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ON TRACES OF HOLOMORPHIC FUNCTIONS ON THE UNIT POLYBALL

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In this paper we completely describe traces of holomorphic Bergman classes and Bloch-type classes on polyballs and obtain related estimates generalizing classical Bergman projection theorem.

1. INTRODUCTION

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $\mathbf{B}^n = \{z \in \mathbb{C}^n \big| \, |z| < 1\}$. The boundary of $\mathbf{B}^n$ will be denoted by $\mathbf{S}^n$, $\mathbf{S}^n = \{z \in \mathbb{C}^n \big| \, |z| = 1\}$.

As usual, we denote by $H(\mathbf{B}^n)$ the class of all holomorphic functions on $\mathbf{B}^n$.

For every function $f \in H(\mathbf{B}^n)$ having a series expansion $f(z) = \sum\limits_{|k| \geq 0} a_k z^k$, we define the operator of fractional differentiation by

$$D^\alpha f(z) = \sum_{|k| \geq 0} (|k| + 1)^\alpha a_k z^k,$$

where α is any real number. It is obvious that for any α, D^α operator is acting from $H(\mathbf{B}^n)$ to $H(\mathbf{B}^n)$.

For $z \in \mathbf{B}^n$ and $r > 0$ set $\mathcal{D}(z, r) = \{w \in \mathbf{B}^n : \beta(z, w) < r\}$ where β is a Bergman metric on $\mathbf{B}^n$, $\beta(z, w) = \dfrac{1}{2} \log \dfrac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}$ is called the Bergman metric ball at z (see [**15**]).

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Let $m > 1$ is a natural number, $M \subset \mathbb{C}^n$ and $K \subset \mathbb{C}^{mn}$, $C^{mn} = C^n \times \cdots \times C^n$, be a hyper surface. Let $X(M)$ be a class of functions on M, $Y(K)$ the same. We say $\operatorname{Trace} Y = X$ or in short $\operatorname{Tr} Y = X$, $K = M^m$, $M^m = M \times \cdots \times M$, if for any $f \in Y(K)$, $f(w, \ldots, w) \in X(M)$, $w \in M$, and for any $g \in X(M)$, there exist a function $f \in Y(K)$ such that $f(w, \ldots, w) = g(w)$, $w \in M$. Traces of various functional spaces in $\mathbb{R}^n$ were described in [**6**] and [**14**]. In polydisk this problem is also known as a problem of diagonal map (see **3**] and references there).

The intention of this paper is to consider the following natural Trace problem for polyballs. Let M be a unit ball and let K be a polyball (product of m balls) in definition we gave above. Let further $H(\mathbf{B} \times \cdots \times \mathbf{B})$ be a space of all holomorphic functions by each z_j, $z_j \in B$, $\;j = 1, \ldots, m : f(z_1, \ldots, z_m)$. Let further Y be a subspace of $H(\mathbf{B} \times \cdots \times \mathbf{B})$. The question we would like to study and solve in this work is the following: Find the complete description of Trace Y in a sense of our definition for several concrete functional classes. We observe that for $n = 1$ this problem completely coincide with the well- known problem of diagonal map. The last problem of description of diagonal of various subspaces of $H(\mathbf{U}^n)$ of spaces of all holomorphic functions in the polydisk was studied by many authors before (see [**2**, **3**, **5**, **8**, **9**, **12**, **13**] and references there).

The goal of this paper is to give a complete description of traces classical Bergman spaces defined on polyballs and traces of some Bloch type classes in polyballs. Let us note that for $n = 1$ traces of Bergman spaces were completely described previously in [**3**] and [**12**] (see also, for example, [**13**] and reference there). In this paper as in case of polydisk estimates for expanded Bergman projection (the operator of polarization) are playing a crucial role during all our proofs.

Trace theorems even for $n = 1$ (case of polydisk) have numerous applications in the theory of holomorphic functions (see for example [**1**, **3**, **10**]).

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

As usual, let $\mathrm{d}\nu$ denote the Lebesgue measure on $\mathbf{B}$ normalized such that $\nu(\mathbf{B}) = 1$. For any real number α, let $\mathrm{d}\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \,\mathrm{d}\nu(z)$ for $|z| < 1$. Here, if $\alpha \le -1$, $c_\alpha = 1$ and if $\alpha > -1$, $c_\alpha = \dfrac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)}$ is the normalizing constant so that ν_α has unit total mass.

2. BERGMAN CLASSES AND BLOCH TYPE SPACES ON THE POLYBALLS

The following estimate is well-known and will used often in the paper. For a proof, see [**15**], Theorem 1.12.

Lemma A. *Suppose $c > 0$ is real and $t > -1$. Then the integral*

$$J_{c,t}(z) = \int_{\mathbf{B}} \frac{(1 - |w|^2)^t \,\mathrm{d}\nu(w)}{|1 - \langle z, w\rangle|^{n+1+t+c}}, \quad z \in \mathbf{B},$$

has the following asymptotic property

$$J_{c,t} \sim (1-|z|^2)^{-c} \quad as \quad |z| \to 1- .$$

We need the following estimate (see [**7**]):

Lemma B. *Let* $0 \le t_1 < s < t_0$, *then*

$$\int_{\mathbf{B}} \frac{(1-|\eta|^2)^s}{|1-\langle z,\eta\rangle|^{n+1+t_0}|1-\langle \xi,\eta\rangle|^{t_1}} \left(\log^k \frac{2}{1-|\eta|^2}\right) d\nu(\eta)$$
$$\le \frac{C}{(1-|z|^2)^{t_0-s}|1-\langle z,\xi\rangle|^{t_1}} \left(\log^k \frac{2}{1-|z|^2}\right),\ z,\ \xi \in \mathbf{B},\ k \in \mathbb{N}.$$

For any integer $k \ge 1$, positive real numbers $r_1, \ldots, r_k$ and function f on

$\mathbf{B} \times \cdots \times \mathbf{B}$, we define

$$\|f\|_{r_1,\ldots,r_k} = \sup_{z_1,\ldots,z_k \in \mathbf{B}} \left\{ |f(z_1,\ldots,z_k)| \prod_{j=1}^{n} \left(1-|z_j|^2\right)^{r_j} \right\}$$

Let $\Lambda(r_1,\ldots,r_k)$ denote the space of all $f \in H(\mathbf{B} \times \cdots \times \mathbf{B})$ such that $\|f\|_{r_1,\ldots,r_k} < \infty$. It can be checked without difficulties that $\Lambda(r_1,\ldots,r_k)$ with the norm $\|f\|_{r_1,\ldots,r_k}$ is a Banach space.

Theorem 1. *Let* $r_j > 0$, $j = 1,\ldots,m$ *and* $r = r_1 + \cdots + r_m$, *then*
Trace $(\Lambda(r_1,\ldots,r_m)) = \Lambda(r)$.

Proof. For every positive large enough b_j we have $F(z,\ldots,z) = f(z)$, where

$$F(z_1,\ldots,z_m) = C \int_{\mathbf{B}} \frac{f(w)(1-|w|)^{\sum_{j=1}^m b_j - n - 1}}{\prod_{j=1}^m (1-\langle \overline{w}, z_j\rangle)^{b_j}} dv(w)$$

by Bergman representation formula (see [**15**]). The proof follows from Hölder's inequality for n-functions and Lemma A. If $f \in \Lambda(r)$, then $|f(w)| \le \|f\|_r (1-|w|^2)^{-r}$. Hence we have by Hölder's inequality

$$|F(z_1,\ldots,z_m)| \le C \int_{\mathbf{B}} \frac{|f(w)|(1-|w|)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |1-\langle z_j, w\rangle|^{b_j}} d\nu(w)$$
$$\le C\|f\|_r \int_{\mathbf{B}} \frac{\prod_{j=1}^m (1-|w|^2)^{b_j - r_j}}{\prod_{j=1}^m |1-\langle z_j, w\rangle|^{b_j}} (1-|w|^2)^{-(n+1)} d\nu(w)$$
$$\le C\|f\|_r \prod_{j=1}^m \left(\int_{\mathbf{B}} \frac{(1-|w|^2)^{s_j}}{|1-\langle z_j, w\rangle|^{mb_j}}\right)^{1/m} \le C\|f\|_r \prod_{j=1}^m (1-|z_j|^2)^{-r_j},$$

where $s_j = m(b_j - r_j) - n - 1$.

Hence $F \in \Lambda(r_1, \dots, r_m)$, $F(z, \dots, z) = f(z)$. The reverse assertion is obvious since if $F \in \Lambda(r_1, \dots, r_n)$, then $F(z, \dots, z) \in \Lambda(r)$. Theorem is proved. $\square$

Let

$$\Lambda_{\log}(r_1, \dots, r_m) = \Bigg\{ f \in H(\mathbf{B^m}) : \sup_{z_j \in \mathbf{B}} |f(z_1, \dots, z_n)| \times \prod_{j=1}^{m} \left(\log \frac{1}{1-|z_j|} \right)^{-1/r_j} (1-|z_j|)^{1/r_j} < \infty,\ \sum_{j=1}^{m} \frac{1}{r_j} = 1,\ r_j > 0 \Bigg\}.$$

Then we have the following theorem. The proof use Lemma B and ideas of Theorem 1.

Theorem 2. Trace $(\Lambda_{\log}(r_1, \dots, r_m)) = \Lambda_{\log}(1)$, *where*

$$\Lambda_{\log}(1) = \left\{ f \in H(\mathbf{B}) : \sup_{z \in \mathbf{B}} |f(z)| \left(\log \frac{1}{1-|z|} \right)^{-1} (1-|z|) < \infty \right\}.$$

Remark 1. Note that Theorem 1 and Theorem 2 are obvious for $m = 1$.

For each real number α and $p \in (0, \infty)$, the Bergman space A^p_α is the intersection of $H(\mathbf{B})$ with $L^p(\mathbf{B}, d\nu_\alpha)$. It is well-known that A^p_α is a closed subspace of $L^p(\mathbf{B}, d\nu_\alpha)$. See [**15**], Chapter 2 for more detail.

To the end of the paper, fix an integer $m \geq 1$. For any two n-tuples of real numbers $\mathbf{a} = (a_1, \dots, a_m)$, and $\mathbf{b} = (b_1, \dots, b_m)$, we define the integral operators

$$(T_{\mathbf{a},\mathbf{b}} f)(z_1, \dots, z_m) = \prod_{j=1}^{m} (1-|z_j|^2)^{a_j} \int_{\mathbf{B}} \frac{f(w)(1-|w|^2)^{-n-1+\sum_{j=1}^{m} b_j}}{\prod_{j=1}^{m} |1-\langle z_j, w\rangle|^{a_j+b_j}} \mathrm{d}\nu(w),$$

and

$$(S_{\mathbf{a},\mathbf{b}} f)(z_1, \dots, z_m) = \prod_{j=1}^{m} (1-|z_j|^2)^{a_j} \int_{\mathbf{B}} \frac{f(w)(1-|w|^2)^{-n-1+\sum_{j=1}^{m} b_j}}{\prod_{j=1}^{m} (1-\langle z_j, w\rangle)^{a_j+b_j}} \mathrm{d}\nu(w),$$

where $z_1, \dots, z_m$ are in $\mathbf{B}$ and f is a function in $L^1\Big(\mathbf{B}, d\nu_{-n-1+\sum_{j=1}^{m} b_j}\Big)$. Note that for such f, the functions $T_{\mathbf{a},\mathbf{b}} f$ and $S_{\mathbf{a},\mathbf{b}} f$ are defined on $\mathbf{B}^m$, the product of m copies of $\mathbf{B}$, and we have $|S_{\mathbf{a},\mathbf{b}} f| \leq T_{\mathbf{a},\mathbf{b}} |f|$.

We will study the boundedness of $T_{\mathbf{a},\mathbf{b}}$ and $S_{\mathbf{a},\mathbf{b}}$ from certain L^p spaces of $\mathbf{B}$ into those of $\mathbf{B}^m$. Consider first the case $1 \leq p < \infty$. Let $s_1, \dots, s_m$ be arbitrary real numbers and put $t = (m-1)(n+1) + \sum_{j=1}^{m} s_j$. The following proposition gives sufficient conditions for the boundedness of $T_{\mathbf{a},\mathbf{b}}$ (and hence, the boundedness of $S_{\mathbf{a},\mathbf{b}}$) from $L^p(\mathbf{B}, d\nu_t)$ into $L^p(\mathbf{B}^m, \mathrm{d}\nu_{s_1} \cdots \mathrm{d}\nu_{s_m})$.

Proposition 1. *Let $1 \le p < \infty$ and $s_j > -1$. Suppose for each $j = 1, \dots, m$, we have $-pa_j < s_j + 1$ and $ms_j + 1 < p(mb_j - n) - (m-1)(n+1)$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |(T_{a,b}f)(z_1, \dots, z_m)|^p \prod_{j=1}^{m} (1 - |z_j|^2)^{s_j} \, \mathrm{d}\nu(z_1) \dots \mathrm{d}\nu(z_m)$$
$$\le C \int_{\mathbf{B}} |f(w)|^p (1 - |w|^2)^{(m-1)(n+1) + \sum\limits_{j=1}^{m} s_j} \, \mathrm{d}\nu(w),$$

for all f in $L^1(\mathbf{B}, \mathrm{d}\nu)$.

Proof. The case $p = 1$ follows from Fubinis theorem and the estimates in Lemma A. Now assume $p > 1$. Let q denote the exponential conjugate of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Choose a positive number such that $p\gamma < \min\{p(mb_j - n) - (m-1)(n+1) - ms_j - 1 : j = 1, \dots, m\}$. Put $\alpha = \frac{1}{m}\Big(\gamma - \frac{1}{q}\Big)$ and $\beta = -n - 1 + \sum\limits_{j=1}^{m} b_j - m\alpha = -n - 1 + \sum\limits_{j=1}^{m} b_j - \gamma + \frac{1}{q}$. For each j, choose e_j such that

$$\frac{n+1}{mq} + \alpha < e_j < \frac{n+1}{mq} + \alpha + \frac{pa_j + s_j + 1}{p}.$$

It is possible to choose such an e_j since $pa_j + s_j + 1 > 0$. Put $d_j = a_j + b_j - e_j$. For any measurable function f on $\mathbf{B}$ and $z_1, \dots, z_m$ in $\mathbf{B}$, using Hölders inequality, we have

$$\int_{\mathbf{B}} \frac{|f(w)|(1 - |w|^2)^{-n-1+\sum\limits_{j=1}^{m} b_j}}{\prod\limits_{j=1}^{m} |1 - \langle z_j, w\rangle|^{a_j + b_j}} \, \mathrm{d}\nu(w)$$
$$= \int_{\mathbf{B}} \left(\frac{|f(w)|(1 - |w|^2)^{\beta}}{\prod\limits_{j=1}^{m} |1 - \langle z_j, w\rangle|^{d_j}} \right) \prod_{j=1}^{m} \frac{(1 - |w|^2)^{\alpha}}{|1 - \langle z_j, w\rangle|^{e_j}} \, \mathrm{d}\nu(w)$$
$$\le \left(\int_{\mathbf{B}} \frac{|f(w)|^p (1 - |w|^2)^{p\beta}}{\prod\limits_{j=1}^{m} |1 - \langle z_j, w\rangle|^{pd_j}} \, \mathrm{d}\nu(w) \right)^{1/p} \prod_{j=1}^{m} \left(\int_{\mathbf{B}} \frac{(1 - |w|^2)^{mq\alpha}}{|1 - \langle z_j, w\rangle|^{mqe_j}} \, \mathrm{d}\nu(w) \right)^{1/(mq)}.$$

For each j, since $mq\alpha = q\gamma - 1 > -1$ and $mqe_j > n + 1 + mq\alpha$, Lemma A shows that

$$\int_{\mathbf{B}} \frac{(1 - |w|^2)^{mq\alpha}}{|1 - \langle z_j, w\rangle|^{mqe_j}} \, \mathrm{d}\nu(w) \le C(1 - |z_j|^2)^{n+1+mq\alpha - mqe_j},$$

where C is independent of $z_1, \ldots, z_m$. Thus we obtain

$$\int_{\mathbf{B}} \frac{|f(w)|(1-|w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |1-\langle z_j, w\rangle|^{a_j+b_j}} \,\mathrm{d}\nu(w)$$
$$\leq C\left(\int_{\mathbf{B}} \frac{|f(w)|^p(1-|w|^2)^{p\beta}}{\prod_{j=1}^m |1-\langle z_j, w\rangle|^{pd_j}} \,\mathrm{d}\nu(w)\right)^{1/p} \prod_{j=1}^m \left(1-|z_j|^2\right)^{\frac{n+1}{mq}+\alpha-e_j}.$$

This implies that

$$|(T_{\mathbf{a},\mathbf{b}}f)(z_1,\ldots,z_m)|^p$$
$$\leq C\left(\int_{\mathbf{B}} \frac{|f(w)|^p(1-|w|^2)^{p\beta}}{\prod_{j=1}^m |1-\langle z_j, w\rangle|^{pd_j}} \,\mathrm{d}\nu(w)\right) \prod_{j=1}^m (1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)}.$$

Now by Fubini theorem,

$$\text{(1)} \qquad \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |(T_{\mathbf{a},\mathbf{b}}f)(z_1,\ldots,z_m))|^p \prod_{j=1}^m (1-|z_j|^2)^{s_j} \,\mathrm{d}\nu(z_1)\cdots \mathrm{d}\nu(z_m)$$
$$\leq C\int_{\mathbf{B}} \left(\prod_{j=1}^m \int_{\mathbf{B}} \frac{(1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j}}{|1-\langle z_j, w\rangle|^{pd_j}} \,\mathrm{d}\nu(z_j)\right)|f(w)|^p(1-|w|^2)^{p\beta}\mathrm{d}\nu(w).$$

For each j, by the choice of e_j and γ, we have $\frac{p(n+1)}{mq}+p\alpha-pe_j+pa_j+s_j > -1$ and $n+1+\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j-pd_j < 0$. Applying Lemma A again, we have

$$\text{(2)} \qquad \int_{\mathbf{B}} \frac{(1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j}}{|1-\langle z_j, w\rangle|^{pd_j}} \,\mathrm{d}\nu(z_j)$$
$$\leq C(1-|w|^2)^{n+1+\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j-pd_j}$$
$$= C(1-|w|^2)^{\frac{p\gamma-p(mb_j-n)+(m-1)(n+1)+(ms_j+1)}{m}},$$

where C independent of w. From (1) and (2) and the fact that

$$\sum_{j=1}^m \frac{p\gamma-p(mb_j-n)+(m-1)(n+1)+(ms_j+1)}{m}$$

$$= (m-1)(n+1) + \sum_{j=1}^{m} s_j - p\Big(\sum_{j=1}^{m} b_j - \gamma - n\Big) + 1$$
$$= (m-1)(n+1) + \sum_{j=1}^{m} s_j - p\beta,$$

the conclusion of the proposition follows. □

Remark 2. Note that for $m = 1$ our assertion in Proposition 1 is well known and has numerous applications (see [**15**]).

For any two n-tuples of real numbers $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$, we consider the integral operator

$$(R_{\mathbf{x},\mathbf{y}}g)(w) = (1-|w|^2)^{-m(n+1)+\sum_{j=1}^{m} y_j} \times \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} g(z_1, \dots, z_m)\left(\prod_{j=1}^{m} \frac{(1-|z_j|^2)^{x_j}}{(1-\langle w, z_j\rangle)^{x_j+y_j}}\right) \mathrm{d}\nu(z_1)\cdots \mathrm{d}\nu(z_m),$$

for $g \in L^1(\mathbf{B}^m, d\nu_{x_1}\cdots d\nu_{x_m})$ and $w \in \mathbf{B}$. Using Proposition 1, we obtain the following proposition which gives conditions for the boundedness of $R_{\mathbf{x},\mathbf{y}}$.

Proposition 2. *Let $1 \le p < \infty$ and $s_j > -1$. Suppose for each j we have $s_j + 1 < p(x_j + 1)$ and $ms_j + 1 > mp(n + 1 - y_j) - (m-1)(n+1)$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbf{B}} |(R_{\mathbf{x},\mathbf{y}}g)(w)|^p (1-|w|^2)^{(m-1)(n+1)+\sum_{j=1}^{m} s_j} \mathrm{d}\nu(w) \le C \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |g(z_1, \dots, z_m)|^p \prod_{j=1}^{m} (1-|z_j|^2)^{s_j} \mathrm{d}\nu(z_1)\cdots \mathrm{d}\nu(z_m).$$

Proof. We first consider the case $p = 1$. We have

$$\int_{\mathbf{B}} |(R_{\mathbf{x},\mathbf{y}}g)(w)|(1-|w|^2)^{(m-1)(n+1)+\sum_{j=1}^{m} s_j} d\nu(w) \tag{3}$$
$$\le \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |g(z_1, \dots, z_m)| \prod_{j=1}^{m} (1-|z_j|^2)^{x_j} \times \left(\int_{\mathbf{B}} \frac{(1-|w|^2)^{-n-1+\sum_{j=1}^{m}(y_j+s_j)}}{\prod_{j=1}^{m} |1-\langle w, z_j\rangle|^{x_j+y_j}} d\nu(w)\right) \mathrm{d}\nu(z_1)\cdots \mathrm{d}\nu(z_m).$$

By Hölders inequality,

$$\int_{\mathbf{B}} \frac{(1-|w|^2)^{-n-1+\sum\limits_{j=1}^{m}(y_j+s_j)}}{\prod\limits_{j=1}^{m}|1-\langle w,z_j\rangle|^{x_j+y_j}}\,\mathrm{d}\nu(w) \le \left(\prod_{j=1}^{m}\int_{\mathbf{B}} \frac{(1-|w|^2)^{-n-1+my_j+ms_j}}{|1-\langle w,z_j\rangle|^{mx_j+my_j}}\,\mathrm{d}\nu(w)\right)^{1/m}.$$

From the assumption of the proposition, we have $-n-1+my_j+ms_j>-1$ and $mx_j+my_j>(-n-1+my_j+ms_j)+(n+1)$ for each j. Lemma A shows that the above product is less than or equal to $\prod\limits_{j=1}^{m}(1-|z_j|^2)^{s_j-x_j}$. From this and (3), the conclusion of the proposition then follows.

Now assume $1<p<\infty$. Put $\mathbf{s}=(s_1,\dots,s_m)$, and let $\mathbf{a}=\mathbf{x}-\mathbf{s}$ and $\mathbf{b}=\mathbf{y}+\mathbf{s}$. Then

$$(S_{\mathbf{a},\mathbf{b}}f)(z_1,\dots,z_m)=\prod_{j=1}^{m}(1-|z_j|^2)^{x_j-s_j}\int_{\mathbf{B}} \frac{f(w)(1-|w|^2)^{-n-1+\sum\limits_{j=1}^{m}(y_j+s_j)}}{\prod\limits_{j=1}^{m}(1-\langle z_j,w\rangle)^{x_j+y_j}}\,\mathrm{d}\nu(w).$$

By the assumption and Proposition 1, $S_{\mathbf{a},\mathbf{b}}$ is a bounded operator from $L^q(\mathbf{B},\mathrm{d}\nu_t)$ into $L^q(\mathbf{B}^m,\mathrm{d}\nu_{s_1}\cdots\mathrm{d}\nu_{s_m})$, where $1<q<\infty$ is the exponential conjugate of p and $t=(m-1)(n+1)+\sum\limits_{j=1}^{m}s_j$. On the other hand, it can be checked easily that $S^*_{\mathbf{a},\mathbf{b}}=R_{\mathbf{x},\mathbf{y}}$. The conclusion of the proposition follows. □

Remark 3. Note that for $m=1$ the assertion of Proposition 2 is well known (see [**15**]).

Proposition 2′. *Let* $p\in(0,\infty,)$ $s_j>-1$, $j=1,\dots,m$, $m\in\mathbb{N}$. *Then the following estimate holds*

$$\begin{aligned}\mathcal{J}&=\int_{\mathbf{B}}|g(w,\dots,w)|^p(1-|w|^2)^{(m-1)(n+1)+\sum\limits_{j=1}^{m}s_j}\,\mathrm{d}\nu(w)\\ &\le C\int_{\mathbf{B}}\cdots\int_{\mathbf{B}}|g(z_1,\dots,z_m)|^p\prod_{j=1}^{m}(1-|z_j|^2)^{s_j}\,\mathrm{d}\nu(z_1)\cdots\mathrm{d}\nu(z_m)=\mathcal{J}_1.\end{aligned}$$

Proof. We have by Lemma 2.24 from [**15**] and properties of r-lattice in Bergman metric (see [**15**], Theorem 2.23)

$$\begin{aligned}\mathcal{J}&\le C\sum_{k\ge0}\sup_{z\in\mathcal{D}(a_k,r)}|g(z,\dots,z)|^p(1-|a_k|^2)^{(m-1)(n+1)+(\sum\limits_{j=1}^{m}s_j)+n+1}\\ &\le C\sum_{k_1\ge0}\cdots\sum_{k_m\ge0}\sup_{\substack{z_1\in\mathcal{D}(a_{k_1},r)\\ \vdots\\ z_m\in\mathcal{D}(a_{k_m},r)}}|g(z_1,\dots,z_m)|^p\\ &\qquad\times(1-|a_{k_1}|^2)^{\tau_1/m}\cdots(1-|a_{k_m}|^2)^{\tau_m/m}\le C_1\mathcal{J}_1,\end{aligned}$$

where $\tau_j = (n+1)m + s_j m$. □

Theorem 3. *Suppose* $1 \le p \le \infty$ *and* $s_1, \ldots, s_m > -1$. *Put* $t = (m-1)(n+1) + \sum_{j=1}^{m} s_j$. *Then there are bounded operators* $S : A^p(\mathbf{B}, \mathrm{d}\nu_t) \to A^p(\mathbf{B}^m, \mathrm{d}\nu_{s_1} \cdots \mathrm{d}\nu_{s_m})$, *and* $R : A^p(\mathbf{B}^m, \mathrm{d}\nu_{s_1} \cdots \mathrm{d}\nu_{s_m}) \to A^p(\mathbf{B}, \mathrm{d}\nu_t)$ *such that* $(Sf)(z, \ldots, z) = f(z)$ *and* $(Rg)(z) = g(z, \ldots, z)$ *for all* $f \in A^p(\mathbf{B}, \mathrm{d}\nu_t)$, *all* $g \in A^p(\mathbf{B}^m, \mathrm{d}\nu_{s_1} \cdots \mathrm{d}\nu_{s_m})$ *and all* $z \in \mathbf{B}$. *In other words, the Trace of* $A^p(\mathbf{B}^m, \mathrm{d}\nu_{s_1} \cdots \mathrm{d}\nu_{s_m})$ *is* $A^p(\mathbf{B}, \mathrm{d}\nu_t)$.

Remark 4. For $n = 1$ Theorem 3 was known before (see [**3**], [**8**], [**12**]).

Proof. If $p = \infty$, then $A^\infty(\mathbf{B}, \mathrm{d}\nu_t) = H^\infty(\mathbf{B})$ and $A^p(\mathbf{B}^m, \mathrm{d}\nu_{s_1} \cdots \mathrm{d}\nu_{s_m}) = H^\infty(\mathbf{B}^m)$. Define $(Sf)(z_1, \ldots, z_m) = f(z_1)$ for $f \in H^\infty(\mathbf{B})$, $z_1, \ldots, z_m \in \mathbf{B}$ and $(Rg)(w) = g(w, \ldots, w)$ for $g \in H^\infty(\mathbf{B}^m)$ and $w \in \mathbf{B}$. Then $\|S\|$, $\|R\| \le 1$ and they satisfy the conclusion of the corollary.

Now suppose $1 \le p < \infty$. Let $\mathbf{a} = (0, \ldots, 0)$, $\mathbf{b} = (b_1, \ldots, b_m)$, where b_j is large enough, and $S = CS_{\mathbf{a},\mathbf{b}}$. It can be checked that $\mathbf{a}$ and $\mathbf{b}$ satisfy the hypothesis of Proposition 1. On the other hand, for $f \in A^p(\mathbf{B}, d\nu_k)$, Sf is holomorphic and for $z \in \mathbf{B}$,

$$\begin{aligned}(Sf)(z, \ldots, z) &= C \int_{\mathbf{B}} \frac{f(w)(1-|w|^2)^{-n-1+\sum_{j=1}^{m} b_j}}{(1-\langle z, w\rangle)^{\sum_{j=1}^{m} b_j}} \, \mathrm{d}\nu(w) \\ &= \int_{\mathbf{B}} \frac{f(w)\, \mathrm{d}\nu_{k+1}(w)}{(1-\langle z, w\rangle)^{n+2+k}} = f(z)\end{aligned}$$

by [**15**], Theorem 2.2.

Let $\mathbf{x} = (x_1, \ldots, x_m)$, where x_j is large enough, $\mathbf{y} = (n+1, \ldots, n+1)$ and $R = c_{x_1} \cdots c_{x_m} R_{\mathbf{x},\mathbf{y}}$. Then $\mathbf{x}$ and $\mathbf{y}$ satisfy the hypothesis of Proposition 2. Furthermore, for $g \in A^p(\mathbf{B}^m, \mathrm{d}\nu_{s_1} \cdots \mathrm{d}\nu_{s_m})$ and $w \in \mathbf{B}$,

$$\begin{aligned}(Rg)(w) &= \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} \frac{g(z_1, \ldots, z_m)\, \mathrm{d}\nu_{x_1}(z_1) \cdots \mathrm{d}\nu_{x_m}(z_m)}{\prod_{j=1}^{m} (1-\langle w, z_j\rangle)^{x_j+y_j}} \\ &= \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} \frac{g(z_1, \ldots, z_m)\, \mathrm{d}\nu_{x_1}(z_1) \cdots \mathrm{d}\nu_{x_m}(z_m)}{\prod_{j=1}^{m} (1-\langle w, z_j\rangle)^{n+1+x_j}} \\ &= g(w, \ldots, w),\end{aligned}$$

by applying [**15**], Theorem 2.2, m times. Therefore, S and R are the required operators.

It is not difficult to see that to get the second part of the Theorem 3 we can also apply Proposition 2′. □

The next lemma shows that if $s_1 = \cdots = s_m$ and $b_1 = \cdots = b_m$ then the converse of Proposition 1 holds true. We do not know if it is also the case for general $s_1, \ldots, s_m$ and $b_1, \ldots, b_m$.

Lemma 1. *Let $a_1, \dots, a_m$, $b_1 = \dots = b_m = b$ and $s_1, \dots, s_m = s$ be real numbers and let $1 < p < \infty$. Put $t = (m-1)(n+1) + ms$. If $S_{\mathbf{a},\mathbf{b}}$ is a bounded operator from $L^p(\mathbf{B}, \mathrm{d}\nu_t)$ into $L^p(\mathbf{B}^m, (\mathrm{d}\nu_s)^m)$, then $-pa_j < s+1$ for all $j = 1, \dots, m$ and $ms + 1 < p(mb-n) - (m-1)(n+1)$.*

Proof. Choose N sufficiently large so that the function $f(w) = (1-|w|^2)^N$ belongs to $L^p(\mathbf{B}, \mathrm{d}\nu_t)$. By the rotation-invariant property of the Lebesgue measure, we see that $S_{\mathbf{a},\mathbf{b}}f$ is a multiple of $\prod_{j=1}^m (1-|zj|)^{a_j}$. Since $S_{\mathbf{a},\mathbf{b}}f$ belongs to $L^p(\mathbf{B}^m, (\mathrm{d}\nu_s)^m)$, we conclude that $pa_j + s_j > -1$ for all $j = 1, \dots, m$. Now let $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The boundedness of $S_{\mathbf{a},\mathbf{b}}$ implies the boundedness of the adjoint $S^*_{\mathbf{a},\mathbf{b}}$ as an operator from $L^q(\mathbf{B}^m, (\mathrm{d}\nu_s)^m)$, into $L^q(\mathbf{B}, \mathrm{d}\nu_t)$. One can check without difficulty that for $g \in L^q(\mathbf{B}^m, (\mathrm{d}\nu_s)^m)$ the adjoint of $S_{\mathbf{a},\mathbf{b}}$ equals

$$\begin{aligned}
&(1-|w|^2)^{-n-1-t+\sum_{j=1}^m b_j} \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} g(z_1, \dots, z_m) \prod_{j=1}^m \frac{(1-|z_j|)^{a_j+s_j}}{(1-\langle w, z_j\rangle)^{a_j+b_j}} \, \mathrm{d}\nu(z_j) \\
&\quad = (1-|w|^2)^{-n-1-t+mb} \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} g(z_1, \dots, z_m) \prod_{j=1}^m \frac{(1-|z_j|)^{a_j+s}}{(1-\langle w, z_j\rangle)^{a_j+b}} \, \mathrm{d}\nu(z_j).
\end{aligned}$$

Letting $g(z_1, \dots, z_m) = \prod_{j=1}^m (1-|z_j|^2)^M$ for some large M, we see that $S^*_{\mathbf{a},\mathbf{b}}g$ is a multiple of $(1-|w|^2)^{-n-1-t+mb}$.

If $1 < p < \infty$, then $1 < q < \infty$. Since $(S^*_{\mathbf{a},\mathbf{b}}g)$ is in $L^q(\mathbf{B}, \mathrm{d}\nu_t)$ we have $q(-n-1-t+mb) + t > -1$, which is equivalent to $t+1 < p(mb-n)$ and hence, $ms + 1 < p(mb-n) - (m-1)(n+1)$. □

Remark 5. Suppose $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{b} = (b, \dots, b)$ and $s_1 = \dots = s_m = s$, where $a_1, \dots, a_m, b, s$ are arbitrary real numbers. Put $t = (m-1)(n+1) + ms$. Suppose $1 < p < \infty$, $s > -1$. Proposition 1 and Lemma 1 show that the following statements are equivalent.

(1) The operator $T_{\mathbf{a},\mathbf{b}}$ is bounded from $L^p(\mathbf{B}, \mathrm{d}\nu_t)$ into $L^p(\mathbf{B}^m, (\mathrm{d}\nu_s)^m)$.

(2) The operator $S_{\mathbf{a},\mathbf{b}}$ is bounded from $L^p(\mathbf{B}, \mathrm{d}\nu_t)$ into $L^p(\mathbf{B}^m, (\mathrm{d}\nu_s)^m)$.

(3) $ms + 1 < p(mb-n) - (m-1)(n+1)$ and $-pa_j < s+1$ for $j = 1, \dots, m$.

We now consider the case $0 < p \le 1$. Let $s_1, \dots, s_m$ be real numbers and let $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. The following proposition gives sufficient conditions for the boundedness of $T_{\mathbf{a},\mathbf{b}}$ from the Bergman space A^p_t into $L^p(\mathbf{B}^m, \mathrm{d}\nu_{s_1} \times \dots \times \mathrm{d}\nu_{s_m})$.

Proposition 3. *Let $p \in (0,1]$, $s_j > -1$. Suppose for each $j = 1, \dots, m$, we have $-pa_j < s_j + 1$ and $s_j + 1 < pb_j - n$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |(T_{\boldsymbol{a},\boldsymbol{b}} f)(z_1, \ldots, z_m)|^p \prod_{j=1}^{m} (1 - |z_j|^2)^{s_j} \mathrm{d}\nu(z_1) \cdots \mathrm{d}\nu(z_m)$$
$$\leq C \int_{\mathbf{B}} |f(w)|^p (1 - |w|^2)^{(m-1)(n+1) + \sum_{j=1}^{m} s_j} \mathrm{d}\nu(w),$$

for all f in $H(\mathbf{B}) \cap\ L^1(\mathbf{B}, d\nu_b)$.

Remark 6. The condition $s_j + 1 < pb_j - n$ is equivalent to $ms_j + 1 < p(mb_j - n) - (m-1)(n+1) - n(1-p)$. This shows that there is an extra summand $(n(1-p))$ in the condition on the s_j's compared to that in Proposition 1. This extra summand vanishes when $p = 1$.

Proof. Let $\mathbf{D}(a, r)$ denote the Bergman disk of radius r centered at a for each $a \in \mathbf{B}$. Fix $0 < r \leq 1$ and choose $\{u_k\}_{k=1}^{m}$ to be any r-lattice in the Bergman metric of $\mathbf{B}$. This means that $\mathbf{B} = \bigcup_{k=1}^{\infty} \mathbf{D}(u_k, r)$, $\mathbf{D}(u_k, r) \cap \mathbf{D}(u_\ell, r) = \emptyset$ if $k \neq \ell$ and there is an integer $N \geq 1$ such that each $z \in \mathbf{B}$ belongs to at most N of the sets $\mathbf{D}(u_k, 2r)$. (See [**15**], Theorem 2.23 and the remark following it for more detail about the existence of such a lattice). For any function $f \in L^1(\mathbf{B}, \mathrm{d}\nu_n)$ and any $z_1, \ldots, z_m \in \mathbf{B}$, we have

$$|(T_{\mathbf{a},\mathbf{b}} f)(z_1, \ldots, z_m)|$$
$$\leq \prod_{j=1}^{m} (1 - |z_j|^2)^{a_j} \sum_{k=1}^{\infty} \int_{\mathbf{D}(u_k, r)} \frac{|f(w)|(1 - |w|^2)^{-n-1+\sum_{j=1}^{m} b_j}}{\prod_{j=1}^{m} |1 - \langle z_j, w\rangle|^{a_j + b_j}} \mathrm{d}\nu(w).$$

By [**15**], Lemma 2.27, there is a constant $C > 0$ so that for each $j = 1, \ldots, m$ and $k \geq 1$, $\frac{1}{C} \leq \left|\frac{1 - \langle z_j, w\rangle}{1 - \langle z_j, u_k\rangle}\right| \leq C$, for all $w \in \mathbf{D}(u_k, r)$. Also by [**15**], Lemma 1.24, $\int_{\mathbf{D}(u_k,r)} (1 - |w|^2)^{-n-1+\sum_{j=1}^{m} b_j} \mathrm{d}\nu(w)$ is comparable with $(1 - |u_k|^2)^{\sum_{j=1}^{m} b_j}$. Thus we have

$$|(T_{\mathbf{a},\mathbf{b}} f)(z_1, \ldots, z_m)|$$
$$\leq C \sum_{k=1}^{\infty} \prod_{j=1}^{m} \frac{(1 - |z_j|^2)^{a_j}}{|1 - \langle z_j, u_k\rangle|^{a_j + b_j}} \int_{\mathbf{D}(u_k, r)} |f(w)|(1 - |w|^2)^{-n-1+\sum_{j=1}^{m} b_j} \mathrm{d}\nu(w)$$
$$\leq C \sum_{k=1}^{\infty} \prod_{j=1}^{m} \frac{(1 - |z_j|^2)^{a_j} (1 - |u_k|^2)^{\sum_{j=1}^{m} b_j}}{|1 - \langle z_j, u_k\rangle|^{a_j + b_j}} \sup\{|f(w)| : w \in \mathbf{D}(u_k, r)\}.$$

Now since $0 < p \leq 1$, using the inequality $(x_1 + x_2 + \cdots)^p \leq x_1^p + x_2^p + \cdots$, which is valid for nonnegative numbers $x_1, x_2, \ldots$, we get

$$|(T_{\mathbf{a},\mathbf{b}}f)(z_1,\ldots,z_m)|^p \leq C\sum_{k=1}^{\infty}\prod_{j=1}^{m}\frac{(1-|z_j|^2)^{pa_j}(1-|u_k|^2)^{p\left(\sum\limits_{j=1}^{m}b_j\right)}}{|1-\langle z_j,u_k\rangle|^{pa_j+pb_j}}\sup\{|f(w)|^p : w\in\mathbf{D}(u_k,r)\}.$$

Integrating with respect to $\mathrm{d}\nu_{s_1}(z_1)\cdots\mathrm{d}\nu_{s_m}(z_m)$ and using Lemma A (note that by assumption, $pa_j+s_j>-1$ and $pa_j+pb_j>n+1+pa_j+s_j$), we obtain

$$\begin{aligned}\int_{\mathbf{B}}\cdots\int_{\mathbf{B}}&|(T_{\mathbf{a},\mathbf{b}}f)(z_1,\ldots,z_m)|^p\prod_{j=1}^{m}(1-|z_j|^2)^{s_j}\,\mathrm{d}\nu(z_1)\cdots\mathrm{d}\nu(z_m)\\ &\leq C\sum_{k=1}^{\infty}\left(\prod_{j=1}^{m}(1-|u_k|^2)^{n+1+s_j-pb_j}\right)\\ &\qquad\qquad\times(1-|u_k|^2)^{p\left(\sum\limits_{j=1}^{m}b_j\right)}\sup\{|f(w)|^p : w\in\mathbf{D}(u_k,r)\}\\ &\leq C\sum_{k=1}^{\infty}(1-|u_k|^2)^{m(n+1)+\sum\limits_{j=1}^{m}s_j}\sup\{|f(w)|^p : w\in\mathbf{D}(u_k,r)\}.\end{aligned}$$

From [**15**], Lemma 2.20, $(1-|u_k|^2)$ is comparable with $(1-|w|^2)$ when $w\in\mathbf{D}(u_k,r)$. This together with [**15**], Lemma 2.24 implies that, if f is holomorphic on $\mathbf{B}$, then

$$\begin{aligned}\int_{\mathbf{B}}\cdots\int_{\mathbf{B}}&|(T_{\mathbf{a},\mathbf{b}}f)(z_1,\ldots,z_m)|^p\prod_{j=1}^{m}(1-|z_j|^2)^{s_j}\,\mathrm{d}\nu(z_1)\cdots\mathrm{d}\nu(z_m)\\ &\leq C\sum_{k=1}^{\infty}\sup\{|f(w)|^p(1-|w|^2)^{m(n+1)+\sum\limits_{j=1}^{m}s_j} : w\in\mathbf{D}(u_k,r)\}\\ &\leq C\sum_{k=1}^{\infty}\int_{\mathbf{D}(u_k,2r)}|f(w)|^p(1-|w|^2)^{(m-1)(n+1)+\sum\limits_{j=1}^{m}s_j}\,\mathrm{d}\nu(w)\\ &\leq C\int_{\mathbf{B}}|f(w)|^p(1-|w|^2)^{(m-1)(n+1)+\sum\limits_{j=1}^{m}s_j}\,\mathrm{d}\nu(w).\end{aligned}$$

To derive the last inequality, we have used the fact that each $z\in\mathbf{B}$ belongs to at most N of the sets $\mathbf{D}(u_k,2r)$. □

REMARK 7. Proposition 3 for $m=1$ is obvious, for $m>1$, $n=1$ it was proved in [**3**].

The following theorem follows directly from Proposition 3 and Proposition 2′.

Theorem 4. *Let* $p\in(0,1]$, $s_1>-1,\ldots,s_m>-1$, $t=(m-1)(n+1)+\sum\limits_{j=1}^{m}s_j$. *Then* Trace $A^p(\mathbf{B}^m,\mathrm{d}\nu_{s_1},\ldots,\mathrm{d}\nu_{s_m})=A^p(\mathbf{B},\mathrm{d}\nu_t)$.

Remark 8. For $n = 1$ Theorem 4 was known before (see [**3**], [**8**], [**12**]).

Remark 9. It is not difficult to notice that some our assertions proved above ($p \leq 1$ case) are true even under general assumption that f is a subharmonic function in the unit ball **B**.

Remark 10. Using approaches we develop in this paper and the previous remark (not sharp) assertions of the type Trace $X \subset Y$ or $Y \subset$Trace X for H^p Hardy classes, weighted Hardy classes, some mixed norm spaces and so-called Bergman-Nevanlinna classes (see [**4**] Chapter 4) can be also obtained.

Remark 11. Traces of mixed norm type analogues of Bergman type classes on polyballs we considered in this paper:

$$\|f\|_{p_1,\dots,p_m} = \left(\int_{\mathbf{B}} \cdots \left(\int_{\mathbf{B}} |f(w_1,...w_m)|^{p_1}(1-|w_1|)^{\alpha_1} \mathrm{d}\nu\right)^{p_2/p_1} \cdots\right)^{p_m/p_{m-1}}$$

can be also described with the help of approaches we develop in this note with some restrictions on $p_1, \dots, p_m$. (See [**11**]).

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