Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 224–235. doi:10.2298/AADM0902224O

FIXED POINT INDEX FOR COMPOSITE TYPE MAPS

Donal O'Regan

An index theory is presented for compact absorbing contractive J^c or SJ^c maps and several new fixed point theorems are given for such maps.

1. INTRODUCTION

In this paper we present the fixed point index for compact absorbing contractive J^c maps. In addition we introduce a new class of maps (motivated from [**7**, **9**]) which we call the SJ^c maps. The index for J^c maps in the literature [**4**] has the advantage that no knowledge of homology theory is involved to construct the index. To construct the index for compact absorbing contractions we will only need a knowledge of homology theory to consider our final property, the normalization property. The results here improve those in the literature; see [**1**–**6**, **8**, **10**–**15**] and the references therein.

In this paper we consider maps with nonempty compact values. Let A be a compact subset of a metric space X. A is called ∞–proximally connected in X if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any $n = 1, 2, \dots,$ and any map $g : \partial \triangle^n \to N_\delta(A)$ there exists a map $g' : \triangle^n \to N_\epsilon(A)$ such that $g(x) = g'(x)$ for $x \in \partial \triangle^n$; here $\triangle^n$ is the n–dimensional standard simplex and $N_\epsilon(A) = \{x \in X : \operatorname{dist}(x, A) < \epsilon\}$. Let X and Y be two metric spaces and $F : X \to 2^Y$. We say $F \in J(X, Y)$ if F is upper semicontinuous with nonempty, compact, ∞–proximally connected values. If Z is another metric space and $F \in J(X, Y)$ with $r : Z \to X$ continuous, then it is well known that $F \circ r \in J(Z, Y)$.

Let X and Y be metric spaces. A decomposition $(F_1, \dots, F_n)$ of a upper semicontinuos map $F : X \to K(Y)$ (here $K(Y)$ denotes the family of nonempty compact subsets of Y) is a sequence of maps

$$X = X_0 \overset{F_1}{\to} X_1 \overset{F_2}{\to} X_2 \overset{F_3}{\to} \cdots \overset{F_{n-1}}{\to} X_{n-1} \overset{F_n}{\to} X_n = Y,$$

2000 Mathematics Subject Classification. 47H10.
Keywords and Phrases. Index theory, compact absorbing contractions.

where $F_i \in J(X_{i-1}, X_i)$, $F = F_n \circ \dots \circ F_1$. One can say that the map F is determined by the decomposition $(F_1, \dots, F_n)$. The number n is said to be the length of the decomposition $(F_1, \dots F_n)$. We will denote the class of decompositions by $\mathcal{D}(X, Y)$. We write $F \in J^c(X, Y)$ if $F : X \to K(Y)$ is upper semicontinuous and is determined by a decomposition $(F_1, \dots, F_n)$.

Remark 1.1. Of course a map in J^c may have different decompositions but for the index theory we will specify the map and the decomposition.

An upper semicontinuous map $F : X \to K(Y)$ is said to belong to $SJ^c(X, Y)$ provided it admits a selector $G \in J^c(X, Y)$ (which is determined by a decomposition $(G_1, \dots, G_n) \in \mathcal{D}(X, Y)$).

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\widetilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\widetilde{f} : \widetilde{E} \to \widetilde{E}$. We call f admissible if $\dim \widetilde{E} < \infty$; for such f we define the generalized trace $\mathrm{Tr}\,(f)$ of f by putting $\mathrm{Tr}\,(f) = \mathrm{tr}\,(\widetilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q \,\mathrm{Tr}\,(f_q).$$

A linear map $f : E \to E$ of a vector space E into itself is called weakly nilpotent provided for every $x \in E$ there exists n_x such that $f^{n_x}(x) = 0$.

Assume that $E = \{E_q\}$ is a graded vector space and $f = \{f_q\} : E \to E$ is an endomorphism. We say that f is weakly nilpotent iff f_q is weakly nilpotent for every q.

It is well known [**9**, pp 53] that any weakly nilpotent endomorphism $f : E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) For each $x \in X$, the set $p^{-1}(x)$ is acyclic,
(ii) p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there is a homeomorphism $f : \Gamma \to \Gamma'$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ f^{-1} = q'$ and $p \circ f^{-1} = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ. A multivalued map $\phi : X \to 2^Y$ is said to be determined by a morphism $\{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\}$ provided $\phi(x) = q\,p^{-1}(x)$ for each $x \in X$; the morphism which determines ϕ is also denoted by ϕ. Note of course if we take (p,q) and (p',q') from $\phi = [(p,q)]$ then since $(p,q) \sim (p',q')$ we have $qp^{-1} = q' \circ f \circ f^{-1} \circ (p')^{-1} = q'(p')^{-1}$ with f as described above. Note a multivalued map determined by a morphism is upper semicontinuous and compact valued. Finally note every morphism determines a multivalued map but not conversely.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p,q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p,q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map,

and

(ii) $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Definition 1.1. *A upper semicontinuous map $\phi : X \to Y$ is said to be strongly admissible* [**9**] (*and we write $\phi \in \mathrm{Ads}\,(X,Y)$*) *provided there exists a selected pair (p,q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$.*

When we talk about $\phi \in \mathrm{Ads}$ it is assumed that we are also considering a specified selected pair (p,q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$.

Remark 1.2. Note the class J^c is contained in the class Ads (see [**4**, pp 463]). In fact J is contained in the class of acyclic maps.

Definition 1.2. *A map $\phi \in \mathrm{Ads}\,(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.*

Remark 1.3. In fact since we specify the pair (p,q) of ϕ it is enough to say ϕ is a Lefschetz map if $\phi_\star = q_\star\, p_\star^{-1} : H(X) \to H(X)$ is a Leray endomorphism. However for the examples of ϕ, X known in the literature [**9**] the more restrictive condition in Definition 1.2 works. We note [**9**, pp 227] that $\phi_\star$ does not depend on the choice of diagram from $[(p,q)]$, so in fact we could specify the morphism.

If $\phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [**9**]) $\mathbf{\Lambda}(\phi)$ (or $\mathbf{\Lambda}_X(\phi)$) by

$$\mathbf{\Lambda}(\phi) = \Lambda(q_\star\, p_\star^{-1}).$$

If we do not wish to specify the selected pair (p, q) of ϕ then we would consider the Lefschetz set $\mathbf{\Lambda}(\phi) = \{\Lambda(q_\star\, p_\star^{-1}) : \phi = q\,(p^{-1})\}$.

Definition 1.3. *A Hausdorff topological space X is said to be a Lefschetz space provided every compact $\phi \in \mathrm{Ads}\,(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies ϕ has a fixed point.*

The following concepts will be needed in Section 3. Let (X, d) be a metric space and S a nonempty subset of X. For $x \in X$ let $d(x, S) = \inf_{y \in S} d(x, y)$. Also $\operatorname{diam} S = \sup\{d(x, y) : x, y \in S\}$. We let $B(x, r)$ denote the open ball in X centered at x of radius r and by $B(S, r)$ we denote $\cup_{x\in S} B(x, r)$. For two nonempty subsets S_1 and S_2 of X we define the generalized Hausdorff distance H to be

$$H(S_1, S_2) = \inf\{\epsilon > 0 : \ S_1 \subseteq B(S_2, \epsilon),\ S_2 \subseteq B(S_1, \epsilon)\}.$$

Now suppose $G : S \to 2^X$. Then G is said to be hemicompact if each sequence $\{x_n\}_{n\in\mathbb{N}}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha\in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\Big\{x = (x_\alpha) \in \prod_{\alpha\in I} E_\alpha : \ x_\alpha = \pi_{\alpha,\beta}(x_\beta)\ \forall\, \alpha, \beta \in I,\ \alpha \leq \beta\Big\}$$

is a closed subset of $\prod\limits_{\alpha\in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha\in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [**1**, **2**] $\bigcap\limits_{\alpha\in I} E_\alpha$.)

2. INDEX THEORY

A map $F \in J^c(X, X)$ determined by $(F_1, \ldots, F_k) \in \mathcal{D}(X, X)$ is said to be a compact absorbing contraction (written $F \in JCAC(X, X)$ or $F \in JCAC(X)$) if there exists $Y \subseteq X$ such that

(i) $F(Y) \subseteq Y$,
(ii) $F|_Y$ (which of course is determined by the restriction of $(F_1, \ldots, F_k) \in \mathcal{D}(X, X)$ to the subset Y) is a compact map with Y an open ANR,
(iii) for every $x \in X$ there exists an integer $n = n(x)$ such that $F^{n(x)}(x) \subseteq Y$.

We are now in a position to define the index. Let X be a metric space, W open in X, $F \in JCAC(X, X)$ (with F determined by $(F_1, \ldots, F_k) \in \mathcal{D}(X, X)$

and Y as described above) and $x \notin F\,x$ for $x \in \partial W$ (here ∂W denotes the boundary of W in X). We now define

$$i\,(X, F, W) = \text{ind}\,(Y, F|_Y, W \cap Y)$$

where ind is as described in [**4**, pp 475] (note $W \cap Y$ is an open subset of the ANR Y and note $x \in F\,x$ for $x \in \partial_Y(W \cap Y)$ since $\partial_Y(W \cap Y) = \overline{(W \cap Y)}^Y \backslash (W \cap Y) \subseteq (\overline{W} \cap Y) \backslash (W \cap Y) \subseteq \overline{W} \backslash W = \partial W$). It is worthwhile remarking that if there exists $x \in X$ with $x \in F\,x$ then from (iii) above we have $x \in Y$.

Our definition is independent of our choice of Y. To see this let (i), (ii) and (iii) above hold with Y replaced by Y_1. First note $Y_1 \cap Y$ is an ANR since it is an open subset of the ANR Y. Also note $F(W \cap Y \cap Y_1) \subseteq Y_1 \cap Y$ since $F(Y_1) \subseteq Y_1$ and $F(Y) \subseteq Y$. Now from the contraction property we have

$$\begin{aligned} \text{ind}\,(Y, F, Y_1 \cap Y \cap W) &= \text{ind}\,(Y_1 \cap Y, F, (Y_1 \cap Y) \cap (Y_1 \cap Y \cap W)) \\ &= \text{ind}\,(Y_1 \cap Y, F, Y_1 \cap Y \cap W) \end{aligned}$$

and from the localization property, since $(Y_1 \cap W) \cap Y$ is open in Y and $(Y_1 \cap W) \cap Y \subseteq Y \cap W \subseteq Y$, we have

$$\text{ind}\,(Y, F, Y \cap W) = \text{ind}\,(Y, F, Y_1 \cap Y \cap W).$$

Thus

$$\text{ind}\,(Y, F, Y \cap W) = \text{ind}\,(Y_1 \cap Y, F, Y_1 \cap Y \cap W). \tag{2.1}$$

Similarly from the contraction property we have

$$\begin{aligned} \text{ind}\,(Y_1, F, Y_1 \cap Y \cap W) &= \text{ind}\,(Y_1 \cap Y, F, (Y_1 \cap Y) \cap (Y_1 \cap Y \cap W)) \\ &= \text{ind}\,(Y_1 \cap Y, F, Y_1 \cap Y \cap W) \end{aligned}$$

and from the localization property, since $(Y \cap W) \cap Y_1$ is open in Y_1 and $(Y \cap W) \cap Y_1 \subseteq Y_1 \cap W \subseteq Y_1$, we have

$$\text{ind}\,(Y_1, F, Y_1 \cap W) = \text{ind}\,(Y_1, F, Y_1 \cap Y \cap W).$$

Thus

$$\text{ind}\,(Y_1, F, Y_1 \cap W) = \text{ind}\,(Y_1 \cap Y, F, Y_1 \cap Y \cap W). \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$\text{ind}\,(Y_1, F, Y_1 \cap W) = \text{ind}\,(Y, F, Y \cap W).$$

Now we discuss some properties of the index.

Property I. (Additivity) *Let W be an open subset of X, $F \in JCAC(X, X)$ (determined by $(F_1, \ldots, F_k) \in \mathcal{D}(X, X)$) and assume $W_1 \subseteq W$, $W_2 \subseteq W$ are disjoint open sets with* $\text{Fix}\,F|_{\overline{W}} \subseteq W_1 \cap W_2$. *Then*

$$i(X, F, W) = i(X, F, W_1) + i(X, F, W_2).$$

Proof. Let Y be as described above. Then the additivity property of ind (see [**4**, pp 476], note $W_1 \cap Y$ and $W_2 \cap Y$ are open in Y and disjoint and $\operatorname{Fix} F|_{\overline{(W\cap Y)}^Y} \subseteq (W_1 \cap Y) \cup (W_2 \cap Y))$ we have

$$\begin{aligned} i(X,F,W) &= \operatorname{ind}(Y,F,W\cap Y) = \operatorname{ind}(Y,F,W_1\cap Y) + \operatorname{ind}(Y,F,W_2\cap Y) \\ &= i(X,F,W_1) + i(X,F,W_2). \end{aligned}$$

□

The following three properties are also immediate.

Property II. (Localization) *Let W and W_1 be open subsets of X with $W_1 \subseteq W$ and let $F \in JCAC(X,X)$ (determined by $(F_1,\dots,F_k) \in \mathcal{D}(X,X)$) with* $\operatorname{Fix} F|_{\overline{W}} \subseteq W_1$. *Then*

$$i(X,F,W) = i(X,F,W_1).$$

Property III. (Existence) *Let W be an open subset of X, $F \in JCAC(X,X)$ (determined by $(F_1,\dots,F_k) \in \mathcal{D}(X,X)$) and* $\operatorname{Fix} F \cap \partial W = \emptyset$. *If $i(X,F,W) \neq 0$ then F has a fixed point in W.*

Property IV. (Excision) *Let W be an open subset of X, $F \in JCAC(X,X)$ (determined by $(F_1,\dots,F_k) \in \mathcal{D}(X,X)$) with* $\operatorname{Fix} F \subseteq W$. *Then*

$$i(X,F,W) = i(X,F,X).$$

Other properties (see [**4**]), for example the homotopy property, could also be formulated. We leave these to the reader.

Finally we discuss the normalization property. For this we need to use homological arguments (see also Remark 1.2).

Property V. (Normalization) *Consider $F \in JCAC(X,X)$ (determined by $(F_1,\dots,F_k) \in \mathcal{D}(X,X)$). Then*

$$i(X,F,X) = \mathbf{\Lambda}(F) = \Lambda(q_\star\, p_\star^{-1})$$

where (p,q) is the selected pair of F, with $F(x) = q\,p^{-1}(x)$, determined by $(F_1,\dots,F_k)$.

Proof. Let Y be as described above. Consider $F|_Y$ and let q', $p' : p^{-1}(Y) \to Y$ be given by $p'(u) = p(u)$ and $q'(u) = q(u)$. Notice (p',q') is a selected pair for $F|_Y$. Now since Y is an ANR (so in particular a Lefschetz space; see Definition 1.3) space then $q'_\star\,(p')_\star^{-1}$ is a Leray endomorphism. For any compact $K \subseteq X$ there exists an n with $F^n(K) \subseteq Y$ (see [**15**]). Now [**9**, Proposition 42.2, pp 208] guarantees (since we consider Čech homology with compact carriers) that the homeomorphism

$$q''_\star\,(p'')_\star^{-1} : H(X,Y) \to H(X,Y)$$

is weakly nilpotent (here p'', $q'' : (\Gamma, p^{-1}(Y)) \to (X,Y)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$). Then [**9**, pp 53] guarantees that $q''_\star\,(p'')_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q''_\star\,(p'')_\star^{-1}) = 0$. Also [**9**, Property 11.5, pp 52] guarantees that $q_\star\, p_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q_\star\, p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$. Now [**4**, pp 477] guarantees that

$$\mathrm{ind}\,(Y, F, Y) = \Lambda\,(q'_\star\,(p')_\star^{-1})$$

so

$$\mathbf{\Lambda}\,(F) = \Lambda\,(q_\star\, p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1}) = \mathrm{ind}\,(Y, F, Y) = i(X, F, X). \qquad \square$$

Let X be a metric space. A map $F \in SJ^c(X, X)$ is said to be a compact absorbing contraction (written $F \in SJAC(X, X)$ or $F \in SJCAC(X)$) if every selector $G \in J^c(X, X)$ of F determined by $(G_1, \ldots, G_n) \in \mathcal{D}(X, X)$ is such that $G \in JCAC(X, X)$.

Note from the proof of Property V we have that $\mathbf{\Lambda}\,(G)$ is defined and as a result we can define the Lefschetz set $\mathbf{\Lambda}\,(F)$ by

$$\mathbf{\Lambda}\,(F) = \{\mathbf{\Lambda}(G) : \; G \in J^c(X, X),\; G \text{ is determined by a decomposition } (G_1, \ldots, G_k) \in \mathcal{D}(X, X) \text{ and } G \text{ is a selector of } F\}.$$

Let W be an open subset of X with $\mathrm{Fix}\, F \cap \partial W = \emptyset$. We now define

$$i(X, F, W) = \{i(X, G, W) : \; G \in J^c(X, X) \text{ is a selector of } F \text{ determined by } (G_1, \ldots, G_n) \in \mathcal{D}(X, X) \text{ and } G \in JCAC(X, X)\}.$$

The following properties (we just list a few) are immediate.

Property I. *Let W be an open subset of X, $F \in SJCAC(X, X)$ and $\mathrm{Fix}\, F \cap \partial W = \emptyset$. If $i(X, F, W) \neq \{0\}$ then F has a fixed point in W.*

Property II. *Let $F \in SJCAC(X, X)$. Then*

$$i(X, F, X) = \mathbf{\Lambda}\,(F).$$

3. FIXED POINT THEORY IN FRÉCHET SPACES.

We now present another approach to establishing fixed points based on projective limits (see [**1**, **2**]). We just state the results in this section since the proofs are essentially those in [**15**]. Let $E = (E, \{|\cdot|_n\}_{n \in \mathbb{N}})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in \mathbb{N}\}$; here $\mathbb{N} = \{1, 2, \ldots\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \cdots \quad \text{for every} \quad x \in E. \tag{3.1}$$

A subset X of E is bounded if for every $n \in \mathbb{N}$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x, r) = \{y \in E : |x - y|_n \leq r\, \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x - y|_n = 0. \tag{3.2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (3.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (3.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now $\mu_{n,m}\,\mu_{m,k} = \mu_{n,k}$ if $n \leq m \leq k$ and $\mu_n = \mu_{n,m}\,\mu_m$ if $n \leq m$. We now assume the following condition holds:

$$(3.3)\qquad \begin{cases} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases}$$

Remark 3.1. (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.
(ii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessarily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally we assume

$$(3.4)\qquad \begin{cases} E_1 \supseteq E_2 \supseteq \cdots \text{ and for each } n \in \mathbb{N}, \\ |j_n\,\mu_{n,n+1}\,j_{n+1}^{-1}\,x|_n \leq |x|_{n+1}\,\forall\, x \in E_{n+1} \end{cases}$$

(here we use the notation from [**1**, **2**] i.e. decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [**1**, **2**]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\,\mu_{n,m}\,j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in \mathbb{N}$ we set $X_n = j_n\,\mu_n(X)$, and we let $\overline{X_n}$, $int\,X_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$\text{pseudo-int}\,(X) = \{x \in X : \ j_n\,\mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } \ n \in \mathbb{N}\}.$$

The set X is pseudo-open if $X = \text{pseudo-int}\,(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x,r) = \{y \in E_n : |x-y|_n \leq r\}$.

Let $M \subseteq E$ and consider the map $F : M \to 2^E$. Assume for each $n \in \mathbb{N}$ and $x \in M$ that $j_n\,\mu_n\,F(x)$ is closed. Let $n \in \mathbb{N}$ and $M_n = j_n\,\mu_n(M)$. Since we first consider Volterra type operators we assume (note this assumption is only needed in Theorems 3.1 and 3.2)

$$(3.5)\qquad \text{if } x, y \in E \text{ with } |x-y|_n = 0 \text{ then } H_n(F\,x, F\,y) = 0;$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively we could assume $\forall n \in \mathbf{N}, \forall x, y \in M$ if $j_n\,\mu_n\,x = j_n\,\mu_n\,y$ then $j_n\,\mu_n\,F\,x = j_n\,\mu_n\,F\,y$ and of course here we do not need to assume that $j_n\,\mu_n\,F(x)$ is closed for each

$n \in \mathbb{N}$ and $x \in M$). Now (3.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y \in M_n$ there exists a $x \in M$ with $y = j_n\, \mu_n(x)$ and we let

$$F_n\, y = j_n\, \mu_n\, F\, x$$

(we could of course call it $F\, y$ since it is clear in the situation we use it); note $F_n : M_n \to C(E_n)$ and note if there exists a $z \in M$ with $y = j_n\, \mu_n(z)$ then $j_n\, \mu_n\, F\, x = j_n\, \mu_n\, F\, z$ from (3.5) (here $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In this paper we assume F_n will be defined on $\overline{M_n}$ i.e. we assume the F_n described above admits an extension (again we call it F_n) $F_n : \overline{M_n} \to 2^{E_n}$ (we will assume certain properties on the extension).

Now we present some fixed point theorems in Fréchet. Our first two results are motivated by Volterra type operators (the proof follows closely that in [**15**]).

Theorem 3.1. *Let E and E_n be as described above, X a subset of E, U a pseudo-open subset of E and $F : Z \to 2^E$ with $Z \subseteq E$, and $X_n \subseteq Z_n$ for each $n \in \mathbb{N}$. Also assume for each $n \in \mathbb{N}$ and $x \in Z$ that $j_n\, \mu_n\, F\,(x)$ is closed and in addition for each $n \in \mathbb{N}$ that $F_n : X_n \to 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

$$\text{for each } n \in \mathbb{N},\ F_n \in SJCAC(X_n, X_n), \tag{3.6}$$

$$\begin{cases} \text{for each } n \in \mathbb{N},\ F_n \text{ has no fixed points in } \partial\, W_n;\ \text{here} \\ W_n = U_n \cap X_n \text{ and } \partial W_n \text{ denotes the boundary of } W_n \text{ in } X_n \end{cases} \tag{3.7}$$

$$\text{for each } n \in \mathbb{N},\ i(X_n, F_n, W_n) \neq \{0\} \tag{3.8}$$

and

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in W_n \text{ solves } y \in F_n\, y \\ \text{in } E_n \text{ then } j_k\, \mu_{k,n}\, j_n^{-1}\,(y) \in W_k \text{ for } k \in \{1, \dots, n-1\}. \end{cases} \tag{3.9}$$

In addition assume either

$$\text{for each } n \in \mathbb{N},\ F_n : W_n \to 2^{E_n} \text{ is hemicompact} \tag{3.10}$$

or

$$\text{for each } n \in \mathbb{N},\ F_n : \overline{W_n} \to 2^{E_n} \text{ is hemicompact} \tag{3.11}$$

hold. Then F has a fixed point in E.

Remark 3.2. We can replace (3.9) in Theorem 3.1 with

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in W_n \text{ solves } y \in F_n\, y \\ \text{in } E_n \text{ then } j_k\, \mu_{k,n}\, j_n^{-1}\,(y) \in \overline{W_k} \text{ for } k \in \{1, \dots, n-1\} \end{cases}$$

provided (3.11) holds.

REMARK 3.3. In Theorem 3.1 it is possible to replace $X_n \subseteq Z_n$ with X_n a subset of the closure of Z_n in E_n provided Z is a closed subset of E so in this case we could have $Z = X$ if X is closed.

Theorem 3.2. *Let E and E_n be as described above, X a subset of E, U a pseudo-open subset of E and $F : Z \to 2^E$ with $Z \subseteq E$, and $\overline{X_n} \subseteq Z_n$ for each $n \in N$. Also assume for each $n \in N$ and $x \in Z$ that $j_n\,\mu_n\,F(x)$ is closed and in addition for each $n \in N$ that $F_n : \overline{X_n} \to 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

$$\text{for each } n \in \mathbb{N},\ F_n \in SJCAC(\overline{X_n}, \overline{X_n}) \tag{3.12}$$

$$\begin{cases} \text{for each } n \in \mathbb{N},\ F_n \text{ has no fixed points in } \partial W_n;\ \text{here} \\ W_n = U_n \cap \overline{X_n} \text{ and } \partial W_n \text{ denotes the boundary of } W_n \text{ in } \overline{X_n} \end{cases} \tag{3.13}$$

and

$$\text{for each } n \in \mathbb{N},\ i(\overline{X_n}, F_n, W_n) \neq \{0\}. \tag{3.14}$$

Also assume (3.9) *and either* (3.10) *or* (3.11) *hold. Then F has a fixed point in E.*

Our next two results are motivated by Urysohn type operators (again the proofs follow closely those in [**15**]). In this case the map F_n will be related to F by the closure property (3.20).

Theorem 3.3. *Let E and E_n be as described above, X a subset of E, U a pseudo-open subset of E and $F : Z \to 2^E$ with $Z \subseteq E$, and $X_n \subseteq Z_n$ for each $n \in \mathbb{N}$. Also for each $n \in \mathbb{N}$ assume there exists $F_n : X_n \to 2^{E_n}$ and suppose the following conditions are satisfied:*

$$\text{for each } n \in \mathbb{N},\ F_n \in SJCAC(X_n, X_n) \tag{3.15}$$

$$\begin{cases} \text{for each } n \in \mathbb{N},\ F_n \text{ has no fixed points in } \partial W_n;\ \text{here} \\ W_n = U_n \cap X_n \text{ and } \partial W_n \text{ denotes the boundary of } W_n \text{ in } X_n \end{cases} \tag{3.16}$$

$$\text{for each } n \in \mathbb{N},\ i(X_n, F_n, W_n) \neq \{0\} \tag{3.17}$$

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in W_n \text{ solves } y \in F_n\,y \text{ in } E_n \\ \text{then } j_k\,\mu_{k,n}\,j_n^{-1}(y) \in W_k \text{ for } k \in \{1, \dots, n-1\} \end{cases} \tag{3.18}$$

$$\begin{cases} \text{for any sequence } \{y_n\}_{n\in\mathbb{N}} \text{ with } y_n \in W_n \\ \text{and } y_n \in F_n\,y_n \text{ in } E_n \text{ for } n \in \mathbb{N} \text{ and} \\ \text{for every } k \in \mathbb{N} \text{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \dots\},\ N_k \subseteq N_{k-1} \text{ for} \\ k \in \{1, 2, \dots\},\ N_0 = N, \text{ and a } z_k \in \overline{W_k} \text{ with} \\ j_k\,\mu_{k,n}\,j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k \end{cases} \tag{3.19}$$

and

$$(3.20)\qquad \begin{cases} \textit{if there exists a } w \in \mathbb{Z} \textit{ and a sequence } \{y_n\}_{n\in\mathbb{N}} \\ \textit{with } y_n \in W_n \textit{ and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in \mathbb{N} \textit{ there exists a subsequence } S \subseteq \\ \{k+1, k+2, \ldots\} \textit{ of } \mathbb{N} \textit{ with } j_k\, \mu_{k,n}\, j_n^{-1}(y_n) \to j_k\, \mu_k(w) \\ \textit{in } E_k \textit{ as } n \to \infty \textit{ in } S, \textit{ then } w \in F\, w \textit{ in } E. \end{cases}$$

Then F has a fixed point in E.

REMARK 3.4. Notice to check (3.19) we need to show that for each $k \in \mathbb{N}$ the sequence $\{j_k\, \mu_{k,n}\, j_n^{-1}(y_n)\}_{n\in N_{k-1}} \subseteq \overline{W_k}$ is sequentially compact.

REMARK 3.5. Condition (3.18) can be removed from the statement of Theorem 3.3. We include it only to explain condition (3.19) (see Remark 3.4).

REMARK 3.6. Note we could replace $X_n \subseteq Z_n$ above with X_n a subset of the closure of Z_n in E_n if Z is a closed subset of E (so in this case we can take $Z = X$ if X is a closed subset of E).

REMARK 3.7. In fact we could replace (3.18) in Theorem 3.3 with

$$\begin{cases} \text{for each } n \in \{2,3,\ldots\} \text{ if } y \in W_n \text{ solves } y \in F_n\, y \text{ in } E_n \\ \text{then } j_k\, \mu_{k,n}\, j_n^{-1}(y) \in \overline{W_k} \text{ for } k \in \{1,\ldots,n-1\} \end{cases}$$

and the result above is again true.

Theorem 3.4. *Let E and E_n be as described above, X a subset of E, U a pseudo-open subset of E and $F : Z \to 2^E$ with $Z \subseteq E$, and $\overline{X_n} \subseteq Z_n$ for each $n \in N$. Also for each $n \in \mathbb{N}$ assume there exists $F_n : \overline{X_n} \to 2^{E_n}$ and suppose the following conditions are satisfied*:

$$(3.21)\qquad \textit{for each } n \in \mathbb{N},\ F_n \in SJCAC(\overline{X_n}, \overline{X_n})$$

$$(3.22)\qquad \begin{cases} \textit{for each } n \in \mathbb{N},\ F_n \textit{ has no fixed points in } \partial W_n;\textit{ here} \\ W_n = U_n \cap \overline{X_n} \textit{ and } \partial W_n \textit{ denotes the boundary of } W_n \textit{ in } \overline{X_n} \end{cases}$$

and

$$(3.23)\qquad \textit{for each } n \in \mathbb{N},\ i(\overline{X_n}, F_n, W_n) \neq \{0\}.$$

In addition assume (3.18)*,* (3.19) *and* (3.20) *hold. Then F has a fixed point in E.*

REFERENCES

1. R. P. AGARWAL, D. O'REGAN: *A Lefschetz fixed point theorem for admissible maps in Fréchet spaces.* Dynamic Systems and Applications, **16** (2007), 1–12.
2. R. P. AGARWAL, D. O'REGAN: *Fixed point theory for compact absorbing contractive admissible type maps.* Applicable Analysis, **87** (2008), 497–508.

3. R. P. Agarwal, D. O'Regan: *Fixed point theory for various classes of permissible maps via index theory.* Comm. Korean Math. Soc., to appear.
4. R. Bader, W. Kryszewski: *Fixed point index for compositions of set valued maps with proximally ∞-connected values on arbitrary ANR's.* Set Valued Analysis, **2** (1994), 459–480.
5. H. Ben-El-Mechaiekh: *Spaces and maps approximation and fixed points.* Jour. Computational and Appl. Mathematics, **113** (2000), 283–308.
6. H. Ben-El-Mechaiekh, P. Deguire: *General fixed point theorems for non–convex set valued maps.* C. R. Acad. Sci. Paris, **312** (1991), 433–438.
7. Z. Dzedzej: *Fixed point index for a class of nonacyclic multivalued maps.* Diss. Math., **253** (1985), 1–58.
8. G. Fournier, L. Gorniewicz: *The Lefschetz fixed point theorem for multi-valued maps of non-metrizable spaces.* Fundamenta Mathematicae, **92** (1976), 213–222.
9. L. Gorniewicz: *Topological Fixed Point Theory of Multivalued Mappings.* Kluwer Acad. Publishers, Dordrecht, 1999.
10. L. Gorniewicz, A. Granas: *Some general theorems in coincidence theory.* J. Math. Pures et Appl., **60** (1981), 361–373.
11. A. Granas: *Fixed point theorems for approximative ANR's.* Bull. Acad. Polon. Sc., **16** (1968), 15–19.
12. A. Granas, J. Dugundji: *Fixed Point Theory.* Springer, New York, 2003.
13. D. O'Regan: *Fixed point theory on extension type spaces and essential maps on topological spaces.* Fixed Point Theory and Applications, **2004** (2004), 13-20.
14. D. O'Regan: *Asymptotic Lefschetz fixed point theory for ANES(compact) maps.* Rendiconti del Circolo Matematico di Palermo, **58** (2009), 87–97.
15. D. O'Regan: *Index and fixed point theory for compact absorbing contractive permissible maps*, to appear.

Department of Mathematics
National University of Ireland,
Galway,
Ireland
E–mail: donal.oregan@nuigalway.ie

(Received May 12, 2009)