be the n-dimensional Euclidean space $\Omega\subset R^n$ a non-empty closed convex set with $0\in int(\Omega)$ and $F:\Omega\to R^n$ a Lipschitz continuous* *mapping. If there exist* $r > 0$ *and* $T > 0$ *such that* $\overline{B_r} \subset \Omega$ *and* $\left\langle (Id_\Omega - Q^T)(x), x \right\rangle > 0$ *for any* $x \in R^n$ *with* $\|x\| = r$*, then there exists* $x_* \in \overline{B_r}$ *such that* $x_*(T) = x_*$*, i.e., the PDS has a periodic orbit.*

Proof. We know [**6**] that the mapping $Id_\Omega - Q^T$ is continuous and hence it is completely continuous (because $(R^n, \langle \cdot\,, \cdot \rangle)$ is finite dimensional). We apply Corollary 3 of Theorem 2, taking $G(\cdot\,, \cdot) = \langle \cdot\,, \cdot \rangle$ and the conclusion of theorem follows. □

Remark. In [**6**] is defined the following open question: *Do periodic cycles for PDS exist in the absence of monotonicity conditions*? From Theorem 5 we obtain the answer *probably yes* in the finite dimensional case.

Now we consider the case of infinite dimensional Hilbert spaces. In this case we consider the mapping $\Psi : \Omega \to H$ defined by

$$\Psi(x) = \int_0^T \Pi_\Omega(x(s);\ -F(x(s)))\,\mathrm{d}s.$$

Definition 2. *We say that the PDS defined by* F *on* Ω *with* $0 \in int(\Omega)$ *has almost periodic orbits if there exist* $r > 0$ *and* $T > 0$ *such that* $\overline{B_r} \subset \Omega$ *and for any* $\varepsilon > 0$ *there exists* $x_\varepsilon \in \overline{B_r}$ *with* $\|x_\varepsilon(T) - x_\varepsilon\| < \varepsilon$*.*

Theorem 6. *Let* $(H, \langle \cdot\,, \cdot \rangle)$ *be a Hilbert space and* $\Omega \subset H$ *a non-empty closed convex set with* $0 \in int(\Omega)$*. Let* $F : \Omega \to H$ *be a Lipschitz continuous mapping. If the following assumption are satisfied*:

(i) *there exist* $r > 0$ *and* $T > 0$ *such that*

$$\overline{B_r} \subseteq \Omega \ \text{ and } \ \langle \Psi(x), x \rangle > 0 \ \text{ for any } x \in S_r,$$

(ii) *the mapping* Ψ *is completely continuous*,

then the PDS has almost periodic orbits.

Proof. We apply Corollary 3 of Theorem 2, taking $G(\cdot\,, \cdot) = \langle \cdot\,, \cdot \rangle$ and we obtain that the equation $\Psi(x) = 0$ is almost solvable in $\overline{B_r}$, which implies the conclusion of theorem. □

In the case when the Hilbert space $(H, \langle \cdot\,, \cdot \rangle)$ is separable, we have the following results.

Theorem 7. *Let* $(H, \langle \cdot\,, \cdot \rangle)$ *be a Hilbert space and* $\Omega \subset H$ *a non-empty closed convex set with* $0 \in int(\Omega)$*. Let* $F : \Omega \to H$ *be a Lipschitz continuous mapping. Suppose that there exists* $r > 0$ *such that* $\overline{B_r} \subseteq \Omega$*. If there exists* $T > 0$ *such that the following assumptions are satisfied*:

(i) *the mapping* Ψ *is bounded*,

(ii) *the mapping* Ψ *satisfies condition* $(S)_0$,

(iii) $\langle \Psi(x), x \rangle > 0$ *for any* $x \in S_r$,

then the PDS has almost periodic orbits.

Proof. Because Ψ is continuous and because our assumptions, we can apply Theorem 3 with $b = 0$ and we obtain that the equation $\Psi(x) = 0$ has a solution x_* in $\overline{B_r}$ and the conclusion of theorem follows. □

5. COMMENTS

We presented in this paper some general solvability theorems in sc Banach spaces and we apply some of our results to the study of the existence of periodic orbits for projected dynamical systems.

We did not studied under what conditions the mappings $Id_\Omega - Q^T$ and Ψ satisfy condition $(S)_0$. We will study this problem in a future paper. We know that the strong monotonicity implies condition $(S)^1_+$ and hence condition $(S)_0$. *Therefore an interesting open subject is to study under what conditions the operators $Id_\Omega - Q^T$ and Ψ satisfy conditions $(S)_+, (S)^1_+$ or $(S)_0$.*

Finally we note that the assumption $0 \in int(\Omega)$ used in the results presented above probably can be removed using some solvability theorems bases on some geometrical conditions like the results presented in [**1**].

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