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COMPANION INEQUALITIES FOR CERTAIN BIVARIATE MEANS

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Dedicated to the Memory of Professor D. S. Mitrinović (1908–1995)

Sharp companion inequalities for certain bivariate means are obtained. In particular, companion inequalities for those discovered by Stolarsky and Sándor are established.

1. INTRODUCTION

The logarithmic and identric means of two positive numbers a and b are defined by

$$L \equiv L(a,b) = \begin{cases} \dfrac{b-a}{\log b - \log a}, & a \neq b \\ a, & a = b \end{cases} \tag{1}$$

and

$$I \equiv I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b \\ a, & a = b, \end{cases} \tag{2}$$

respectively. Also, let

$$A_k \equiv A_k(a,b) = \left(\frac{a^k+b^k}{2}\right)^{1/k} \tag{3}$$

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denote the power mean of order $k \neq 0$ of a and b. In particular, the arithmetic and geometric mean of a and b are

$$A \equiv A_1(a,b) = \frac{a+b}{2}\,, \quad G \equiv G(a,b) = \lim_{k\to 0} A_k(a,b) = \sqrt{ab}\,. \tag{4}$$

All means defined above have been studied extensively by many researchers. Many remarkable inequalities and identities have been established. For more details the interested reader is referred to [**1**]–[**6**] and [**13**]–[**16**].

In this paper we shall use the weighted geometric mean S of a and b with weights $a/(a+b)$ and $b/(a+b)$:

$$S \equiv S(a,b) = a^{a/(a+b)} b^{b/(a+b)}. \tag{5}$$

This mean is a special case of GINI's mean (see, e.g., [**6**]) and is related to the identric mean as follows (cf. [**9**])

$$S(a,b) = \frac{I(a^2,b^2)}{I(a,b)}\,. \tag{6}$$

For more properties of the mean S see, e.g., [**8**], [**10**], and [**14**].

Also, we will deal with the Heronian mean denoted by He (see [**3**]) and defined as follows

$$He \equiv He(a,b) = \frac{a+\sqrt{ab}+b}{3} = \frac{2A+G}{3}\,. \tag{7}$$

We recall now some inequalities which are of interest in this paper.

In 1980, K. B. STOLARSKY [**17**] proved that for all $a \neq b$ one has

$$A_{2/3} < I, \tag{8}$$

and that the order $2/3$ of the power mean is the best one, i.e., that the number $2/3$ in (8) cannot be replaced by a bigger one.

In 1991, J. SÁNDOR [**10**] proved that

$$He < A_{2/3} \tag{9}$$

while in [**14**] it has been shown that

$$A_2 < S \tag{10}$$

and also that inequalities (9) and (10) are sharp in a certain sense.

For later use, let us mention three inequalities

$$L < I < A \tag{11}$$

(see, e.g., [**3**]),

$$A < \frac{e}{2}\, I \tag{12}$$

(see [**11**]) and

$$I < \frac{2}{e}\,(A+G) \tag{13}$$

(see [**7**]).

The goal of this paper is to obtain companion inequalities of type (12) for the inequalities (8)–(10). One of our results will give an improvement of (13) . Also, new proofs of the inequalities (8)–(10) are offered.

It is worth mentioning that other means and their connections with functional equations are studied in [**4**] and [**18**].

2. MAIN RESULTS

The double inequality in Theorem 1 is known and follows from (11) and (12) . We will give a new proof of this result which also shows that the associated constants are optimal.

In what follows we will assume, without loss of generality, that $b > a > 0$.

Theorem 1. *We have*

$$I < A < \frac{e}{2}\,I, \tag{14}$$

where the constants 1 and $\frac{e}{2}$ are best possible.

Proof. Let $x = b/a$. Consider the function

$$f_1(x) = \frac{A(x,1)}{I(x,1)}\,.$$

Its logarithmic derivative is

$$\frac{f_1'(x)}{f_1(x)} = \frac{2\log x}{(x-1)^2(x+1)}\left(\frac{x+1}{2} - \frac{x-1}{\log x}\right).$$

This in conjunction with

$$\frac{x+1}{2} - \frac{x-1}{\log x} = A(x,1) - L(x,1) > 0,$$

gives $f_1'(x) > 0$ for $x > 1$. Thus $f_1(x)$ is strictly increasing on the stated domain. Hence $f_1(x) > \lim\limits_{x\to 1} f_1(x) = 1$ and $f_1(x) < \lim\limits_{x\to\infty} f_1(x) = \frac{e}{2}$. The proof of the inequality (14) is complete. Since $f_1(x)$ is continuous for $x > 1$, it follows that the constants 1 and $\frac{e}{2}$ are best possible. □

A companion inequality to (8) is contained in the following.

Theorem 2. *The following inequalities*

$$A_{2/3} < I < \frac{2\sqrt{2}}{e}\,A_{2/3} \tag{15}$$

are valid. Moreover, the constants 1 and $\dfrac{2\sqrt{2}}{e}$ *are best possible.*

Proof. Let $x^3 = b/a$ and let $f_2(x) = \dfrac{A_{2/3}(x^3,1)}{I(x^3,1)}$. Logarithmic differentiation gives $\dfrac{f_2'(x)}{f_2(x)} = \dfrac{3x^2}{(x^3-1)^2}\,k(x)$, where $k(x) = 3\log x - \dfrac{(x+1)(x^3-1)}{x(x^2+1)}$. Letting $t = x^3$ in the inequality

$$\frac{\log t}{t-1} < \frac{1+t^{1/3}}{t+t^{1/3}}$$

(see [**5**, p. 272]) we obtain $k(x) < 0$ for $x > 1$. Thus $f_2(x)$ is strictly decreasing when $x > 1$. Easy computations give

$$\lim_{x\to 1} f_2(x) = 1 \text{ and } \lim_{x\to\infty} f_2(x) = \frac{e}{2\sqrt{2}}\,.$$

Since $f_2(x)$ is continuous and strictly decreasing on its domain, we conclude that the constants 1 and $\dfrac{2\sqrt{2}}{e}$ in (15) are best possible. □

Remark 1. The second inequality in (15) and the following one $\sqrt{2}A_{2/3} < A+G$, which is easy to prove, provide a refinement of inequality (13)

$$I < \frac{2\sqrt{2}}{e}\,A_{2/3} < \frac{2}{e}\,(A+G).$$

We shall now establish inequality (9) together with its companion inequality.

Theorem 3. *We have*

$$He < A_{2/3} < \frac{3}{2\sqrt{2}}\,He, \tag{16}$$

where the constants 1 *and* $\dfrac{3}{2\sqrt{2}}$ *are best possible.*

Proof. Let $x^3 = b/a$ and let

$$f_3(x) = \frac{He(x^3,1)}{A_{2/3}(x^3,1)}\,.$$

Logarithmic differentiation gives

$$\frac{f_3'(x)}{f_3(x)} = -\frac{3}{2}\cdot\frac{x^{1/2}(x-1)(x^{1/2}-1)^2}{2(x^3+x^{3/2}+1)(x^2+1)} < 0.$$

Thus $f_3(x)$ is strictly decreasing for $x > 1$. This in conjunction with

$$\lim_{x\to 1} f_3(x) = 1 \text{ and } \lim_{x\to\infty} f_3(x) = \frac{2\sqrt{2}}{3}$$

gives the assertion (16) . □

Corollary. *The following inequalities*

$$He < I < \frac{3}{e} He \tag{17}$$

are valid. Moreover, the constants 1 *and* $3/e$ *are best possible.*

Proof. For the proof of (17) we use (15) and (16) together with

$$He/I = (He/A_{2/3})(A_{2/3}/I).$$

Taking into account that the product of two positive strictly decreasing functions is also strictly decreasing we conclude that the constants 1 and $3/e$ are best possible. □

We close this section with the following.

Theorem 4. *One has*

$$A_2 < S < \sqrt{2}\, A_2\,, \tag{18}$$

where the constants 1 *and* $\sqrt{2}$ *are best possible.*

Proof. Let $x = \frac{b}{a}$ and let $f_4(x) = \frac{A_2(x,1)}{S(x,1)}$. Then $\frac{f_4'(x)}{f_4(x)} = \frac{h(x)}{(x+1)^2(x^2+1)}$, where $h(x) = x^2 - 1 - (x^2+1)\log x$.

Taking into account that

$$\frac{x-1}{\log x} = L(x,1) < A(x,1) = \frac{x+1}{2} < \frac{x^2+1}{x+1}\,,$$

we obtain $h(x) < 0$ for $x > 1$. Thus $f_4(x)$ is a strictly decreasing function for all $x > 1$. Moreover, $\lim_{x\to 1} f_4(x) = 1$ and $\lim_{x\to\infty} f_4(x) = 1/\sqrt{2}$. The assertion (18) follows. □

REMARK 2. Among known refinements of the STOLARSKY inequality (8) the following one (see [**6**]) $A_{2/3} < \sqrt{I_{5/6} I_{7/6}} < I$, where $I_t \equiv I_t(a,b) = \big(I(a^t,b^t)\big)^{1/t}$ $(t \neq 0)$, seems to be an exotic one.

Strong inequalities connecting the identric mean I with other means (e.g., the GAUSS arithmetic-geometric mean) are established in [**12**]. Inequalities connecting means L, I and the SIEFFERT mean are obtained in [**13**].

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