Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 359–370. doi:10.2298/AADM0902359A

FRACTIONAL EDGE DOMINATION IN GRAPHS

S. Arumugam, Sithara Jerry

Let $G = (V, E)$ be a graph. A function $f : E \to [0, 1]$ is called an *edge dominating function* if $\sum_{x \in N[e]} f(x) \geq 1$ for all $e \in E(G)$, where $N[e]$ is the closed neighbourhood of the edge e. An edge dominating function f is called minimal (MEDF) if for all functions $g : E \to [0, 1]$ with $g < f$, g is not an edge dominating function. The fractional edge domination number γ'_f and the upper fractional edge domination number Γ'_f are defined by

$$\gamma'_f(G) = \min\{|f| : f \text{ is an MEDF of } G\} \text{ and}$$
$$\Gamma'_f(G) = \max\{|f| : f \text{ is an MEDF of } G\},$$

where $|f| = \sum_{e \in E} f(e)$. Further we introduce the fractional parameters corresponding to edge irredundance and edge independence, leading to the fractional edge domination chain. We also consider topological properties of the set of all edge dominating functions of G.

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [**4**]. The order and size of G are denoted by n and m respectively.

A dominating set of $G = (V, E)$ is a subset S of V such that every vertex of $V - S$ is adjacent to a vertex in S. A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)\big(\Gamma(G)\big)$. A subset S of V is called a 2-packing if $N[u] \cap N[v] = \emptyset$ for all $u, v \in S$. The maximum cardinality of

2000 Mathematics Subject Classification. 05C69.
Keywords and Phrases. Edge dominating function, edge irredundant function, edge independent function.

a 2-packing in G is called the 2-packing number of G and is denoted by $P_2(G)$. A comprehensive treatment of the fundamentals of domination is given in Haynes et al. [**13**].

Hedetniemi et al. [**14**] introduced the concept of dominating function and fractional domination in graphs. A dominating function (DF) of a graph $G = (V, E)$ is a function $f : V \to [0, 1]$ such that

$$\sum_{x \in N[v]} f(x) \geq 1$$

for all $v \in V$, where $N[v]$ is the closed neighbourhood of v. A DF f is called minimal (MDF) if there is no function $g : V \to [0, 1]$ such that $g < f$ and g is a DF. For a DF f of G we denote $\sum\limits_{x \in N[v]} f(x)$ by $f(N[v])$. The boundary set B_f and the positive set P_f of a DF f are defined by

$$B_f = \{v \in V : f(N[v]) = 1\} \text{ and } P_f = \{v \in V : f(v) > 0\}.$$

Let A and B be subsets of V. We say that A dominates B and write $A \to B$ if every vertex in $B - A$ is adjacent to some vertex in A. The following theorem gives a simple necessary and sufficient condition for a DF to be an MDF.

Theorem 1.1. [**5**] *A* DF f *of* G *is an* MDF *if and only if* $B_f \to P_f$.

For any DF f, let $|f| = \sum\limits_{v \in V} f(v)$. The fractional domination number $\gamma_f(G)$ is defined by

$$\gamma_f(G) = \min\{|f| : f \text{ is an MDF of } G\}.$$

Theorem 1.2. [**10**, **12**] *If* G *is an* r*-regular graph with* n*-vertices,* $\gamma_f(G) = \dfrac{n}{r+1}$.

The concept of edge domination was introduced by Mitchell and Hedetniemi [**15**]. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G. Further results on edge domination are given in Arumugam and Velammal [**3**].

A subset S of E is called a 2-edge packing if $N[e] \cap N[f] = \emptyset$ for all $e, f \in S$. The maximum cardinality of a 2-edge packing in G is called the 2-edge packing number of G and is denoted by $P_2'(G)$. Functional generalizations for vertex subsets have been extensively studied in literature [**6**, **9**, **16**]. Cockayne and Mynhardt [**8**] have indicated that edge subsets may also be embedded into sets of functions and an analogous concept of convexity could also be developed.

In this paper we introduce the concept of fractional edge domination and initiate a study of the fractional edge domination number. Further we introduce the fractional parameters corresponding to edge irredundance and edge independence, leading to the fractional edge domination chain. We also consider topological properties of the set of all edge dominating functions of G.

We need the following definitions and theorems.

A subset S of V is called a 2-packing in G if $N[u] \cap N[v] = \emptyset$ for all $u, v \in S$. The maximum cardinality of a 2-packing in G is called the 2-packing number of G and is denoted by $P_2(G)$. We observe that for any graph G, $P_2'(G) = P_2\big(L(G)\big)$, where $L(G)$ is the line graph of G.

A graph G is called a block graph if each block of G is a complete subgraph.

Theorem 1.3. [**10**] *For any graph* $G, P_2(G) \leq \gamma(G)$.

Theorem 1.4. [**11**] *For any block graph* $G, P_2(G) = \gamma(G)$.

2. EDGE DOMINATING FUNCTIONS IN GRAPHS

Definition 2.1. *Let* $G = (V, E)$ *be a connected graph. A function* $f : E(G) \to [0,1]$ *is called an edge dominating function* (EDF) *of* G *if* $\sum_{x \in N[e]} f(x) \geq 1$ *for all* $e \in E(G)$. *An* EDF g *of* G *is called a minimal edge dominating function* (MEDF) *if any function* $f : E(G) \to [0,1]$ *with* $f < g$ *is not an* EDF *of* G. *The fractional edge domination number* γ_f' *and the upper fractional edge domination number* Γ_f' *of* G *are defined by*

$$\gamma_f'(G) = \min\{|f| : f \text{ is an MEDF of } G\} \text{ and}$$
$$\Gamma_f'(G) = \max\{|f| : f \text{ is an MEDF of } G\}.$$

Observation 2.2. *Since the characteristic function of a* γ'*-set of* G *is an* MEDF *of* G, *it follows that* $\gamma_f' \leq \gamma'$.

Observation 2.3. *The problem of finding the fractional edge domination number is equivalent to finding the optimal solution of the following linear programming problem.*

Minimize $z = \sum_{e \in E(G)} f(e)$

Subject to $\sum_{x \in N[e]} f(x) \geq 1$ *for all* $e \in E(G)$ *and* $0 \leq f(e) \leq 1$ *for all* $e \in E(G)$.

Definition 2.4. *A function* $g : E \to [0,1]$ *is called an edge packing function if for every* $e \in E(G)$, $g(N[e]) \leq 1$. *An edge packing function* g *of* G *is called a maximal edge packing function if any function* $f : E(G) \to [0,1]$ *with* $f > g$ *is not an edge packing function of* G. *The fractional edge packing number,* $p_f'(G)$ *and the upper fractional edge packing number* $P_f'(G)$ *are defined as follows.*

$$p_f'(G) = \min\{|g| : g \text{ is a maximal edge packing function of } G\} \text{ and}$$
$$P_f'(G) = \max\{|g| : g \text{ is a maximal edge packing function of } G\}.$$

Observation 2.5. *The problem of finding the upper fractional edge packing number is equivalent to finding the optimal solution of the following linear programming problem.*

Maximize $z = \sum_{i=1}^{m} f(e_i)$

Subject to $\sum_{x \in N[e]} f(x) \leq 1$ *for all* $e \in E(G)$ *and* $0 \leq f(e) \leq 1$ *for all* $e \in E(G)$.

We observe that this L.P.P is the dual of the L.P.P corresponding to the fractional edge domination number γ'_f. Since the optimal solution of the primal problem and its dual are equal, $\gamma'_f(G) = P'_f(G)$ for all graphs G. This fact is very useful in determining the values of these parameters. In fact if we can find an MEDF g and a maximal edge packing function h such that $|g| = |h|$, then $\gamma'_f(G) = P'_f(G) = |g| = |h|$.

Observation 2.6. *For any graph* G, *we have* $\gamma'_f(G) = \gamma_f(L(G))$. *Hence it follows that*

$$\gamma'_f(C_n) = \gamma_f(C_n) = \frac{n}{3} \quad \text{and} \quad \gamma'_f(P_n) = \gamma_f(P_{n-1}) = \left\lceil \frac{n-1}{3} \right\rceil.$$

Observation 2.7. *If* G *is* r*-regular, then its line graph* $L(G)$ *is* $(2r-2)$ *regular and hence it follows from Theorem* 1.2. *that* $\gamma'_f(G) = \frac{m}{2r-1}$. *In particular,* $\gamma'_f(K_n) = \frac{n(n-1)}{2(2n-3)}$.

Observation 2.8. *Since the line graph of any bipartite graph* G *with bipartition* V_1, V_2 *such that* $\deg(v) = r$ *for all* $v \in V_1$ *and* $deg(u) = s$ *for all* $u \in V_2$ *is* $(r+s-2)$ *regular, we have* $\gamma'_f(G) = \frac{m}{r+s-1}$. *In particular* $\gamma'_f(K_{r,s}) = \frac{rs}{r+s-1}$.

Observation 2.9. *For the complete bipartite graph* $K_{r,r}$, *we have* $\gamma' = r$ *and* $\gamma'_f = \frac{r^2}{2r-1}$. *Hence it follows that the difference between* γ' *and* γ'_f *can be made arbitrarily large.*

Theorem 2.10. *If* G *has a* 2*-edge packing* M *such that* $\{N[e] : e \in M\}$ *is a partition of* $E(G)$, *then* $\gamma'_f = P'_f = |M|$.

Proof. Consider the characteristic function $f = \chi_M : E \to \{0, 1\}$. Clearly $f(N[e]) = 1$ for all $e \in E(G)$ and hence f is an MEDF as well as a maximal edge packing function. Hence it follows from Observation 2.5 that $\gamma'_f = P'_f = |f| = |M|$. □

Corollary 2.11. *If* G *is the friendship graph consisting of* k *blocks,* $\gamma'_f = k$.

Proof. Let $V_i = (uv_iw_iu)$ be the blocks of G. Then $M = \{v_iw_i : 1 \leq i \leq k\}$ is a 2-edge packing of G such that $\{N[e] : e \in M\}$ is partition of $E(G)$. □

We now proceed to prove that for a tree T, $\gamma'(T) = P'_2(T)$, which implies $\gamma'(T) = \gamma'_f(T)$. In fact we give two different proofs for this result.

Definition 2.12. *Let* T *be a tree with* $\Delta \geq 3$. *A path* $P = (v_1, v_2, \ldots, v_r)$ *is called a hanging path at* v_r *if* $\deg v_1 = 1, \deg v_r \geq 3$ *and* $\deg v_i = 2$ *for all* $i, 2 \leq i \leq r-1$.

We observe that any tree T with $\Delta \geq 3$ has at least k hanging paths where k is the number of leaves in T. Also there exists a vertex w such that the tree T_1

obtained from T by deleting the vertices of all the hanging paths at w, but retaining w, is either trivial or w is a leaf of T_1.

Theorem 2.13. *For any tree* $T, \gamma'(T) = P_2'(T)$.

Proof. Let T be any tree. Since $L(T)$ is a block graph, it follows from Theorem 1.4 that $\gamma\big(L(T)\big) = P_2\big(L(T)\big)$. Hence $\gamma'(T) = P_2'(T)$.

Alternate proof. If T is the path P_n, then $\gamma'(T) = P_2'(T) = \left\lfloor \dfrac{n+1}{3} \right\rfloor$. Now, suppose $\Delta(T) \geq 3$. Then $|V(T)| \geq \Delta + 1$. We prove the result by induction on $|V(T)|$. If $|V(T)| = \Delta + 1$ and $\Delta \geq 3$, then $T = K_{1,\Delta}$ and $\gamma'(T) = P_2'(T) = 1$. We now assume that result is true for all trees T with $\Delta(T) \geq 3$ and $|V(T)| < n$. Let T be a tree such that T is not a star, $\Delta(T) \geq 3$ and $|V(T)| = n$. If there is a hanging path $P = (v_1, v_2, \ldots, v_r)$ of length at least 3, for the tree $T_1 = T - \{v_1, v_2, v_3\}$, $\gamma'(T_1) = \gamma'(T) - 1$ and $P_2'(T_1) = P_2'(T) - 1$ and hence the result follows by induction.

Now, suppose every hanging path in T has length at most 2. Let w be a vertex of T such that the tree T_1 obtained from T by deleting all the vertices of the hanging paths at w, but retaining w, is either trivial or w is a leaf of T_1.

If T_1 is trivial, then $\gamma'(T) = P_2'(T) = a$, where a is the number of hanging paths of length 2 at w.

Suppose T_1 is nontrivial. If all the hanging paths at w have length 1, then for the tree $T_2 = T - \{x\}$ where x is a leaf adjacent to w, we have $\gamma'(T_2) = \gamma'(T)$ and $P_2'(T_2) = P_2'(T)$. If there exists at least one hanging path of length 2 at w, then for the tree $T_2 = T_1 - \{w\}$, we have $\gamma'(T) \leq \gamma'(T_2) + a$ and $P_2'(T) \geq P_2'(T_2) + a$, where a is the number of hanging paths of length 2 at w. Now by induction $\gamma'(T_2) = P_2'(T_2)$ and hence $\gamma'(T) \leq P_2'(T)$. Also it follows from Theorem 1.3 that $P_2'(T) \leq \gamma'(T)$ and hence $\gamma'(T) = P_2'(T)$. □

Corollary 2.14. *For any tree* T, *we have* $\gamma_f'(T) = \gamma'(T)$.

Proof. Since $\gamma'(T) = P_2'(T) \leq P_f'(T) = \gamma_f'(T) \leq \gamma'(T)$, the result follows. □

Remark 2.15. The second proof of Theorem 2.13 gives a recursive algorithm for determining γ', and hence γ_f', for any tree T. Since for the subtree constructed in the proof both γ' and P_2' get equally reduced, we can continue the process until we reach a path, thus determining $\gamma'(T)$.

We now proceed to determine γ_f' for several classes of graphs. The following lemma is useful in this regard.

Lemma 2.16. *Let* G *be any connected graph and* $G \neq K_{1,n-1}$. *Then there exists an* MEDF f *of* G *such that* $|f| = \gamma_f'(G)$ *and* $f(e) = 0$ *for every pendant edge* e.

Proof. Let g be any MEDF of G with $|g| = \gamma_f'(G)$. Suppose there exists a pendant edge e_1 with $g(e_1) > 0$. Since $G \neq K_{1,n-1}$, there exists a non-pendent edge $f_1 \in N(e_1)$. Now define $g_1 : E(G) \to [0,1]$ by

$$g_1(e_1) = 0, \;\; g_1(f_1) = g(f_1) + g(e_1) \;\text{ and }\; g_1(x) = g(x) \text{ for all } x \in E - \{e_1, f_1\}.$$

If $g_1(f_1) = g(f_1) + g(e_1) > 1$, then $g_2 : E(G) \to [0,1]$ defined by $g_2(f_1) = 1$ and $g_2(x) = g_1(x)$ for all $x \in E - \{f_1\}$ is an EDF of G and $|g_2| < |g_1|$, which is a contradiction. Hence $g_1(f_1) \leq 1$ and g_1 is an EDF of G with $|g_1| = |g| = \gamma'_f(G)$. Continuing this process, we obtain an MEDF f of G such that $|f| = \gamma'_f(G)$ and $f(e) = 0$ for every pendant edge e. □

Theorem 2.17. *For any r-regular graph G of order n with $r > 0$, $\gamma'_f(G \circ K_1) = \frac{n}{2}$, where $G \circ K_1$ is the corona of G obtained by attaching a pendant edge at every vertex of G.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let e_i be the pendant edge of $G \circ K_1$ incident at v_i. We define $f : E(G) \to [0,1]$ by

$$f(e_i) = 0 \text{ and } f(x) = \frac{1}{r} \text{ for all } x \in E(G).$$

Clearly, f is an EDF of $G \circ K_1$ and $|f| = \frac{n}{2}$. Hence $\gamma'_f(G \circ K_1) \leq \frac{n}{2}$.

Now, let f be any minimum EDF of $G \circ K_1$ with $f(e_i) = 0$, $1 \leq i \leq n$. Then $\sum_{i=1}^{n} f(N[e_i]) = \sum_{i=1}^{n} f(e_i) + 2 \sum_{e \in E(G)} f(e) \geq n$. Hence $2|f| \geq n$, so that $|f| \geq \frac{n}{2}$. Thus $\gamma'_f(G \circ K_1) \geq \frac{n}{2}$ and hence $\gamma'_f(G \circ K_1) = \frac{n}{2}$. □

Remark 2.18. In general, for any graph G, $\gamma'_f(G \circ K_1) \geq \frac{n}{2}$. Also there exist non-regular graphs G for which $\gamma'_f(G \circ K_1) > \frac{n}{2}$. For example $\gamma'_f(P_n \circ K_1) = \gamma'(P_n \circ K_1) = \left\lceil \frac{n}{2} \right\rceil > \frac{n}{2}$ if n is odd.

Problem 2.19. *Characterize graphs G for which $\gamma'_f(G \circ K_1) = \frac{|V(G)|}{2}$.*

Let G_1 and G_2 be two graphs with disjoint vertex sets. The graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1)$ and $v \in V(G_2)\}$ is called the join of G_1 and G_2 and is denoted by $G_1 + G_2$.

Theorem 2.20. *For the wheel $W_n = K_1 + C_{n-1}$, we have $\gamma'_f(W_n) = \frac{(n-1)(n-2)}{(3n-7)}$.*

Proof. Let $V(W_n) = \{v_1, v_2, \ldots, v_{n-1}, v\}$ and $E(W_n) = \{x_i = vv_i : 1 \leq i \leq n-1\} \cup \{e_i = v_i v_{i+1} : 1 \leq i \leq n-2\} \cup \{e_{n-1} = v_{n-1} v_1\}$. We define $f : E(W_n) \to [0,1]$ by

$$f(e_i) = \frac{n-3}{3n-7} \text{ and } f(x_i) = \frac{1}{3n-7} \text{ for all } i,\ 1 \leq i \leq n-1.$$

It can be easily verified that $f(N[e]) = 1$ for all $e \in E(W_n)$ and hence f is both an MEDF and a maximal edge packing function of W_n. Hence $\gamma'_f = |f| = \frac{(n-1)(n-2)}{3n-7}$. □

In the following theorem we give a sharp upper bound for γ'_f.

Theorem 2.21. *For any graph* $G, \gamma'_f(G) \le \dfrac{m}{2\delta - 1}$ *and the bound is sharp.*

Proof. We define $f : E(G) \to [0,1]$ by $f(e) = \dfrac{1}{2\delta - 1}$ for all $e \in E(G)$. Clearly, f is an EDF of G, so that $\gamma'_f(G) \le |f| = \dfrac{m}{2\delta - 1}$. Also, if G is any r-regular graph, $\gamma'_f(G) = \dfrac{m}{2r - 1}$ and hence the bound is sharp. □

Observation 2.22. *Let* $G = (V, E)$ *be a graph and let* $A, B \subseteq E$. *We say that* A *dominates* B *if every edge in* $B - A$ *is adjacent to an edge in* A *and we write* $A \to B$. *Now, let* f *be any* EDF *of* G. *The boundary* B'_f *and the positive set* P'_f *of* f *are defined by* $B'_f = \left\{e : \sum_{x \in N[e]} f(x) = 1\right\}$ *and* $P'_f = \{e : f(e) > 0\}$. *It follows from Theorem* 1.1 *and Observation* 2.6 *that an* EDF f *is an* MEDF *of* G *if and only if* $B'_f \to P'_f$.

Definition 2.23. *A function* $g : E \to [0,1]$ *is called an edge irredundant function if for every edge* $e \in E(G)$ *with* $g(e) > 0$ *there exists an edge* $x \in N[e]$ *such that* $g(N[x]) = 1$. *An edge irredundant function* g *is maximal if for all functions* $f : E \to [0,1]$ *with* $f > g$, f *is not an edge irredundant function. The fractional edge irredundance number* $ir'_f(G)$ *and the upper fractional edge irredundance number* $IR'_f(G)$ *are defined by*

$$ir'_f(G) = \min\{|g| : g \text{ is a maximal edge irredundant function on } G\} \text{ and}$$
$$IR'_f(G) = \max\{|g| : g \text{ is a maximal edge irredundant function on } G\}.$$

Lemma 2.24. *Any* MEDF *of a graph* G *is a maximal edge irredundant function.*

Proof. Let g be any MEDF of G. Since $B'_f \to P'_f$, it follows that g is an edge irredundant function. Now, let $h : E \to [0,1]$ be such that $h > g$. Then there exists $e \in E$ such that $g(e) < h(e) \le 1$. Clearly, $h(e) > 0$ and for any $x \in N[e], h(N[x]) > g(N[x]) \ge 1$. Hence h is not an edge irredundant function and thus g is a maximal edge irredundant function. □

Corollary 2.25. *For any graph* G, $ir'_f(G) \le \gamma'_f(G) \le \Gamma'_f(G) \le IR'_f(G)$.

Definition 2.26. *Let* f, g *be two* EDF*s of a graph* G *and let* $0 < \lambda < 1$. *Then* $h_\lambda = \lambda f + (1 - \lambda) g$ *is called a convex combination of* f *and* g.

Observation 2.26. *It can be easily verified that any convex combination of two* EDF*s is again an* EDF. *However a convex combination of two* MEDF*s need not be an* MEDF.

For example, consider the graph G given in Figure 1.

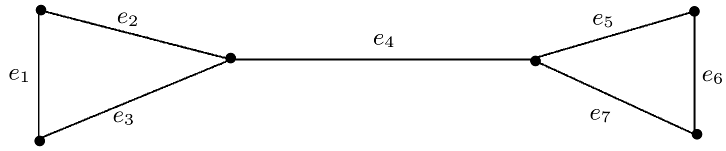

Figure 1

Let

$$\big(f(e_1), f(e_2), f(e_3), f(e_4), f(e_5), f(e_6), f(e_7)\big) = \Big(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\Big) \text{ and}$$

$$\big(g(e_1), g(e_2), g(e_3), g(e_4), g(e_5), g(e_6), g(e_7)\big) = \Big(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}\Big).$$

It can be easily verified that f and g are EDFs and $B'_f \to P'_f$ and $B'_g \to P'_g$, so that f and g are MEDFs of G.

Now, $h_{1/2} = \Big(\frac{14}{30}, \frac{8}{30}, \frac{8}{30}, \frac{1}{10}, \frac{8}{30}, \frac{14}{30}, \frac{8}{30}\Big)$. Clearly, $B'_{h_\lambda} = \{e_1, e_6\}$ and $P'_{h_\lambda} = E(G)$. Since B'_{h_λ} does not dominate $P'_{h_\lambda}, h_\lambda$ is not an MEDF of G.

As in the case of dominating functions, the minimality of $h_\lambda = \lambda f + (1-\lambda)g$, where f and g are MEDFs is an all or nothing situation in the sense that either all convex combinations of f and g are MEDFs or no convex combination of f and g is an MEDF.

Theorem 2.28. *f and g be* MEDF*s of a graph G and $0 < \lambda < 1$. Then h_λ is an* MEDF *of G if and only if $B'_f \cap B'_g$ dominates $P'_f \cup P'_g$.*

Proof. Since $B'_{h_\lambda} = B'_f \cap B'_g$ and $P'_{h_\lambda} = P'_f \cup P'_g$, the result follows. □

Arumugam and Rejikumar [**2**] have introduced the concept of independent functions. We now proceed to introduce the edge analogue of independent functions.

Definition 2.29. *Let $G = (V, E)$ be a graph. A function $f : E \to [0, 1]$ is called an edge independent function if for every edge e with $f(e) > 0$, we have $\sum_{x \in N[e]} f(x) = 1$. An edge independent function f is called maximal edge independent function* (MEIF) *if for every $e \in E$ with $f(e) = 0$, we have $\sum_{x \in N[e]} f(x) \geq 1$.*

Observation 2.30. *Lef f be an* MEIF *of G. It follows from definition that $\sum_{x \in N[e]} f(x) \geq 1$ for all $e \in E$. Hence f is an* EDF. *Further, since f is an edge independent function, $P'_f \subseteq B'_f$ and hence f is an* MEDF *of G.*

Definition 2.31. *The fractional edge independence number β'_{0_f} and the fractional edge independent domination number i'_f are defined by*

$\beta'_{0_f}(G) = \max\{|f| : f$ *is an* MEIF *of* $G\}$ *and*

$i'_f(G) = \min\{|f| : f$ *is an* MEIF *of* $G\}$.

Since every MEIF *is an* MEDF, *we have $\gamma'_f(G) \leq i'_f(G) \leq \beta'_{0_f}(G) \leq \Gamma'_f(G)$. Hence we obtain the following analogue of domination chain for the fractional edge domination*:

$$ir'_f(G) \leq \gamma'_f(G) \leq i'_f(G) \leq \beta'_{0_f}(G) \leq \Gamma'_f(G) \leq IR'_f(G).$$

Remark 2.32. The convex combination of two edge independent functions need not be an edge independent function. Consider the path $P_4 = (v_1, v_2, v_3, v_4)$.

Let $e_i = v_i v_{i+1}$, $i = 1, 2, 3$. Define $f_1(e_1) = f_1(e_3) = 0$, $f_1(e_2) = 1$, $f_2(e_1) = f_2(e_3) = 1$, $f_2(e_2) = 0$ and $f_3(e_1) = f_3(e_2) = f_3(e_3) = \frac{1}{2}$. Clearly f_1 and f_2 are edge independent functions, $f_3 = \frac{1}{2} f_1 + \frac{1}{2} f_2$ and f_3 is not an edge independent function. Further f_1 and f_2 are both MEIFs and hence a convex combination of two MEIFs need not be an edge independent function.

REMARK 2.33. Let f and g be two edge independent functions. Then $h_\lambda = \lambda f + (1-\lambda)g$, where $0 < \lambda < 1$, is an edge independent function if and only if $P'_f \cup P'_g \subseteq B'_f \cap B'_g$. Hence either all convex combinations of f and g are edge independent functions or none of them is an edge independent function. We now prove that similar result is true for MEIFs.

Theorem 2.34. *Let f and g be two* MEIF*s. Then either all convex combinations of f and g are* MEIF*s or none of them is an* MEIF*s.*

Proof. Let $h_\lambda = \lambda f + (1-\lambda)g$ where $0 < \lambda < 1$. Suppose that h_{λ_1} is an MEIF and let $\lambda \neq \lambda_1$. We claim that h_λ is an MEIF. Let $e \in E$. Suppose $h_\lambda(e) = 0$. Then $f(e) = g(e) = 0$. Since f and g are MEIFs, we have $f(N[e]) \geq 1$ and $g(N[e]) \geq 1$. Hence it follows that $h_\lambda(N[e]) \geq 1$.

Now, suppose $h_\lambda(e) > 0$. Then either $f(e) > 0$ or $g(e) > 0$. Hence $h_{\lambda_1} > 0$. Since h_{λ_1} is an MEIF, $h_{\lambda_1}(N[e]) = 1$, so that $f(N[e]) = g(N[e]) = 1$. Hence $h_\lambda(N[e]) = 1$. Thus $P_{h_\lambda} \subseteq B_{h_\lambda}$ and $h_\lambda(N[e]) \geq 1$ for all $e \in E$. Hence h_λ is an MEIF. □

REMARK 2.35. ALLAN and LASKAR [**1**] have proved that if G is $K_{1,3}$-free, then $\gamma(G) = i(G)$. In particular, since the line graph $L(G)$ is $K_{1,3}$-free, it follows that $\gamma'(G) = i'(G)$ for all graphs. However, a similar result is not true for the corresponding fractional parameters as shown in the following theorem.

Theorem 2.36. *If $G = C_5 \circ K_1$, $i'_f(G) > \gamma'_f(G)$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5\}$ and $E(G) = \{x_i = v_i v_{i+1} : 1 \leq i \leq 4\} \cup \{e_i = v_i u_i : 1 \leq i \leq 5\} \cup \{x_5 = v_5 v_1\}$. Let f be any MEIF of G with $|f| = i'_f(G)$.

Case i. There exists a pendant edge, say e_1, with $f(e_1) = 0$.

Without loss of generality we may assume that $f(x_5) = \alpha > 0$, so that $f(x_5) + f(x_1) = 1$. Since $f(N[x_5]) = 1$, it follows that $f(e_5) = f(x_4) = 0$, so that $f(N[e_5]) = \alpha$. Hence $\alpha = 1$, so that $f(x_1) = 0$. Now $f(e_4) + f(x_4) + f(x_3) \geq 1$ and $f(e_2) + f(x_1) + f(x_2) \geq 1$ and hence $|f| \geq 3$.

Case ii. $f(e_i) > 0$ for all $i, 1 \leq i \leq 5$.

Then $f(N[e_i]) = 1$ for all $i, 1 \leq i \leq 5$ and adding these equations we get $2|f| - \sum_{i=1}^{5} f(e_i) = 5$, so that $|f| > \frac{5}{2}$. Hence $i'_f(G) > \frac{5}{2}$. Further it follows from Theorem 2.17. that $\gamma'_f(C_5 \circ K_1) = \frac{5}{2}$ and hence $i'_f(G) > \gamma'_f(G)$. □

PROBLEM 2.37. *Characterize the class of graphs for which $\gamma'_f(G) = i'_f(G)$.*

3. TOPOLOGY ON EDF

Let $G = (V, E)$ be a graph of order n and size m. Let $E = \{e_1, e_2, \ldots, e_m\}$ and let $f : E \to [0, 1]$ be an EDF of G. Clearly f can be represented as an ordered m-tuple $(f(e_1), f(e_2), \ldots, f(e_m))$ and hence $f \in [0, 1]^m$. Thus the set $\mathcal{F}_1$ of all EDFs of G is a subset of $[0, 1]^m$. Hence $(\mathcal{F}_1, \tau)$ is a topological space where τ is the subspace topology of $[0, 1]^m$ with usual topology. Convergence in this topological space $(\mathcal{F}_1, \tau)$ is the same as pointwise convergence. Now let (f_n) be a sequence in $\mathcal{F}_1$ converging pointwise to a function f, so that $\lim\limits_{n\to\infty} f_n(e_i) = f(e_i)$, where $1 \le i \le m$. Let $e \in E$. Since each f_n is an EDF, we have $\sum\limits_{x\in N[e]} f_n(x) \ge 1$. Taking limit as $n \to \infty$, we get $\sum\limits_{x\in N[e]} f(x) \ge 1$. Thus the pointwise limit of a sequence of EDFs is again an EDF and hence $\mathcal{F}_1$ is a closed subset of $[0, 1]^m$. Thus $(\mathcal{F}_1, \tau)$ is a compact topological space.

The following theorem shows that a similar result is true for the set of all MEDFs.

Theorem 3.1. *Let $\mathcal{F}_{1,m}$ denote set of all* MEDF*s of G. Then $\mathcal{F}_{1,m}$ is a closed subspace of $\mathcal{F}_1$ and hence is compact.*

Proof. Let (f_n) be a sequence in $\mathcal{F}_{1,m}$ such that $f_n \to f$. We claim that $f \in \mathcal{F}_{1,m}$. Let $e \in E$ be such that $f(e) > 0$. Then there exists a positive integer n_0 such that $f_n(e) > 0$ for all $n \ge n_0$. Since each f_i is an MEDF it follows that there exists an edge $x_i \in N[e]$ such that $\sum\limits_{y\in N[x_i]} f_i(y) = 1$ for all $i \ge n_0$. Since $E(G)$ is finite, it follows that at least one edge x_i is repeated infinitely many times and hence we can find a subsequence (f_{n_i}) of (f_n) and an edge x such that $\sum\limits_{y\in N[x]} f_{n_i}(y) = 1$ for all $i \ge n_0$. Taking limit as $i \to \infty$, we get $\sum\limits_{y\in N[x]} f(y) = 1$ and hence f is an MEDF of G. Thus $\mathcal{F}_{1,m}$ is closed. □

The proof of the following theorem is similar to that of Theorem 3.1.

Theorem 3.2. *Let $\mathcal{IR}_1(G)$ denote the collection of all edge irredundant functions of G. Then $\mathcal{IR}_1(G)$ is a compact space.*

However, the set of all maximal edge irredundant functions of a graph G need not be compact as seen in following example.

Example. Consider $G = P_8 = (v_1, v_2, \ldots, v_8)$.

Define $f_n : E \to [0, 1]$ by $f_n(v_1v_2) = \frac{1}{n}$, $f_n(v_2v_3) = 0$, $f_n(v_3v_4) = 1 - \frac{1}{n}$, $f_n(v_4v_5) = \frac{2}{n}$, $f_n(v_5v_6) = 0$, $f_n(v_6v_7) = 1 - \frac{2}{n}$ and $f_n(v_7v_8) = \frac{2}{n}$. It can be easily verified that each f_n is a maximal edge irredundant function of P_8 and $(f_n) \to f$ where f is given by $f(v_1v_2) = 0$, $f(v_2v_3) = 0$, $f(v_3v_4) = 1$, $f(v_4v_5) = 0$, $f(v_5v_6) = 0$, $f(v_6v_7) = 1$ and $f(v_7v_8) = 0$. Clearly f is not a maximal edge irredundant function. Hence the set of all maximal edge irredundant functions of P_8 is not a compact space.

The above example leads to the following problems.

PROBLEM 3.4. *Characterize graphs G for which the space of all maximal edge irredundant functions is compact.*

PROBLEM 3.5. *Characterize edge irredundant functions which can be obtained as the limit of a sequence of maximal edge irredundant functions.*

Conclusion and Scope. In this paper we have introduced the fractional parameters corresponding to edge domination, edge irredundance and edge independence, leading to the fractional edge domination chain. Further investigation of this new domination chain and the study of the class of graphs for which some of these fractional parameters are equal will definitely yield many new results. One can also study the concept of universal minimal edge dominting functions and convexity graph corresponding to MEDFs which have been introduced by COCKAYNE et al. [**5**, **7**] for vertex domination.

Acknowledgement. The authors are thankful to the Department of Science and Technology, New Delhi for its support through the Project SR/S4/MS/282/05. We are also thankful to the anonymous referee for helpful suggestions, which led to substantial improvement in the paper.

REFERENCES

1. R. B. ALLAN, R. C. LASKAR: *On domination and independent domination number of a graph.* Discrete Math., **23** (1978), 73–76.
2. S. ARUMUGAM, K. REJIKUMAR: *Fractional independence and fractional domination chain in graphs.* AKCE J Graphs. Combin., **4** (2) (2007), 161–169.
3. S. ARUMUGAM, S. VELAMMAL: *Edge domination in graphs.* Taiwanese Journal of Mathematics, **2** (2) (1998), 173–179.
4. G. CHARTRAND, L. LESNIAK: *Graphs and Digraphs.* Chapman and Hall, CRC, 4^{th} edition, 2005.
5. E. J. COCKAYNE, G. FRICKE, S. T. HEDETNIEMI, C. M. MYNHARDT: *Properties of minimal dominating functions of graphs.* Ars Combin., **41** (1995), 107–115.
6. E. J. COCKAYNE, G. MCGILLIVRAY, C. M. MYNHARDT: *Convexity of minimal dominating functions of trees.* Utilitas Mathematica, **48** (1995), 129–144.
7. E. J. COCKAYNE, G. MCGILLIVRAY, C. M. MYNHARDT: *Convexity of minimal dominating functions of trees*-II. Discrete Math., **125** (1994), 137–146.
8. E. J. COCKAYNE, C. M. MYNHARDT: *Convexity of extremal domination-related functions of graphs.* In Domination in Graphs - Advanced Topics, (Ed. T. W. Haynes, S. T. Hedetniemi, P. J. Slater), Marcel Dekker, Inc., New York, (1998), 109–131.
9. E. J. COCKAYNE, C. M. MYNHARDT, B. YU: *Universal minimal total dominating functions in graphs.* Networks, **24** (1994), 83–90.
10. G. S. DOMKE, S. T. HEDETNIEMI, R. C. LASKAR: *Fractional packings, coverings and irredundance in graphs.* Congr. numer., **66** (1988), 227–238.

11. G. S. Domke, S. T. Hedetniemi, R. C. Laskar, R. Allan: *Generalized packings and coverings of graphs.* Congr. Numer., **62** (1988), 259–270.
12. D. Grinstead, P. J. Slater: *Fractional domination and fractional packing in graphs.* Congr. numer., **71** (1990), 153–172.
13. T. W. Haynes, S. T. Hedetniemi, P. J. Slater: *Fundamentals of Domination in Graphs.* Marcel Dekker, Inc., 1998.
14. S. M. Hedetniemi, S. T. Hedetniemi, T. V. Wimer: *Linear time resource allocation algorithms for trees.* Technical report URI-014, Department of Mathematics, Clemson University, 1987.
15. S. Mitchell, S. T. Hedetniemi: *Edge domination in trees.* Congr. Numer., **19** (1977), 489–509.
16. Bo Yu: *Convexity of minimal total dominating functions in graphs.* J. Graph Theory, **24** (4) (1997), 313–321.

Core Group Research Facility (CGRF),
National Centre for Advanced Research
in Discrete Mathematics (n-CARDMATH)
Kalasalingam University,
Anand Nagar, Krishnankoil-626 190, India
E–mail: s.arumugam.klu@gmail.com

(Received September 10, 2008)
(Revised March 6, 2009)