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FAMILIES OF OPTIMAL MULTIPOINT METHODS FOR SOLVING NONLINEAR EQUATIONS: A SURVEY

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Multipoint iterative root-solvers belong to the class of the most powerful methods for solving nonlinear equations since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. Although the construction of these methods has occurred in the 1960s, their rapid development have started in the first decade of the 21-st century. The most important class of multipoint methods are optimal methods which attain the convergence order 2^n using $n+1$ function evaluations per iteration. In this paper we give a review of optimal multipoint methods of the order four ($n=2$), eight ($n=3$) and higher ($n>3$), some of which being proposed by the authors. All of them possess as high as possible computational efficiency in the sense of the Kung-Traub hypothesis (1974). Numerical examples are included to demonstrate a very fast convergence of the presented optimal multipoint methods.

1. INTRODUCTION

Almost a half century ago, J. F. Traub proved [**19**, Theorem 5-3] that one-point iterative methods for solving single nonlinear equations of the form $f(x)=0$, which require the evaluation of a given function f and the first $p-1$ derivatives of f, can reach the order of convergence at most p. For this reason, a great attention was paid to multipoint iterative methods since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. Beside Traub's research presented in his fundamental book [**19**], this class of methods was also extensively studied in some papers published in the 1970s (see,

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e.g., [**5**]–[**7**], [**9**]). Surprisingly, the interest for multipoint methods has again grown in the first decade of this century. However, some of the newly developed methods were represented by new iterative formulae but without any improvement compared to the existing methods, others were only rediscovered methods of the 1960s (see, e.g., [**13**] for more details), and only a few new methods have brought a genuine advance in the theory and practice of iterative processes.

The main goal and motivation in the construction of new methods is as high as possible computational efficiency. In other words, it is desirable to attain as high as possible convergence order with fixed number of function evaluations per iteration. In the case of multipoint methods without memory, this demand is closely connected with the optimal order considered in the Kung-Traub conjecture [**9**] from 1974:

Kung-Traub's conjecture [**9**]. *Multipoint iterative methods without memory, requiring* $n+1$ *function evaluations per iteration, have the order of convergence at most* 2^n.

Multipoint methods which satisfy the Kung-Traub conjecture (still unproved) are usually called *optimal methods*; consequently, $r = 2^n$ is the *optimal order*. A class of optimal n-point methods, reaching the order 2^n with $\theta = n+1$ function evaluations per iteration, will be denoted with Ψ_{2^n} $(n \geq 1)$.

The computational efficiency of an iterative method (IM) of the order r, requiring θ function evaluations per iteration, is most frequently calculated by Ostrowski-Traub's efficiency index

$$E(\mathrm{IM}) = r^{1/\theta}.$$

(see [**12**], [**19**]). Following the Kung-Traub conjecture, the *optimal computational efficiency* would be $E_n^{(o)} = 2^{n/(n+1)}$. We note that this conjecture is supported by two families of multipoint methods presented in [**9**] and the family of multipoint methods proposed in [**15**] for arbitrary n. In particular, it is also confirmed in the case of several two-point methods (see, e.g., [**3**]–[**9**], [**14**]), three-point methods (e.g., [**1**], [**2**], [**14**], [**18**]) and the four-point method [**11**]. The aim of this paper is to give a review of the main results concerned with optimal multipoint methods for solving nonlinear equations, presented in some old and some very recent papers and books. These results can serve not only as a review of contributions achieved in the last fifty years but also as reference material for further investigation in this topic.

The paper is organized as follows. First, in Section 2 we point to Traub's investigation in this area and the first multipoint method derived by Ostrowski in 1960. A wide class of optimal two-point methods of the fourth order and some special cases are given in Section 3. A family of optimal three-point methods of the order eight, where a number of function evaluations is reduced by applying the Hermite interpolation polynomial of the third order, is considered in Section 4. Another optimal three-point methods are also presented in this section. General classes of optimal n-point methods for arbitrary $n \geq 3$, derived by combining

Newton's method and an economization scheme based on the Hermite interpolation polynomial, are the subject of Section 5. This section also presents two optimal n-point families proposed by KUNG and TRAUB [**9**]. Section 6 contains some numerical examples which demonstrate exceptional convergence speed of the presented optimal multipoint methods.

2. SOME EARLY MULTIPOINT METHODS

Let f be a real sufficiently smooth analytic function, defined on an interval $I_f \subset \boldsymbol{R}$ which contains a simple root α of f. Throughout this paper we will often use the following quantities and abbreviations:

$$u(x) = \frac{f(x)}{f'(x)}, \quad c_k = c_k(\alpha) = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)} \ \ (k = 2, 3, \ldots), \quad f[y, x] = \frac{f(y) - f(x)}{y - x}.$$

For simplicity, we will sometimes omit iteration indices in iterative formulae and write $\hat{x} = \phi(x)$ instead of $x_{k+1} = \phi(x_k)$.

The following theorem considers a generalized Newton's method and gives a simple way for constructing multipoint methods.

Theorem 1. (TRAUB [**19**, Th. 8.1]) *Let α be a simple root of a function f and let $\phi(x)$ define the iterative method of the order p. Then the composite iterative function $\psi(x)$ introduced by Newton's method*

$$\psi(x) = \phi(x) - \frac{f\big(\phi(x)\big)}{f'(x)}, \tag{1}$$

defines the iterative method of the order $p + 1$.

EXAMPLE 1. Let $\phi(x)$ be given by Newton's method $\phi(x) = x - u(x)$. Then, according to (1), the iterative function

$$\psi(x) = x - u(x) - \frac{f(x - u(x))}{f'(x)} \tag{2}$$

defines the method of third order.

Consider an iterative function constructed by combining Newton's method and the secant method in the following manner:

$$y = x - \frac{f(x)}{f'(x)}, \quad \hat{x} = y - f(y)\frac{y - x}{f(y) - f(x)}.$$

Here the derivative $f'(y)$ is replaced by the corresponding divided difference $f[y, x] = (f(y) - f(x))/(y - x)$. Hence we obtain the two-point method of third order

$$\hat{x} = x - u(x) + \frac{u(x)f\big(x - u(x)\big)}{f\big(x - u(x)\big) - f(x)} = x + \frac{u(x)f(x)}{f\big(x - u(x)\big) - f(x)}. \tag{3}$$

This two-point method may be visualized as the intersection with the x-axis of the secant line through the points $\big(x, f(x)\big)$ and $\big(y, f(y)\big)$, $y = x - u(x)$, represented by the dashed line in Figure 1, where $x' = \hat{x}$.

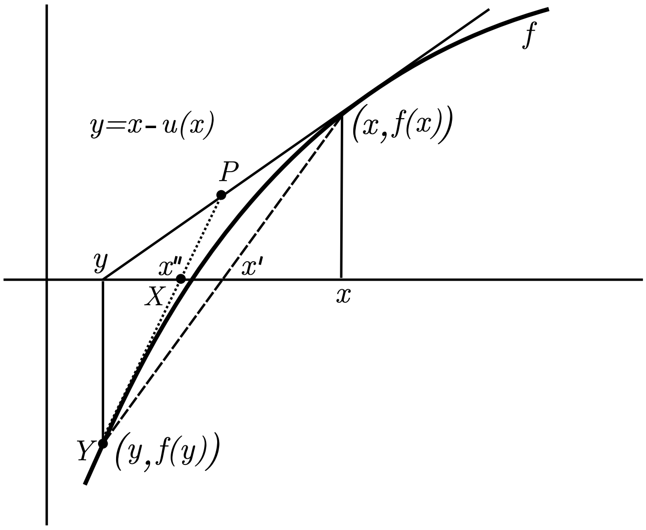

Figure 1.

The both two-point methods (2) and (3) have a cubic convergence and require three function evaluations. Therefore, they are not optimal in the sense of the Kung-Traub conjecture. The first optimal two-point method was constructed by Ostrowski [**12**], several years before Traub's extensive investigation in this area. Ostrowski derived his method replacing $f'(x_k)$ by a linear combination of $f(x_{k-1})$ and $f(x_k)$ at every second step in this way,

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \quad x_{k+1} = y_k - \frac{f(y_k)(y_k - x_k)}{2f(y_k) - f(x_k)}. \tag{4}$$

More precisely, this two-point method was derived using the interpolation by linear fraction $w(x) = (x+b)/(cx+d)$ $(d - bc \neq 0)$ satisfying

$$w(x_k) = f(x_k),\ w(y_k) = f(y_k),\ w'(x_k) = f'(x_k),$$

see [**12**, Ch. 11].

Ostrowski's method can be derived by a geometric approach using Fig. 1. Let the point $P\Big(\frac{1}{2}(y+x), \frac{1}{2}f(x)\Big)$ bisect the segment determined by the points $(y, 0)$ and $\big(x, f(x)\big)$, where $y = x - f(x)/f'(x)$ is the Newton approximation. In fact, this segment is a part of the tangent line at $\big(x, f(x)\big)$. A new approximation x'' is the intersection with the x-axis of the secant line through the points P and $\big(y, f(y)\big)$ (drawn by doted line in Fig. 1). From the similarity of the right-angle triangles (with hypotenuses $\overline{PX}$ and $\overline{XY}$), from Fig. 1 we observe that the ratios

$$\frac{\frac{1}{2}f(x)}{x'' - \frac{1}{2}(y+x)} = \frac{f(y)}{x'' - y}$$

hold. Solving the last equation in x'', we obtain Ostrowski's method (4) setting $x'' = x_{k+1},\ y = y_k,\ x = x_k$. TRAUB [**19**, p. 184] gave the following error relation (without the proof)

$$\frac{x_{k+1} - \alpha}{(x_k - \alpha)^4} \to c_2(\alpha)\big[c_2^2(\alpha) - c_3(\alpha)\big],$$

pointing to the fourth order of the method (4).

A generalization of Ostrowski's method (4) was proposed by KING [**7**] in the form

$$K_f(\beta; x) = x - u(x) - \frac{f\big(x - u(x)\big)}{f'(x)} \cdot \frac{f(x) + \beta f\big(x - u(x)\big)}{f(x) + (\beta - 2)f\big(x - u(x)\big)}, \tag{5}$$

where β is a parameter. King's method (5) is optimal and has the order four. It is easy to show that Ostrowski's method (4) is obtained from (5) for $\beta = 0$.

3. FAMILY OF OPTIMAL TWO-POINT METHODS

Consider a composite iterative two-point scheme

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f'(y_k)}, \end{cases} \qquad (k = 0, 1, \ldots) \tag{6}$$

assuming that an initial approximation x_0 is reasonably good. The presented scheme is simple and its rate of convergence is equal to four, which is a consequence of the fact that Newton's method is of the second order and the following generalization of Traub's theorem [**19**, Th. 2.4]:

Theorem 2. *Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_s(x)$ be iterative functions with the orders $r_1, r_2, \ldots, r_s$, respectively. Then the composition of iterative functions*

$$\varphi(x) = \varphi_1\big(\varphi_2\big(\cdots\big(\varphi_s(x)\big)\cdots\big)\big)$$

defines the iterative method of the order $r_1 r_2 \cdots r_s$.

However, the two-point method (6) requires four function evaluations per iteration step so that it is *not* optimal in the sense of Kung-Traub's conjecture. To reduce the number of evaluations and thus increase the computational efficiency, we will use CHUN's approach [**4**] to approximate $f'(y)$ by

$$f'(y_k) \approx \frac{f'(x_k)}{g(t_k)}, \quad t_k = \frac{f(y_k)}{f(x_k)},$$

assuming that a real function g and its derivatives g' and g'' are continuous in the neighborhood of 0. Now the two-step scheme (6) becomes

$$\text{(7)} \qquad \begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - g(t_k)\dfrac{f(y_k)}{f'(x_k)}, \quad t_k = \dfrac{f(y_k)}{f(x_k)}, \end{cases} \qquad (k = 0, 1, \ldots).$$

The function g in (7) has to be determined so that the two-point method (7) attains the optimal order four using only three function evaluations $f(x_k)$, $f'(x_k)$ and $f(y_k) = f(x_k - u(x_k))$, which is the subject of the following theorem (see [**14**]).

Theorem 3. *Let $\alpha \in I_f$ be a simple root of real single-valued function $f : I_f \to \boldsymbol{R}$ possessing a certain number of continuous derivatives in the neighborhood of $\alpha \in I_f$, where I_f is an open interval. Let g be a function satisfying $g(0) = 1$, $g'(0) = 2$ and $|g''(0)| < \infty$. If x_0 is sufficiently close to α, then the order of convergence of the family of two-step methods (7) is four and the error relation*

$$\text{(8)} \qquad \varepsilon_{k+1} = \Big[c_2^3(5 - g''(0)/2) - c_2c_3\Big]\varepsilon_k^4 + O(\varepsilon_k^5)$$

holds.

Proof. Let $c_k = f^{(k)}(\alpha)/(k!f'(\alpha))$ and let us introduce the errors

$$\varepsilon_k = x_k - \alpha, \quad \eta_k = y_k - \alpha, \quad \varepsilon_{k+1} = x_{k+1} - \alpha.$$

Using the Taylor series we find

$$f(x_k) = f'(\alpha)\Big(\varepsilon_k + c_2\varepsilon_k^2 + c_3\varepsilon_k^3 + c_4\varepsilon_k^4 + O(\varepsilon_k^5)\Big)$$

and

$$\text{(9)} \qquad f'(x_k) = f'(\alpha)\Big(1 + 2c_2\varepsilon_k + 3c_3\varepsilon_k^2 + 4c_4\varepsilon_k^3 + O(\varepsilon_k^4)\Big).$$

Hence, applying again the Taylor series to $1/f'(x_k)$, we get

$$\text{(10)} \quad \eta_k = \varepsilon_k - \frac{f(x_k)}{f'(x_k)} = c_2\varepsilon_k^2 + (2c_3 - 2c_2^2)\varepsilon_k^3 + (4c_2^3 - 7c_2c_3 + 3c_4)\varepsilon_k^4 + O(\varepsilon_k^5).$$

Furthermore, we have

$$\text{(11)} \qquad f(y_k) = f'(\alpha)\Big(\eta_k + c_2\eta_k^2 + c_3\eta_k^3 + c_4\eta_k^4 + O(\eta_k^5)\Big).$$

Let us represent the function g by its Taylor's polynomial of the second order at the point $t=0$,

$$g(t_k) = g(0) + g'(0)t_k + \frac{g''(0)}{2}\, t_k^2, \quad t_k = f(y_k)/f(x_k). \tag{12}$$

Now, using (9)–(12), we obtain

$$\begin{aligned}
\varepsilon_{k+1} =& x_{k+1} - \alpha = \eta_k - g(t_k)\frac{f(y_k)}{f'(x_k)} \\
=& \Big[-2c_3\big(g(0)-1\big) + c_2^2\big(4g(0)-g'(0)-2\big)\Big]\varepsilon_k^3 + \\
& \Big[-3c_4\big(g(0)-1\big) + c_2c_3\big(-7+14g(0)-4g'(0)\big) \\
& + c_2^3\big(4-13g(0)+7g'(0)-g''(0)/2\big)\Big]\varepsilon_k^4 + O(\varepsilon_k^5).
\end{aligned}$$

Substituting the conditions $g(0)=1$ and $g'(0)=2$ of the theorem in the last expression of ε_{k+1}, we obtain

$$\varepsilon_{k+1} = \Big[c_2^3(5-g''(0)/2) - c_2c_3\Big]\varepsilon_k^4 + O(\varepsilon_k^5),$$

which is exactly the error relation (8). Hence we conclude that the order of the family (7) is four. □

Remark 1. From (8) we observe that the asymptotic error constant (AEC) of the method (7) is

$$\mathrm{AEC}(7) = \lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n^4} = c_2^3\big(5-g''(0)/2\big) - c_2c_3.$$

In the case of a particular method obtained by choosing a specific function g, we take $g''(0)$ in AEC(7) to obtain the corresponding expression of AEC.

In what follows we consider some special cases of the family of two-point methods (7). In addition to the new methods, it is shown that most of existing optimal two-point methods appear as special cases of the presented family (7). The chosen function g in the subsequent examples satisfies the conditions $g(0) = 1,\ g'(0)=2$ and $|g''(0)|<\infty$, given in Theorem 3.

Method 1. For g given by

$$g(t) = \frac{1+\beta t}{1+(\beta-2)t} \quad (\beta \in \boldsymbol{R})$$

we obtain King's fourth order family of two-point methods given by (5). Let us note that King's family produces the following special cases:

Ostrowski's method [**12**], $\beta = 0$:

$$K(0;x) = x - u(x) - \frac{u(x)f(x-u(x))}{f(x)-2f(x-u(x))}; \tag{13}$$

Kou's method [**8**], $\beta = 1$:

$$K(1;x) = x - \frac{f(x)^2 + [f(x-u(x))]^2}{f'(x)[f(x) - f(x-u(x))]}; \tag{14}$$

Chun's method [**4**], $\beta = 2$:

$$K(2;x) = x - u(x)\left\{1 + \frac{f(x-u(x))}{f(x)} + \frac{2\big[f(x-u(x))\big]^2}{f(x)^2}\right\}. \tag{15}$$

Method 2. Choosing g in the form

$$g(t) = \Big(1 + \frac{2}{m}t\Big)^m \quad (m \in \boldsymbol{N}),$$

one obtains the fourth order method

$$\phi_m(x) = x - \frac{f(x-u(x))}{f'(x)}\left(1 + \frac{2}{m}\cdot\frac{f(x-u(x))}{f(x)}\right)^m. \tag{16}$$

Taking $m = 1$ we get Chun's method (15). The iterative method obtained from (16) for $m = 2$

$$\phi_2(x) = x - \frac{f(x-u(x))}{f'(x)}\Big(1 + \frac{f(x-u(x))}{f(x)}\Big)^2 \tag{17}$$

is the new one.

Method 3. For g given by

$$g(t) = \frac{1+\gamma t^2}{1-2t},$$

where γ is a real parameter, we obtain a new one parameter family of two-point methods

$$\psi_\gamma(x) = x - \frac{f(x-u(x))}{f'(x)f(x)}\cdot\frac{f(x)^2 + \gamma\big[f(x-u(x))\big]^2}{f(x) - 2f(x-u(x))}. \tag{18}$$

Setting $\gamma = 0$ in (18), one obtains Ostrowski's method (13).

Method 4. The choice

$$g(t) = \frac{1}{1-2t+at^2} \quad (a \in \boldsymbol{R})$$

gives a new family of optimal two-point methods defined by the iterative function

$$\eta_a(x) = x - u(x) - \frac{f(x-u(x))}{f'(x)}\cdot\frac{1}{1 - \dfrac{2f(x-u(x))}{f(x)} + a\Big[\dfrac{f(x-u(x))}{f(x)}\Big]^2}. \tag{19}$$

Ostrowski's method (13) is obtained as a special case putting $a = 0$ in (19).

Method 5. Choosing

$$g(t) = \frac{t^2 + (c-2)t - 1}{ct - 1} \quad (c \in \boldsymbol{R}),$$

we construct the iterative function $\omega_c(x, y)$,

$$\text{(20)} \qquad \begin{cases} y = x - \dfrac{f(x)}{f'(x)}, \\ \omega_c(x, y) = y - \dfrac{f(y)}{f'(x)} \left[1 + \dfrac{f(y)\big(f(y) - 2f(x)\big)}{f(x)\big(cf(y)) - f(x)\big)} \right], \end{cases}$$

which defines a one parameter family of two-point methods $x_{n+1} = \omega_c(x_k, y_k)$ of the order four. Taking $c = 1$ in (20), we obtain Maheshvari's method [**10**] as a special case,

$$\text{(21)} \qquad M(x) = x - u(x) \left\{ \frac{\big[f(x - u(x))\big]^2}{f(x)^2} - \frac{f(x)}{f(x - u(x)) - f(x)} \right\}.$$

Method 6. The function

$$g(t) = \frac{1}{t}\Big(\frac{2}{1 + \sqrt{1 - 4t}} - 1 \Big)$$

satisfies the conditions of Theorem 3 in a limit case when $t \to 0$ and produces the fourth order method

$$\text{(22)} \qquad E(x) = x - \frac{2u(x)}{1 + \sqrt{1 - \dfrac{4f(x - u(x))}{f(x)}}}$$

proposed in [**16**].

REMARK 2. The efficiency index for optimal two-point methods is $E_4^{(o)} = 4^{1/3} \approx 1.587$.

4. FAMILY OF OPTIMAL THREE-POINT METHODS

In this section we consider some classes of optimal three-point methods with the optimal order eight requiring four function evaluations. First of them relies on optimal two-point methods belonging to the class Ψ_4. Some two-point methods from the class Ψ_4 are given in Section 3.

Let f be a real analytic function defined on an interval $I_f \subset \boldsymbol{R}$ and f' does not vanish on I_f. Let us assume that a simple root α of f is isolated in the interval I_f and let $\varphi_f \in \Psi_4$ denote an iterative function from the class of optimal two-point iterative methods. Then the improved approximation $\hat{x}$ to α can be found by the following three-point iterative scheme:

$$
(23) \qquad \begin{cases} (1) \quad y = x - \dfrac{f(x)}{f'(x)}, \\ (2) \quad z = \varphi_f(x, y), \quad \varphi_f \in \Psi_4, \\ (3) \quad \hat{x} = z - \dfrac{f(z)}{f'(z)}. \end{cases}
$$

We note that the first two steps define an optimal two-point method from the class Ψ_4 with the order $r_1 = 4$ using the Newton method in the first step, while the third step is Newton's method of the order $r_2 = 2$. The presented scheme is simple and, according to Theorem 2, its rate of convergence is equal to $r_1 \cdot r_2 = 8$.

However, the three-point method (23) requires five function evaluations per iteration step so that it is *not* optimal in the sense of Kung-Traub's conjecture. To reduce the number of evaluations and increase the computational efficiency, we will approximate $f'(z)$ using available data. Since we have four values $f(x), f'(x), f(y)$ and $f(z)$, it is convenient to approximate f by the Hermite interpolation polynomial h of degree 3 in the nodes $x,\ y,\ z$ and utilize the approximation $f'(z) \approx h'(z)$ in the third step of the iterative scheme (23). This idea was employed in [**15**] for a general class of optimal multipoint methods, see Section 5.

The Hermite interpolation polynomial of third order for the given data has the form

$$
(24) \qquad h(t) = a_0 + a_1(t-x) + a_2(t-x)^2 + a_3(t-x)^3,
$$

and its derivative is

$$
(25) \qquad h'(t) = a_1 + 2a_2(t-x) + 3a_3(t-x)^2.
$$

The unknown coefficients will be determined from the conditions:

$$
h(x) = f(x), \quad h(y) = f(y), \quad h(z) = f(z), \quad h'(x) = f'(x).
$$

Putting $t = x$ into (24) and (25) we immediately get $a_0 = f(x)$ and $a_1 = f'(x)$. The next two coefficients a_2 and a_3 are obtained from the system of two linear equations formed by using the remaining two conditions putting y and z into (24). We get

$$
(26) \qquad a_2 = \frac{(z-x)f[y,z]}{(z-y)(y-x)} - \frac{(y-x)f[z,x]}{(z-y)(z-x)} - f'(x)\left(\frac{1}{z-x} + \frac{1}{y-x}\right),
$$

and

$$
(27) \qquad a_3 = \frac{f[z,x]}{(z-y)(z-x)} - \frac{f[y,x]}{(z-y)(y-x)} + \frac{f'(x)}{(z-x)(y-x)}.
$$

Replacing the obtained coefficients into (25) and putting $t = z$, we get the explicit formula for $h'(z)$,

$$h'(z) = f[z,x]\left(2 + \frac{z-x}{z-y}\right) - \frac{(z-x)^2}{(y-x)(z-y)} f[y,x] + f'(x)\frac{z-y}{y-x}. \tag{28}$$

Remark 3. The derivative $f'(z)$ can be approximated using available data in different ways. For example, Bi et al. [**1**] used an approximation of the second derivative f'' and divided differences. Neta [**11**] applied inverse interpolation to derive optimal methods of higher order.

We now use the relation (28) to approximate $f'(z) \approx h'(z)$. Replacing $f'(z)$ in (23) by the expression (28), we state the following class of three-point methods: Given an initial approximation x_0, the improved approximations x_k $(k = 1, 2, \ldots)$ are calculated by the three-step procedure

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = \varphi_f(x_k, y_k), \\ x_{k+1} = z_k - \dfrac{f(z_k)}{h'(z_k)} \end{cases} \quad (k = 0, 1, \ldots). \tag{29}$$

The proposed class of root-solvers requires only four function evaluations and possesses the eighth order of convergence, which is the subject of Theorem 5. The proof of this theorem requires the estimation of quality of the approximation $f'(x) \approx h'(x)$.

Let H_m be the Hermite interpolation polynomial of the degree m satisfying

$$H_m^{(k)}(t_j) = f^{(k)}(t_j) \quad (j = 0, \ldots, s;\ k = 0, \ldots, \gamma_j - 1), \tag{30}$$

where $t_0, t_1, \ldots, t_s$ are interpolation nodes and $\gamma_0, \gamma_1, \ldots, \gamma_s$ are their respective multiplicities. We will use the following well-known expression for the error of the Hermite interpolation (see, e.g., [**19**, p. 244]).

Theorem 4. *Let f and its derivatives $f', \ldots, f^{(m+1)}$ be continuous in the interval (a, b) determined by interpolation nodes $t_0, t_1, \ldots, t_s$. Then*

$$f(t) - H_m(t) = \frac{f^{(m+1)}(\xi)}{(m+1)!} \prod_{j=0}^{s} (t - t_j)^{\gamma_j}, \tag{31}$$

where $\xi \in (a, b)$ and $\gamma_0 + \gamma_1 + \cdots + \gamma_s = m + 1$.

Now we state the following convergence theorem.

Theorem 5. *If an initial approximation x_0 is sufficiently close to the root α of a function f, then the convergence order of the class of three-point methods* (29) *is equal to eight.*

Proof. For brevity, we omit the iteration index and consider the values $x,\ y,\ z$ in one iteration step, denoting the improved approximation with $\hat{x}=x_{k+1}$. Let us introduce the errors

$$\varepsilon=x-\alpha,\quad u=y-\alpha,\quad v=z-\alpha,$$

where α is the root of f. Then the error in the new iterative step is $\hat{\varepsilon}=\hat{x}-\alpha$. Since the iterative function $\varphi_f\in\Psi_4$ is of the fourth order, we have the estimation

$$v=q\varepsilon^4+O(\varepsilon^5), \tag{32}$$

where q is the asymptotic error constant of the two-step method φ_f applied in (29). It is easy to find that

$$y-x=u-\varepsilon=O(\varepsilon),\quad z-x=v-\varepsilon=O(\varepsilon),\quad z-y=v-u=O(\varepsilon^2). \tag{33}$$

Let us find the convergence order of the modified Newton method in the third step in (29), where $f'(z)$ is replaced with $h'(z)$ (given by (28)). To do that, compare the corresponding iterative function

$$g(z):=\hat{x}=z-\frac{f(z)}{h'(z)} \tag{34}$$

to the Newton iterative function $N(z)=z-f(z)/f'(z)$. Consider the special case of (30) when $m=3$, and $\gamma_0=2,\ \gamma_1=\gamma_2=1$ are the multiplicities of the nodes $t_0=x,\ t_1=y$ and $t_2=z$, respectively. Then, in regard to (31), the error of the Hermitian interpolation is given by

$$f(t)-h(t)=\frac{f^{(4)}(\xi)}{4!}(t-x)^2(t-y)(t-z). \tag{35}$$

Differentiating (35) and taking $t=z$, in regard to (32) and (33) we obtain

$$f'(z)-h'(z)=\frac{f^{(4)}(\xi)}{4!}(z-x)^2(z-y)=\frac{f^{(4)}(\xi)}{4!}(v-\varepsilon)^2(v-u)=O(\varepsilon^4)=O(v),$$

whence

$$f'(z)=h'(z)\big(1+O(\varepsilon^4)\big)=h'(z)\big(1+O(v)\big).$$

This relation yields

$$\hat{\varepsilon}=\hat{x}-\alpha=z-\frac{f(z)}{h'(z)}-\alpha=z-\frac{f(z)}{f'(z)}+f(z)O(v)-\alpha=N(z)-\alpha+O(v^2)=O(v^2),$$

since $f(z)=O(z-\alpha)=O(v)$ and $N(z)-\alpha=O(v^2)$. Hence

$$g(z)-\alpha=\hat{x}-\alpha=O(v^2), \tag{36}$$

which means that the modified Newton method (14) also possesses the quadratic convergence. Finally, according to (32) and (36), we get

$$\hat{\varepsilon} = \hat{x} - \alpha = O(v^2) = O(\varepsilon^8),$$

which completes the proof of Theorem 5. □

REMARK 4. By virtue of (32) and (36), from Theorem 2 it follows that the order of convergence of the composite iteration $g(\varphi_f(x, y))$ is $2 \cdot 4 = 8$.

REMARK 5. Since the convergence order is $2^3 = 8$ and the number of function evaluations is $\theta = 4$ for the considered class of three-point methods Ψ_8, we conclude that the Kung-Traub conjecture is supported for $n = 3$.

Using the Taylor series and symbolic computation in the programming package *Mathematica* (*Maple* or *Matlab* are also convenient), we can determine the asymptotic error constant of the three-point methods (29). The following abbreviations are used in the program given below.

ck= $f^{(k)}(\alpha)/(k!f'(\alpha))$, e= $x - \alpha$, e1= $\hat{x} - \alpha$,
fx= $f(x)$, fy= $f(y)$, fz= $f(z)$, f1x= $f'(x)$, f1a= $f'(\alpha)$,
f1z= $f'(z)$ (calculated by (28)).

Program (written in *Mathematica*):

```
fx=f1a*(e+c2*e^2+c3*e^3+c4*e^4); f1x=D[fx,e];
u=e-Series[fx/f1x, {e,0,4}]; fy=f1a*(u+c2*u^2+c3*u^3 +c4*u^4);
v=q*e^4; fz=f1a*(v+c2*v^2+c3*v^3+c4*v^4);
fxy=(fx-fy)/(e-u); fxz=(fx-fz)/(e-v); fyz=(fy-fz)/(u-v);
f1z=fxz*(2+(v-e)/(v-u))-(v-e)^2/((u-e)*(v-u))*fxy+f1x*(v-u)/(u-e);
e1=v-fz/f1z//Simplify
```

Out$[e1] = c_2 q(c_4 + q)e^8 + O[e^9]$

Therefore, the asymptotic error constant (AEC) of the class of methods (29) is given by

$$\text{AEC}(29) = c_2 q(c_4 + q).$$

The AEC q should be determined for each particular two-point method φ_f applied in the iterative scheme (29). For example, $q = \text{AEC}(13) = c_2(c_2^2 + 2\beta c_2^2 - c_3)$ for King's two-point method (5) so that the AEC of the three-point method (29) in this particular case is

$$\text{AEC}\big((29)\text{–}(5)\big) = c_2^2(c_2^2 + 2\beta c_2^2 - c_3)(c_2^3 + 2\beta c_2^3 - c_2 c_3 + c_4).$$

REMARK 6. Note that the output Out[e1] of the above program points to the eighth order of convergence. In fact, this program determines the rate of convergence of the three-point scheme (29) in an easy way, using symbolic computation.

In what follows we will present in short two families of optimal three-point methods, both based on King's two-point method (5). Bi et al. [**1**] used this method in the third step and constructed the family of optimal three-point methods

$$(37)\qquad \begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - h(\mu_k)\dfrac{f(y_k)}{f'(x_k)}, \\ x_{k+1} = z_k - \dfrac{f(x_k)+\beta f(z_k)}{f(x_k)+(\beta-2)f(z_k)} \cdot \dfrac{f(z_k)}{f[z_k,y_k]+f[z_k,x_k,x_k](z_k-y_k)}, \end{cases}$$

where $h(t)$ is a real-valued function, $\mu_k = f(y_k)/f(x_k)$ and

$$f[z,x,x] = \frac{f[z,x]-f'(x)}{z-x}.$$

Another family of optimal three-point methods was recently proposed by Thurkal and Petković [**18**]. King's method makes the first two steps of the following three-step scheme

$$(38)\qquad \begin{cases} y = x - \dfrac{f(x)}{f'(x)}, \\ z = y - \dfrac{f(y)}{f'(x)} \cdot \dfrac{f(x)+bf(y)}{f(x)+(b-2)f(y)}, & (a,b \in \boldsymbol{R}) \\ g(x) = z - \dfrac{f(z)}{f'(x)}\Big[\varphi\Big(\dfrac{f(y)}{f(x)}\Big) + \dfrac{f(z)}{f(y)-af(z)} + \dfrac{4f(z)}{f(x)}\Big], \end{cases}$$

where φ is arbitrary real function satisfying the conditions

$$(39)\qquad \varphi(0)=1,\ \varphi'(0)=2,\ \varphi''(0)=10-4b,\ \varphi'''(0)=12b^2-72b+72,$$

and a and b are real parameters. The iterative method is defined by

$$(40)\qquad x_{k+1} = g(x_k) \quad (k=0,1,\ldots),$$

starting with an initial approximation x_0 to the root α of f. Some examples of the choice of the function φ in (38) are given in Table 2.

All families (29), (37) and (38) have the order eight and need four function evaluations. Therefore, their efficiency index is $8^{1/4} \approx 1.682$, which is better than the efficiency index $4^{1/3} \approx 1.587$ of any two-point method of the fourth order requiring three function evaluations (see Remark 2). We note that the efficiency index of the Newton method is only $2^{1/2} \approx 1.414$.

5. GENERAL CLASSES OF OPTIMAL MULTIPOINT METHODS

At the beginning of this paper we have mentioned that Kung and Traub [**9**] constructed two optimal n-point families of iterative methods for arbitrary $n \geq 3$. We present these families, called here K-T family for brevity, in the form given in [**9**].

K-T (41): For any n, define iterative function $p_j(f)$ $(j = 0, \ldots, n)$ as follows: $p_0(f)(x) = x$ and for $n > 0$,

$$\left\{\begin{array}{l} p_1(f)(x) = x + \gamma f(x), \ \gamma \text{ is a nonzero constant}, \\ \quad \vdots \\ p_{j+1}(f)(x) = R_j(0), \end{array}\right. \tag{41}$$

for $j = 1, \ldots, n-1$, where $R_j(y)$ is the inverse interpolatory polynomial of degree at most j such that $R_j(f(p_\lambda(f)(x)) = p_\lambda(f)(x)$ $(\lambda = 0, \ldots, j)$. The iterative method is defined by $x_{k+1} = p_n(f)(x_k)$ starting with an initial approximation x_0. Let us note that the family K-T (41) requires no evaluation of derivatives of f.

K-T (42): For any n, define iterative function $q_j(f)$ $(j = 0, \ldots, n)$ as follows: $q_1(f)(x) = x$ for $n > 1$,

$$\left\{\begin{array}{l} q_2(f)(x) = x - f(x)/f'(x), \\ \quad \vdots \\ q_{j+1}(f)(x) = S_j(0), \end{array}\right. \tag{42}$$

for $j = 2, \ldots, n-1$, where $S_j(y)$ is the inverse interpolatory polynomial of degree at most j such that

$$S_j(f(x)) = x, \quad S_j'(f(x)) = 1/f'(x), \quad S_j(f(q_\lambda(f)(x))) = q_\lambda(x) \quad (\lambda = 2, \ldots, j).$$

The iterative method is defined by $x_{k+1} = q_n(f)(x_k)$ starting with an initial approximation x_0.

For a fixed n, the methods K-T (41) and K-T (42) can be easily constructed using a recursive procedure on a computer, see [**9**]. In Section 6, Example 3, we have taken $n = 4$ to obtain the three-point methods of the eighth order.

Now we present a class of optimal n-point methods for arbitrary $n \geq 3$, proposed recently in [**15**]. Let $\psi_f \in \Psi_4$ and let us define a class of n-point methods

$$\begin{array}{ll} (1) & \phi_1(x) = N(x) = x - \dfrac{f(x)}{f'(x)}, \\ (2) & \phi_2(x) = \psi_f(x, \phi_1(x)), \\ (3) & \phi_3(x) = N\big(\phi_2(x)\big) = \phi_2(x) - \dfrac{f\big(\phi_2(x)\big)}{f'\big(\phi_2(x)\big)}, \\ & \vdots \end{array} \tag{43}$$

$$(n-1) \quad \phi_{n-1}(x) = N\big(\phi_{n-2}(x)\big) = \phi_{n-2}(x) - \frac{f\big(\phi_{n-2}(x)\big)}{f'\big(\phi_{n-2}(x)\big)},$$

$$(n) \quad \phi_n(x) = N(\phi_{n-1}(x)) = \phi_{n-1}(x) - \frac{f\big(\phi_{n-1}(x)\big)}{f'\big(\phi_{n-1}(x)\big)},$$

Sometimes, we will formally write $x = \phi_0(x)$. For an initial approximation $x = x_0$, from (43) we obtain the family of iterative methods

$$x_{k+1} = \phi_n(x_k) \quad (k = 0, 1, \ldots).$$

Applying Theorem 2 we find that the n-point methods (43), written in the composite form,

$$\varphi_n(x) = N\Big(N\Big(\cdots\Big(N\Big(\psi_f\big(x, \phi_1(x)\big)\Big)\Big)\cdots\Big)\Big),$$

have the order of convergence $r = \overbrace{2\cdot 2\cdots 2}^{n-2\ times}\cdot 4 = 2^n$ since the iterative function $\psi_f \in \Psi_4$ defines the iterative method of the fourth order. However, note that the iterative scheme (43) requires $2(n-2)+3 = 2n-1$ function evaluations per iterations, which is rather inefficient. Therefore, it is ultimately necessary to reduce the number of function evaluations using a suitable approximation of all derivatives $f'(\varphi_2(x)), \ldots, f'(\varphi_{n-1}(x))$ that appear in the steps (3)–(n).

For simplicity, we drop the argument of iterative functions φ_λ whenever it does not make a confusion. To carry out the substitution procedure of the derivatives $f'(\varphi_\lambda)$, we form the Hermite interpolation polynomial of the third order for the index $\lambda \in \{2, \ldots, n-1\}$,

$$(44) \quad h_{(\lambda)}(t) = a_1^{(\lambda)} + a_2^{(\lambda)}\big(t-\varphi_{\lambda-2}\big) + a_3^{(\lambda)}\big(t-\varphi_{\lambda-2}\big)^2 + a_4^{(\lambda)}\big(t-\varphi_{\lambda-2}\big)^3 \quad (\phi_0 = x),$$

satisfying the following conditions

$$(45) \qquad h_{(\lambda)}\big(\varphi_{\lambda-2}\big) = f\big(\varphi_{\lambda-2}\big),$$

$$(46) \qquad h_{(\lambda)}\big(\varphi_{\lambda-1}\big) = f\big(\varphi_{\lambda-1}\big),$$

$$(47) \qquad h_{(\lambda)}\big(\varphi_{\lambda}\big) = f\big(\varphi_{\lambda}\big),$$

and

$$(48) \qquad h'_{(\lambda)}\big(\varphi_{\lambda-2}\big) = f'\big(\varphi_{\lambda-2}\big).$$

This approach was already applied in Section 4. The subscript index λ in the denotation of the Hermite interpolation polynomial h should not be mixed with the degree of h (which is always three). This index points to the use of the polynomial h instead of the derivative f' at the point φ_λ in the $(\lambda+1)$-st step.

We have exactly four conditions (45)–(48) for determining four unknown coefficients $a_1^{(\lambda)}$, $a_2^{(\lambda)}$, $a_3^{(\lambda)}$, $a_4^{(\lambda)}$ in (44). From the conditions (45) and (48) we immediately find

$$(49) \qquad a_1^{(\lambda)} = f(\varphi_{\lambda-2}), \quad a_2^{(\lambda)} = f'(\varphi_{\lambda-2}).$$

Putting $\varphi_{\lambda-1}$ and φ_λ in (44) and employing the already determined values of $a_1^{(\lambda)}$ and $a_2^{(\lambda)}$ (given by (49)), from (46) and (47) we form the system of two linear equations and find its solutions $a_3^{(\lambda)}$ and $a_4^{(\lambda)}$ given by (26) and (27) (in different notation).

Substituting the coefficients $a_1^{(\lambda)}$, $a_2^{(\lambda)}$ (given by (49)) and $a_3^{(\lambda)}$ and $a_4^{(\lambda)}$ (found from the mentioned linear system) in (44) we completely determine the Hermite interpolation polynomial $h_{(\lambda)}$ which fits the function f. By differentiating (44) we obtain

$$h'_{(\lambda)}(t) = a_2^{(\lambda)} + 2a_3^{(\lambda)}\big(t-\varphi_{\lambda-2}\big) + 3a_4^{(\lambda)}\big(t-\varphi_{\lambda-2}\big)^2 \quad (\varphi_0 = x). \tag{50}$$

Rewriting the expression (28) for specific points, we find

$$\begin{aligned} h'_{(\lambda)}(\varphi_\lambda) = {} & 2\Big(f[\varphi_{\lambda-2},\varphi_\lambda] - f[\varphi_{\lambda-2},\varphi_{\lambda-1}]\Big) + f[\varphi_{\lambda-1},\varphi_\lambda] \\ & + \frac{\varphi_{\lambda-1}-\varphi_\lambda}{\varphi_{\lambda-1}-\varphi_{\lambda-2}}\Big(f[\varphi_{\lambda-2},\varphi_{\lambda-1}] - f'(\varphi_{\lambda-2})\Big) \end{aligned} \tag{51}$$

for $\lambda \in \{2,\dots,n-1\}$, assuming that $\varphi_0(x) = x$.

Starting from the iterative scheme (43), we substitute derivatives $f'(\varphi_2)$, $\dots, f'(\varphi_{n-1})$ appearing in the steps (3)–(n) by $h'_{(2)}(\varphi_2), \dots, h'_{(n-1)}(\varphi_{n-1})$, which are calculated by (51). This constructive way gives the following class of n-point iterative methods

$$\begin{aligned} &(1) \quad \phi_1(x) = N(x) = x - \frac{f(x)}{f'(x)}, \\ &(2) \quad \phi_2(x) = \psi_f(x), \\ &(3) \quad \phi_3(x) = \widetilde{N}\big(\phi_2(x)\big) := \phi_2(x) - \frac{f\big(\phi_2(x)\big)}{h'_{(2)}\big(\phi_2(x)\big)}, \\ &\qquad \vdots \\ &(n-1) \quad \varphi_{n-1}(x) = \widetilde{N}_{n-2}\big(\varphi_{n-2}(x)\big) := \varphi_{n-2}(x) - \frac{f\big(\varphi_{n-2}(x)\big)}{h'_{(n-2)}\big(\varphi_{n-2}(x)\big)}, \\ &(n) \quad \varphi_n(x) = \widetilde{N}_{n-1}\big(\varphi_{n-1}(x)\big) := \varphi_{n-1}(x) - \frac{f\big(\varphi_{n-1}(x)\big)}{h'_{(n-1)}\big(\varphi_{n-1}(x)\big)}. \end{aligned} \tag{52}$$

For an initial approximation $x = x_0$, from (52) we obtain the family of n-point iterative methods

$$x_{k+1} = \phi_n(x_k) = \widetilde{N}_{n-1}\Big(\widetilde{N}_{n-2}\Big(\cdots\Big(\widetilde{N}_2\Big(\psi_f(x_k)\Big)\Big)\cdots\Big)\Big) \quad (k = 0,1,\dots). \tag{53}$$

Remark 7. In the above scheme the modified Newton method defined by

$$\widetilde{N}_\lambda(\varphi_\lambda) := \varphi_\lambda - \frac{f(\varphi_\lambda)}{h'_{(\lambda)}(\varphi_\lambda)} \quad (\lambda = 2,\dots,n-1)$$

is involved. It was proved in [**15**] that this method has also the quadratic convergence. In addition, note that the computational cost of the evaluation of $h'_{(\lambda)}(\varphi_\lambda)$ is very cheap since we deal with the quadratic polynomial $h'_{(\lambda)}$ (given by (50)).

The convergence order of general class of multipoint methods (52) was considered in the following theorem, stated in [**15**].

Theorem 6. *The class of n-point iterative methods* (52) *has the order of convergence* $r = 2^n$.

The proof of this theorem goes by induction and using Theorem 4, see [**15**]. In essence, it is similar to the proof of Theorem 5.

Remark 8. Since the modified Newton methods $\widetilde{N}_2, \ldots, \widetilde{N}_{n-1}$ in the iterative scheme (52) are of the second order (see Remark 7), we can apply Theorem 2 to the class of n-point methods (52), written in the composite form

$$x_{k+1} = \varphi_n(x_k) = \widetilde{N}_{n-1}\Big(\widetilde{N}_{n-2}\Big(\cdots\Big(\widetilde{N}_2\Big(\psi_f\big(x_k, \phi_1(x_k)\big)\Big)\Big)\cdots\Big)\Big),$$

to obtain the order of convergence $r = \overbrace{2 \cdot 2 \cdots 2}^{n-2\ times} \cdot 4 = 2^n$.

Remark 9. We speak about the class of n-point methods (52) since the choice of various iterative functions $\psi_f \in \Psi_4$ gives a variety of n-point methods.

Remark 10. It is worth mentioning that the class of n-point iterative methods (52) requires $3 + (n-2) = n+1$ function evaluations per iteration. Indeed, all derivatives $h'_{(2)}, \ldots, h'_{(n-1)}$ in (52) are evaluated using the already calculated values and do not require any additional function calculations.

Therefore, according to Theorem 6 and Remark 10, it follows that the presented general class of n-point methods (52) is *optimal* and has the *optimal computational efficiency* $E_n^{(o)} = 2^{n/(n+1)}$, that is, $\varphi_n \in \Psi_{2^n}$ $(n \geq 3)$. This result supports the Kung-Traub conjecture. For example, for $n = 3,\ 4$ and 5 we evaluate the computational efficiency

$$E_3^{(o)} = 2^{3/4} \approx 1.682, \quad E_4^{(o)} = 2^{4/5} \approx 1.741, \quad E_5^{(o)} = 2^{5/6} \approx 1.782.$$

Obviously, the sequence $\{E_n^{(o)}\}$ is monotonically increasing and tends to 2.

6. NUMERICAL EXAMPLES

In this section we present the convergence behavior of some optimal multipoint methods considered in the previous sections. For demonstration, among many numerical examples we selected four examples implemented in the programming package *Mathematica* by the use of multi-precision arithmetic. The tables of results also contain the computational order of convergence, evaluated by the following formula (see [**20**])

$$\tilde{r} \approx \frac{\log |(x_{k+1} - \alpha)/(x_k - \alpha)|}{\log |(x_k - \alpha)/(x_{k-1} - \alpha)|}. \tag{54}$$

EXAMPLE 2. We applied two-step methods (7) considered in Section 3 to the test function

$$f(x) = \ln (x^2 + x + 2) - x + 1$$

for finding its root which lies in the interval $[2, 5]$. The exact root is $\alpha = 4.1525907\ldots$ and we chose the initial approximation $x_0 = 3$. The absolute values of the errors of approximations x_k in the first three iterations are displayed in Table 1, where $A(-t)$ means $A \times 10^{-t}$.

TABLE 1
$f(x) = \ln (x^2 + x + 2) - x + 1, \quad \alpha = 4.1525907\ldots, \quad x_0 = 3$

Two-point methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	$\tilde{r}$ (54)
(13)	2.51(−3)	2.46(−14)	2.27(−58)	4.0000
(14)	5.63(−3)	1.06(−12)	1.34(−51)	3.9997
(15)	9.50(−3)	1.21(−11)	3.21(−47)	3.9996
(17)	7.34(−3)	3.68(−12)	2.35(−49)	3.9995
(18), $\gamma = 1$	7.66(−4)	1.37(−16)	1.41(−67)	3.9998
(19), $a = 1$	3.91(−3)	1.95(−13)	1.21(−54)	3.9998
(20), $c = 1$	7.53(−3)	4.16(−12)	3.84(−49)	3.9994
(21)	1.35(−3)	1.33(−15)	1.25(−63)	4.0001

EXAMPLE 3. The three-point methods (38) (four variants), (41), (42) and (37) (four variants) were applied to find the root $\alpha = -1$ of the function

$$f(x) = e^{-x^2+x+2} - \cos(x + 1) + x^3 + 1,$$

starting from $x_0 = -0.7$. The absolute errors $|x_k - \alpha|$ in the first three iterations are given in Table 2.

TABLE 2
$f(x) = e^{-x^2+x+2} - \cos(x + 1) + x^3 + 1, \quad \alpha = -1, \quad x_0 = -0.7.$

Three-point methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	$\tilde{r}$ (54)
(38), $\varphi(t) = 12t^3 + 5t^2 + 2t + 1$	1.65(−7)	4.74(−58)	2.15(−462)	8.00019
(38), $\varphi(t) = \frac{5 - 2t + t^2}{5 - 12t}$	9.15(−7)	2.89(−52)	2.87(−416)	7.99997
(38), $\varphi(t) = \left(1 + \frac{t}{1 - 2t}\right)^2$	8.84(−7)	2.06(−52)	1.76(−417)	8.00017
(38), $\varphi(t) = \frac{1}{1 - 2t - t^2}$	9.21(−7)	3.11(−52)	5.20(−416)	9.00010
K-T (41), $\gamma = 0.01$	2.82(−7)	2.18(−55)	2.81(−440)	7.99990
K-T (42)	2.45(−7)	5.73(−56)	5.07(−445)	8.00010
(37), $h(t) = 1 + \frac{4t}{2 - 5t}$, $\beta = 3$	7.86(−7)	4.47(−52)	4.86(−414)	8.00006
(37), $h(t) = 1 + 2t + 5t^2 + t^3$, $\beta = 3$	1.19(−6)	1.69(−50)	2.92(−401)	7.99957
(37), $h(t) = \frac{1}{1 - 2t - t^2 + t^3}$, $\beta = 3$	8.83(−7)	1.19(−51)	1.32(−410)	7.99981
(37), $h(t) = (1 - 3t)^{-2/3}$, $\beta = 3$	7.12(−7)	1.95(−52)	6.17(−417)	8.00000

EXAMPLE 4. We applied the three-point methods $(N, \psi_f, \widetilde{N}_2)$ of the form (52) ($n = 3$) to the function

$$f(x) = e^x \sin 5x - 2$$

to find sufficiently close approximation to the root of f in the interval $[1, 1.6]$. We chose $x_0 = 1.2$ as the initial approximation. The sought zero on this interval with 20 accurate decimal digits is $\alpha = 1.36397318026371268918\ldots$. The results of the first three iterations are given in Table 3.

TABLE 3

$f(x) = e^x \sin 5x - 2, \quad \alpha = 1.3639731\ldots, \quad x_0 = 1.2.$

Three-point methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	$\tilde{r}$ (54)
$\{$Ostrowski's IM (4), $\widetilde{N}_2\}$	1.30(−5)	1.86(−39)	3.25(−310)	8.0001
$\{$King's IM $(5)_{\beta=-1}, \widetilde{N}_2\}$	7.01(−6)	1.20(−41)	9.05(−328)	7.9997
$\{$King's IM $(5)_{\beta=1}, \widetilde{N}_2\}$	2.33(−5)	2.18(−37)	1.29(−293)	7.9999
$\{$Euler-like's IM (21), $\widetilde{N}_2\}$	1.03(−5)	2.68(−40)	5.74(−317)	7.9998
$\{$Maheshwari's IM (22), $\widetilde{N}_2\}$	3.22(−5)	3.04(−36)	1.92(−284)	7.9999

EXAMPLE 5. The four-point methods $(N, \psi_f, \widetilde{N}_2, \widetilde{N}_3)$ of the form (52) ($n = 4$) of the sixteenth order were applied to the function

$$f(x) = (x - 2)(x^{10} + x + 1)e^{-x-1}.$$

The initial approximation $x_0 = 2.1$ and the same two-point methods were taken. The absolute errors of approximations are given in Table 4.

TABLE 4

$f(x) = (x - 2)(x^{10} + x + 1)e^{-x-1}, \quad \alpha = 2, \quad x_0 = 2.1.$

Four-point methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	$\tilde{r}$ (54)
$\{$Ostrowski's IM (4), $\widetilde{N}_2, \widetilde{N}_3\}$	5.41(−10)	6.13(−141)	6.99(−2236)	15.9986
$\{$King's IM $(5)_{\beta=-1}, \widetilde{N}_2, \widetilde{N}_3\}$	1.14(−8)	2.74(−118)	3.32(−1872)	16.0001
$\{$King's IM $(5)_{\beta=1}, \widetilde{N}_2, \widetilde{N}_3\}$	2.97(−9)	1.85(−110)	9.51(−1746)	16.0000
$\{$Euler-like's IM (21), $\widetilde{N}_2, \widetilde{N}_3\}$	3.01(−9)	1.46(−129)	5.85(−2055)	16.0031
$\{$Maheshwari's IM (22), $\widetilde{N}_2, \widetilde{N}_3\}$	4.66(−8)	8.26(−107)	8.49(−1687)	15.9997

From the results displayed in Tables 1–4 and a number of numerical experiments, it can be concluded that the convergence of the tested multipoint methods is remarkably fast. Although two iterative steps are quite satisfactory in solving most practical problems when the initial approximation is reasonably good. We have displayed results of the third iteration to demonstrate remarkably fast convergence of the tested methods. Since the approximations of great accuracy are obtained using only a few function evaluations per iteration, it is clear that these methods possess a very high computational efficiency. Actually, the convergence behavior of the considered multipoint methods strongly depends of the structure of tested functions and the accuracy of initial approximations.

From the last columns of Tables 1–4 we also observe that the computational order of convergence $\tilde{r}$ and the theoretical results for all considered methods perfectly match. However, this is the case when initial approximations are reasonably close to the roots. Discussing this subject we should note that the choice of good initial approximations is of great importance in the application of iterative methods, including multipoint methods. We note that an efficient method for finding initial approximations of great accuracy was recently proposed in [**17**], see also [**21**] and [**22**].

7. CONCLUSIONS

The primary aim of this survey paper is to present general classes of very efficient multipoint methods and to check the Kung-Traub conjecture for various values of $n \geq 2$. We end this paper with a natural question of practical interest: does the construction of faster and faster multipoint methods always have a justification? Certainly not if sufficiently good initial approximations are not provided. In those cases it is not possible, in practice, to attain the expected convergence speed (determined in a theoretical analysis), at least at the beginning of iterative process. It is always more profitable to put an effort into the determination of sufficiently close initial approximations to the roots, instead of applying very fast root-solvers with bad starting approximations.

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