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SUPERSTABILITY OF ADJOINTABLE MAPPINGS ON HILBERT C^*-MODULES

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Dedicated to the Memory of Professor D. S. Mitrinović (1908–1995)

We define the notion of φ-perturbation of a densely defined adjointable mapping and prove that any such mapping f between Hilbert $\mathcal{A}$-modules over a fixed C^*-algebra $\mathcal{A}$ with densely defined corresponding mapping g is $\mathcal{A}$-linear and adjointable in the classical sense with adjoint g. If both f and g are everywhere defined then they are bounded. Our work concerns with the concept of Hyers–Ulam–Rassias stability originated from the Th. M. Rassias' stability theorem that appeared in his paper [*On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300]. We also indicate complementary results in the case where the Hilbert C^*-modules admit non-adjointable C^*-linear mappings.

1. INTRODUCTION

We say a functional equation $(\mathcal{E})$ is *stable* if any function g approximately satisfying the equation $(\mathcal{E})$ is near to an exact solution of $(\mathcal{E})$. The equation $(\mathcal{E})$ is called *superstable* if every approximate solution of $(\mathcal{E})$ is indeed a solution (see **5**] for another notion of superstability namely *superstability modulo the bounded functions*). More than a half century ago, S. M. Ulam [**23**] proposed the first stability problem which was partially solved by D. H. Hyers [**10**] in the framework of Banach spaces. Later, T. Aoki [**3**] proved the stability of the additive mapping and Th. M. Rassias [**20**] proved the stability of the linear mapping for mappings f from a normed space into a Banach space such that the norm of the Cauchy difference $f(x+y)-f(x)-f(y)$ is bounded by the expression $\varepsilon(\|x\|^p+\|y\|^p)$ for some $\varepsilon \geq 0$, for some $0 \leq p < 1$ and for all x, y. The terminology "Hyers–Ulam–Rassias stability" was indeed originated from Th. M. Rassias's paper [**20**]. In 1994, a further generalization was obtained by P. Găvruţa [**9**], in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. This

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terminology can be applied to functional equations and mappings on various generalized notions of HILBERT spaces; see [**1**, **2**, **4**]. We refer the interested reader to monographs [**6**, **7**, **11**, **13**, **19**, **22**] and references therein for more information.

The notion of HILBERT C^*-module is a generalization of the notion of HILBERT space. This object was first used by I. KAPLANSKY [**14**]. Interacting with the theory of operator algebras and including ideas from non-commutative geometry it progresses and produces results and new problems attracting attention, see [**8**, **15**, **18**].

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{X}$ be a complex linear space, which is a right $\mathcal{A}$-module with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in \mathcal{X}, a \in \mathcal{A}, \lambda \in \mathbb{C}$. The space $\mathcal{X}$ is called a (right) pre-HILBERT $\mathcal{A}$-module if there exists an $\mathcal{A}$-inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \quad$ if and only if $\quad x = 0$;

(ii) $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$;

(iii) $\langle x, ya \rangle = \langle x, y \rangle a$;

(iv) $\langle x, y \rangle^* = \langle y, x \rangle$;

for all $x, y, z \in \mathcal{X}$, $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$. The pre-HILBERT module $\mathcal{X}$ is called a (right) HILBERT $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. Left HILBERT $\mathcal{A}$-modules can be defined in a similar way. Two typical examples are

(I) Every inner product space is a left pre-Hilbert $\mathbb{C}$-module.

(II) Let $\mathcal{A}$ be a C^*-algebra. Then every norm-closed right ideal I of $\mathcal{A}$ is a HILBERT $\mathcal{A}$-module if one defines $\langle a, b \rangle = a^* b \quad (a, b \in I)$.

A mapping $f : \mathcal{X} \to \mathcal{Y}$ between HILBERT $\mathcal{A}$-modules is called adjointable if there exists a mapping $g : \mathcal{Y} \to \mathcal{X}$ such that $\langle f(x), y \rangle = \langle x, g(y) \rangle$ for all $x \in \mathcal{D}(f) \subseteq \mathcal{X}, y \in \mathcal{D} \subseteq \mathcal{Y}$. Throughout the paper, we assume that f and g are both everywhere defined or both densely defined. The unique mapping g is denoted by f^* and is called the adjoint of f.

An $\mathcal{A}$-linear bounded operator K on a Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called "compact" if it belongs to the norm-closed linear span of the set of all elementary operators $\theta_{x,y}$ $(x, y \in \mathcal{X})$ defined by $\theta_{x,y}(z) = x\langle y, z \rangle$ $(z \in \mathcal{X})$.

In this paper, we prove the superstability of adjointable mappings on Hilbert C^*-modules in the spirit of HYERS–ULAM–RASSIAS and indicate interesting complementary results in the case where the HILBERT C^*-modules admit non-adjointable C^*-linear mappings.

2. MAIN RESULTS

Throughout this section, $\mathcal{A}$ denotes a C^*-algebra, $\mathcal{X}$ and $\mathcal{Y}$ denote HILBERT $\mathcal{A}$-modules, and $\varphi : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ is a function. We start our work with the following definition.

Definition 2.1. *A (not necessarily linear) mapping $f : \mathcal{X} \to \mathcal{Y}$ is called a φ-perturbation of an adjointable mapping if there exists a (not necessarily linear) corresponding mapping $g : \mathcal{Y} \to \mathcal{X}$ such that*

$$\|\langle f(x), y\rangle - \langle x, g(y)\rangle\| \le \varphi(x,y) \qquad \big(x \in \mathcal{D}(f) \subseteq \mathcal{X}, y \in \mathcal{D}(g) \subseteq \mathcal{Y}\big)\,. \tag{2.1}$$

To prove our main result, we need the following known lemma (cf. [**15**,p. 8]) that we prove it for the sake of completeness.

Lemma 2.2. *Every densely defined adjointable mapping between Hilbert C^*-modules over a fixed C^*-algebra $\mathcal{A}$ is $\mathcal{A}$-linear. If the adjointable mapping is everywhere defined then it is bounded.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{X}$ be a pair of densely defined adjointable mappings between two Hilbert C^*-modules $\mathcal{X}$ and $\mathcal{Y}$. For every $x_1, x_2, x_3 \in \mathcal{D}(f) \subseteq \mathcal{X}$, every $y \in \mathcal{D}(g) \subseteq \mathcal{Y}$, every $\lambda \in \mathbb{C}$, every $a \in \mathcal{A}$ the following equality holds:

$$\begin{aligned}\langle f(\lambda x_1 + x_2 + x_3 a), y\rangle &= \langle \lambda x_1 + x_2 + x_3 a, g(y)\rangle \\ &= \lambda\langle x_1, g(y)\rangle + \langle x_2, g(y)\rangle + a^*\langle x_3, g(y)\rangle \\ &= \lambda\langle f(x_1), y\rangle + \langle f(x_2), y\rangle + a^*\langle f(x_3), y\rangle \\ &= \langle \lambda f(x_1) + f(x_2) + f(x_3)a, y\rangle\,.\end{aligned}$$

By the density of the domain of g in $\mathcal{Y}$ the equality yields the $\mathcal{A}$-linearity of f.

Now, suppose f and g to be everywhere defined on $\mathcal{X}$ and $\mathcal{Y}$, respectively. For each x in the unit sphere of $\mathcal{X}$, define $\tau_x : \mathcal{Y} \to \mathcal{A}$ by $\tau_x(y) = \langle f(x), y\rangle = \langle x, g(y)\rangle$. Then $\|\tau_x(y)\| \le \|x\|\|g(y)\| \le \|g(y)\|$ for any x from the unit ball. By the Banach–Steinhaus theorem we conclude that the set $\{\|\tau_x\| : x \in \mathcal{X}, \|x\| \le 1\}$ is bounded. Due to the equality $\|f(x)\| = \sup_{\|y\|\le 1} \|\langle f(x), y\rangle\| = \sup_{\|y\|=1} \|\tau_x(y)\| = \|\tau_x\|$ the mapping f has to be bounded. □

Theorem 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a φ-perturbation of an adjointable mapping with corresponding mapping $g : \mathcal{Y} \to \mathcal{X}$. Suppose that the mappings f and g are everywhere defined on the respective Hilbert C^*-modules. Furthermore, suppose that for some sequence $\{c_n\}$ of non-zero complex numbers either both the conditions* (2.2) *and* (2.3) *or both the conditions* (2.4) *and* (2.5) *below hold for the perturbation bound mapping $\varphi(x, y)$:*

$$\lim_{n\to+\infty} |c_n|^{-1}\varphi(c_n x, y) = 0 \qquad (x \in \mathcal{X}, y \in \mathcal{Y}), \tag{2.2}$$

$$\lim_{n\to+\infty} |c_n|^{-1}\varphi(x, c_n y) = 0 \qquad (x \in \mathcal{X}, y \in \mathcal{Y}), \tag{2.3}$$

$$\lim_{n\to+\infty} |c_n|\varphi(c_n^{-1}x, y) = 0 \qquad (x \in \mathcal{X}, y \in \mathcal{Y}), \tag{2.4}$$

$$\lim_{n\to+\infty} |c_n|\varphi(x, c_n^{-1}y) = 0 \qquad (x \in \mathcal{X}, y \in \mathcal{Y})\,. \tag{2.5}$$

Then f is adjointable. In particular, f is bounded, continuous and $\mathcal{A}$-linear, as well as its adjoint is g.

Proof. Let $\lambda \in \mathbb{C}$ be an arbitrary number. Replacing x by λx in (2.1), we get

$$\|\langle f(\lambda x), y\rangle - \langle \lambda x, g(y)\rangle\| \leq \varphi(\lambda x, y),$$

and since a multiplication of (2.1) by $|\lambda|$ yields

$$\|\langle \lambda f(x), y\rangle - \langle \lambda x, g(y)\rangle\| \leq |\lambda|\varphi(x, y)$$

we obtain

$$\|\langle f(\lambda x), y\rangle - \langle \lambda f(x), y\rangle\| \leq \varphi(\lambda x, y) + |\lambda|\varphi(x, y). \tag{2.6}$$

If (2.3) holds, we take $c_n y$ instead y in (2.6) to get

$$\|\langle f(\lambda x), y\rangle - \langle \lambda f(x), y\rangle\| \leq |c_n|^{-1}\varphi(\lambda x, c_n y) + |\lambda||c_n|^{-1}\varphi(x, c_n y)$$

and, as $n \to \infty$, we obtain

$$\langle f(\lambda x), y\rangle = \langle \lambda f(x), y\rangle \qquad (x \in \mathcal{X}, y \in \mathcal{Y})\,. \tag{2.7}$$

If (2.5) holds, we take $c_n^{-1}y$ instead y in (2.6) and we arrive also at (2.7). Therefore,

$$f(\lambda x) = \lambda f(x) \qquad (x \in \mathcal{X}, \lambda \in \mathbb{C})\,. \tag{2.8}$$

If (2.2) holds, we take $c_n x$ instead x in (2.1) to get

$$\|\langle f(c_n x), y\rangle - \langle c_n x, g(y)\rangle\| \leq \varphi(c_n x, y)$$

and, by (2.7), we obtain

$$\|\langle f(x), y\rangle - \langle x, g(y)\rangle\| \leq |c_n|^{-1}\varphi(c_n x, y).$$

Taking the limit as $n \to \infty$ we conclude that

$$\langle f(x), y\rangle = \langle x, g(y)\rangle \qquad (x \in \mathcal{X}, y \in \mathcal{Y})\,. \tag{2.9}$$

Hence f is adjointable and admits the mapping g as its adjoint.

Alternatively, if (2.4) holds, we take $c_n^{-1}x$ instead x in (2.6) and arrive at the same conclusion (2.9). By Lemma 2.2 the mapping f is $\mathcal{A}$-linear and bounded with the adjoint g. □

Using the sequence $c_n = 2^n$ we get the following results.

Corollary 2.4. *If $f : \mathcal{X} \to \mathcal{Y}$ is an everywhere defined φ-perturbation of an adjointable mapping, where $\varphi(x, y) = \varepsilon\,\|x\|^p\,\|y\|^q$ $(\alpha > 0, p \neq 1, q \neq 1)$, then f is adjointable and hence a bounded C^*-linear mapping.*

Corollary 2.5. *If $f : \mathcal{X} \to \mathcal{Y}$ is an everywhere defined φ-perturbation of an adjointable mapping, where $\varphi(x, y) = \varepsilon_1 \|x\|^p + \varepsilon_2 \|y\|^q$ $(\varepsilon_1 \geq 0, \varepsilon_2 \geq 0, p \neq 1, q \neq 1)$, then f is adjointable and hence a bounded C^*-linear mapping.*

We would like to point out that the proof of Theorem 2.3 works equally well in the case that the functions f and g are well-defined merely on norm-dense subsets of $\mathcal{X}$ and $\mathcal{Y}$, respectively. This case covers the situation of pairs of adjoint to each other, densely defined $\mathcal{A}$-linear operators between pairs of Hilbert $\mathcal{A}$-modules. However, since boundedness cannot be demonstrated, in general, in that case we arrive at the following statement:

Theorem 2.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a φ-perturbation of an adjointable mapping with corresponding mapping $g : \mathcal{Y} \to \mathcal{X}$. Suppose, that the mappings f and g are densely defined on the respective Hilbert C^*-modules. Furthermore, suppose that for the perturbation bound mapping $\varphi(x, y)$ either both the conditions* (2.2) *and* (2.3), *or both the conditions* (2.4) *and* (2.5) *hold. Then f is adjointable. In particular, f is $\mathcal{A}$-linear, as well as its adjoint is g.*

Corollary 2.7 *The equation $f(x)^* y = x g(y)^*$ $\quad (x \in \mathcal{I}, y \in \mathcal{J})$ is superstable, where $f : \mathcal{I} \to \mathcal{J}$ and $g : \mathcal{J} \to \mathcal{I}$ are adjoint to each other, densely defined $\mathcal{A}$-linear mappings between right ideals $\mathcal{I}, \mathcal{J}$ of $\mathcal{A}$.*

The critical case of φ-perturbations is that one where the function φ satisfies neither the pair of conditions (i) and (ii), nor the pair of conditions (i') and (ii'). We demonstrate that there may exist φ-perturbed bounded C^*-linear mappings f on certain types of Hilbert C^*-modules $\mathcal{X}$ over certain C^*-algebras $\mathcal{A}$ which are not adjointable. Moreover, any non-adjointable bounded C^*-linear mapping f on suitably selected Hilbert C^*-modules $\mathcal{X}$ can be φ-perturbed by "compact" operators on $\mathcal{X}$ using this type of perturbation functions.

Proposition 2.8. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module over a given C^*-algebra $\mathcal{A}$. Suppose there exists a non-adjointable bounded $\mathcal{A}$-linear mapping $f : \mathcal{X} \to \mathcal{X}$, (so $\mathcal{X}$ cannot be self-dual by* [**8, 15**]*). Then there exist (at least countably many) positive constants c_α and respective "compact" $\mathcal{A}$-linear operators $K_\alpha : \mathcal{X} \to \mathcal{X}$ $(\alpha \in I)$ such that f is ϕ-perturbed for a function $\phi(x, y) = c_\alpha \cdot \|x\| \cdot \|y\|$ and for $g = K_\alpha^*$.*

Proof. By results of Huaxin Lin [**16**] and [**17**, Theorem 1.5], the Banach algebra $End_{\mathcal{A}}(\mathcal{X})$ of all bounded $\mathcal{A}$-linear mappings on $\mathcal{X}$ is the left multiplier algebra of the C*-algebra $K_{\mathcal{A}}(\mathcal{X})$ of all "compact" $\mathcal{A}$-linear operators on $\mathcal{X}$. Since $End_{\mathcal{A}}(\mathcal{X})$ is the completion of $K_{\mathcal{A}}(\mathcal{X})$ with respect to the left strict topology defined by the set of semi-norms $\{\| \cdot K\| : K \in K_{\mathcal{A}}(\mathcal{X})\}$, there exists a bounded net $\{K_\alpha : \alpha \in I\}$ of "compact" operators such that the set $\{K_\alpha K : \alpha \in I\}$ converges with respect to the operator norm to fK for any single "compact" operator K. Therefore,

$$0 = \lim_{\alpha \in I} \|\langle (fK - K_\alpha K)(x), y\rangle\| = \lim_{\alpha \in I} \|\langle (f - K_\alpha)K(x), y\rangle\|$$

for any "compact" operator K. However, the set $\{K(x) : K \in K_{\mathcal{A}}(\mathcal{X}), x \in \mathcal{X}\}$ is norm-dense in $\mathcal{X}$, hence

$$\|\langle f(x), y\rangle - \langle K_\alpha(x), y\rangle\| \leq \|f - K_\alpha\| \cdot \|x\| \cdot \|y\|$$

for any $x, y \in \mathcal{X}$ and any $\alpha \in I$. Setting $c_\alpha = \|f - K_\alpha\|$ for any fixed index α and taking into account the adjointability of the operators $\{K_\alpha\}$ we arrive at the desired result. □

Corollary 2.9. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module over a given C^*-algebra $\mathcal{A}$. Suppose there exists a non-adjointable bounded $\mathcal{A}$-linear mapping $f : \mathcal{X} \to \mathcal{X}$. Then there does not exist any φ-perturbation of f such that $\varphi(x, y)$ satisfies either both the conditions* (2.2) *and* (2.3) *or both the conditions* (2.4) *and* (2.5).

REFERENCES

1. M. Amyari: *Stability of C^*-inner products.* J. Math. Anal. Appl., **322** (2006), 214–218.
2. M. Amyari, M. S. Moslehian: *Stability of derivations on Hilbert C^*-modules.* Topological Algebras and Applications, 31–39, Contemp. Math. 724, 2007.
3. T. Aoki: *On the stability of the linear transformation in Banach spaces.* J. Math. Soc. Japan, **2** (1950), 64–66.
4. C. Baak, H. Y. Chu, M. S. Moslehian: *On linear n-inner product preserving mappings.* Math. Inequal. Appl., **9**, No. 3 (2006), 453–464.
5. J. Baker: *The stability of the cosine equation.* Proc. Amer. Math. Soc., **74** (1979), 242–246.
6. S. Czerwik (ed.): *Stability of Functional Equations of Ulam–Hyers–Rassias Type*, Hadronic Press Inc., Palm Harbor, Florida, 2003.
7. S. Czerwik: *Functional equations and inequalities in several variables.* World Scientific Publishing Co., Inc., River Edge, NJ, 2002.
8. M. Frank: *Geometrical aspects of Hilbert C*-modules.* Positivity, **3** (1999), 215–243.
9. P. Găvruţa: *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings.* J. Math. Anal. Appl., **184** (1994), 431–436.
10. D. H. Hyers: *On the stability of the linear functional equation.* Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.
11. D. H. Hyers, G. Isac, Th. M. Rassias: *Stability of Functional Equations in Several Variables.* Birkhauser, Boston, Basel, Berlin, 1998.
12. D. H. Hyers, Th. M. Rassias: *Approximate homomorphisms.* Aequationes Math., **44**, No. 2–3 (1992), 125–153.
13. S.-M. Jung: *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis.* Hadronic Press, Palm Harbor, Florida, 2001.
14. I. Kaplansky: *Modules over operator algebras.* Amer J. Math., **75** (1953), 839–858.
15. E. C. Lance: *Hilbert C^*-Modules.* LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
16. H. Lin: *Hilbert C^*-modules and their bounded module maps.* Sci. China, Ser. A, **34**, No. 12 (1991), 1409–1421.
17. H. Lin: *Bounded module maps and pure completely positive mappings.* J. Operator Theory, **26** (1991), 121–138.

18. V. M. Manuilov, E. V. Troitsky: *Hilbert C^*-modules.* Translations of Mathematical Monographs, 226. American Mathematical Society, Providence, RI, 2005.
19. D. S. Mitrinović: *Analytic Inequalities* (in cooperation with P. M. Vasić). Die Grundlehren der mathematischen Wissenschaften, Band 165, Springer-Verlag, New York-Berlin, 1970.
20. Th. M. Rassias: *On the stability of the linear mapping in Banach spaces.* Proc. Amer. Math. Soc., **72** (1978), 297–300.
21. Th. M. Rassias: *On the stability of functional equations and a problem of Ulam.* Acta Appl. Math., **62**, No. 1 (2000), 23–130.
22. Th. M. Rassias (ed.): *Functional Equations, Inequalities and Applications.* Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.
23. S. M. Ulam: *Problems in Modern Mathematics.* Chapter VI, Science Editions, Wiley, New York, 1964.

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