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CHROMATICITY OF COMPLETE 4-PARTITE GRAPHS WITH CERTAIN STAR OR MATCHING DELETED

G. C. Lau, Y. H. Peng

Let $P(G, \lambda)$ be a chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H \mid H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 4-partite graphs G accordingly to the number of 5-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, we obtain new families of chromatically unique complete 4-partite graphs with certain star or matching deleted.

1. INTRODUCTION

All graphs considered in this paper are finite and simple. For a graph G, we denote by $P(G; \lambda)$ (or $P(G)$), the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (simply *χ-equivalent*), denoted $G \sim H$ if $P(G) = P(H)$. A graph G is said to be *chromatically unique* (simply *χ-unique*), if $H \sim G$ implies that $H \cong G$. A family $\mathcal{G}$ of graphs is said to be chromatically-closed (simply *χ-closed*) if for any graph $G \in \mathcal{G}$, $P(H) = P(G)$ implies that $H \in \mathcal{G}$. Many families of χ-unique graphs are known (see [**3, 4**]).

For a graph G, let $e(G)$, $v(G)$, $t(G)$ and $\chi(G)$ respectively be the number of vertices, edges, triangles and chromatic number of G. By $\overline{G}$, we denote the complement of G. Let O_n be an edgeless graph with n vertices. Also let $Q(G)$ and $K(G)$ be the number of induced subgraphs C_4 and complete subgraphs K_4 in G. Let S be a set of $s(\geq 1)$ edges of G. Denote by $G - S$ the graph obtained from

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G by deleting all edges in S, and by $\langle S\rangle$ the graph induced by S. For $t \geq 2$ and $1 \leq p_1 \leq p_2 \leq \cdots \leq p_t$, let $K(p_1, p_2, \ldots, p_t)$ be a complete t-partite graph with partition sets V_i such that $|V_i| = p_i$ for $i = 1, 2, \ldots, t$. In [**5, 8, 9, 10**], Lau and Peng, and Zhao et al. proved that certain families of complete t-partite graphs $(t = 3, 4, 5)$ with a matching or a star deleted are χ-unique. In this paper, we first characterize certain complete 4-partite graphs G accordingly to the number of 5-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, we obtain new families of chromatically unique complete 4-partite graphs with certain star or matching deleted.

2. PRELIMINARY RESULTS AND NOTATIONS

Let $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ denote the family $\{K(p_1, p_2, \ldots, p_t) - S \mid S \subset E(K(p_1, p_2, \ldots, p_t))$ and $|S| = s\}$. For $p_1 \geq s + 1$, we denote by $K_{i,j}^{-K(1,s)}(p_1, p_2, \ldots, p_t)$ the graph in $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ where the s edges in S induced a $K(1, s)$ with center in V_i and all the end-vertices in V_j, and by $K_{i,j}^{-sK_2}(p_1, p_2, \ldots, p_t)$ the graph in $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$ where the s edges in S induced a matching with end-vertices in V_i and V_j.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G, k)$ denote the number of k-independent partitions in G. If G is of order n, then $P(G, \lambda) = \sum_{k=1}^{n} \alpha(G, k)(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda - 1)\cdots(\lambda - k + 1)$ (see [**6**]). Therefore, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \ldots,$ if $G \sim H$.

For a graph G with n vertices, the polynomial $\sigma(G, x) = \sum_{k=1}^{n} \alpha(G, k)x^k$ is called the *σ-polynomial* of G (see [**1**]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H; $G \vee H$ denotes the graph whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{xy|x \in V(G)$ and $y \in V(H)\} \cup E(G) \cup E(H)$. Throughout this paper, all the t-partite graphs G under consideration are 2-connected with $\chi(G) = t$. For terms used but not defined here we refer to [**7**].

Lemma 2.1. (Koh and Teo [**3**]) *Let G and H be two graphs with $H \sim G$, then $v(G) = v(H)$, $e(G) = e(H)$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \ldots,$ and*

$$-Q(G) + 2K(G) = -Q(H) + 2K(H).$$

Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.

Lemma 2.2. (Brenti [**1**]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, \ldots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x).$$

Lemma 2.3. *Let G be a connected t-partite graph. If $H \sim G$, then there exists a complete t-partite graph $F = K(x_1, x_2, \ldots, x_t)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G)$.*

Proof. Since $V(G)$ has a t-independent partition, then $V(H)$ also has a t-independent partition with independent sets $V_1, V_2, \ldots, V_t$ such that $|V_i| = x_i$. Hence, H is a t-partite graph and there exists a complete t-partite graph $F = K(x_1, x_2, \ldots, x_t)$ such that $H = F - S'$. Since $H \sim G$, by Lemma 2.1, we have $s' = e(F) - e(G)$. □

Let $H = K(x_1, x_2, x_3, \ldots, x_t)$ and $H' = K(x_1, x_2, \ldots, x_i{+}1, \ldots, x_j{-}1, \ldots, x_t)$. If $i < j$ and $x_j - x_i \geq 2$, then H' is called an *improvement* of H.

Lemma 2.4. *Suppose $H' = K(x_1, x_2, \ldots, x_i + 1, \ldots, x_j - 1, \ldots, x_t)$ is an improvement of $H = K(x_1, x_2, x_3, \ldots, x_t)$, then $\alpha(H, t+1) > \alpha(H', t+1)$.*

Proof. Note that $\alpha(H', t+1) = \sum_{k=1}^{t} 2^{x_k - 1} + 2^{x_i - 1} - 2^{x_j - 2}$ and $\alpha(H, t+1) = \sum_{k=1}^{t} 2^{x_k - 1}$. Hence, $\alpha(H, t+1) - \alpha(H', t+1) = 2^{x_j - 2} - 2^{x_i - 1} \geq 2^{x_i - 1} > 0$. □

Suppose $G = K(p_1, p_2, \ldots, p_t)$ and $H = G - S$ for a set S of s edges of G. Define $\alpha_k(H) = \alpha(H, k) - \alpha(G, k)$ for $k \geq t + 1$.

Lemma 2.5. (Zhao [**9**]) *Let $G = K(p_1, p_2, \ldots, p_t)$ and $H = G - S$. If $p_1 \geq s + 1$, then*

$$s \leq \alpha_{t+1}(H) = \alpha(H, t+1) - \alpha(G, t+1) \leq 2^s - 1,$$

$\alpha_{t+1}(H) = s$ if and only if the subgraph induced by any $r \geq 2$ edges in S is not a complete multipartite graph, and $\alpha_{t+1}(H) = 2^s - 1$ if and only if $\langle S \rangle = K(1, s)$.

Lemma 2.6. (Dong et al. [**2**]) *Let p_1, p_2 and s be positive integers with $3 \leq p_1 \leq p_2$, then*

(i) $K_{1,2}^{-K(1,s)}(p_1, p_2)$ *is χ-unique for $1 \leq s \leq p_2 - 2$,*

(ii) $K_{2,1}^{-K(1,s)}(p_1, p_2)$ *is χ-unique for $1 \leq s \leq p_1 - 2$, and*

(iii) $K^{-sK_2}(p_1, p_2)$ *is χ-unique for $1 \leq s \leq p_1 - 1$.*

The following lemma is easily proved by induction.

Lemma 2.7. *Let s_i $(1 \leq i \leq t)$ be positive integers. Then*

$$\sum_{i=1}^{t} \binom{s_i}{2} = \binom{\sum_{i=1}^{t} s_i}{2} - \sum_{j=1}^{t-1} \left[s_j \sum_{i=j+1}^{t} s_i \right].$$

For a graph $G \in \mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively Type 2, and Type 3) if the vertices of the induced C_4 are in exactly two (respectively three, and four) partite sets of $V(G)$. An example of induced C_4 of Type 1, 2 and 3 is shown in Figure 1.

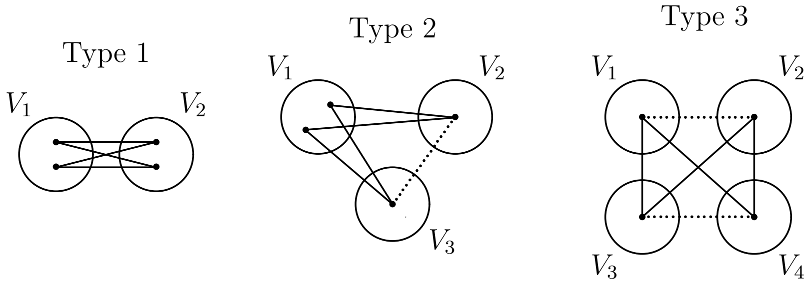

Figure 1: Three types of induced C_4

Suppose G is a graph in $\mathcal{K}^{-s}(p_1, p_2, \ldots, p_t)$. Let S_{ij} $(1 \le i \le t, 1 \le j \le t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \ge 0$. By Lemma 2.7, we have

Lemma 2.8. *Let $F = K(p_1, p_2, p_3, p_4)$ be a complete 4-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) = Q(F) - \sum_{1 \le i < j \le 4} (p_i - 1)(p_j - 1)s_{ij} + \binom{s}{2} - s_{12}(s_{13} + s_{14} + s_{23} + s_{24}) \\ - s_{13}(s_{14} + s_{23} + s_{34}) - s_{14}(s_{24} + s_{34}) - s_{23}(s_{24} + s_{34}) - s_{24}s_{34} \\ + \sum_{1 \le i < j \le 4} \left[s_{ij} \sum_{k \notin \{i,j\}} \binom{p_k}{2} \right], \end{aligned}$$

and

$$K(G) = K(F) - \sum_{\substack{i<j,\, k<\ell \\ \{i,j,k,\ell\} = \{1,2,3,4\}}} s_{ij} p_k p_\ell + (s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23}).$$

Moreover,

$$\max\{Q(G)\} = Q(F) - s(p_1 - 1)(p_2 - 1) + \binom{s}{2} + s\left(\binom{p_3}{2} + \binom{p_4}{2}\right)$$

and

$$\min\{K(G)\} = K(F) - sp_3p_4$$

if and only if each edge in S joins vertices in the same two partite sets of smallest size in F. In particular, $\max\{Q(G) - 2K(G)\}$ is attained if and only if each edge in S joins vertices in the same two partite sets of the smallest size in F.

Proof. Note that G has induced C_4 of Type 1, Type 2 or Type 3. Let $Q_1(G)$ (respectively, $Q_2(G)$ and $Q_3(G)$) be the number of induced C_4 of Type 1 (respectively, Type 2 and Type 3) in G. Observe that $S = \bigcup_{1\le i<j\le 4} S_{ij}$ with $s_{ij} \ge 0$. Hence,

$$\begin{aligned}Q_1(G) &= \sum_{1\le i<j\le 4} \binom{p_i}{2}\binom{p_j}{2} - \sum_{1\le i<j\le 4} (p_i-1)(p_j-1)s_{ij} + \sum_{1\le i<j\le 4} \binom{s_{ij}}{2} \\ &= Q(F) - \sum_{1\le i<j\le 4} (p_i-1)(p_j-1)s_{ij} + \binom{s}{2} - s_{12}(s_{13}+s_{14}+s_{23} \\ &\quad + s_{24}+s_{34}) - s_{13}(s_{14}+s_{23}+s_{24}+s_{34}) - s_{14}(s_{23}+s_{24}+s_{34}) \\ &\quad - s_{23}(s_{24}+s_{34}) - s_{24}s_{34}.\end{aligned}$$

We now find $Q_2(G)$. Since the number of 2-element subsets of V_k is $\binom{p_k}{2}$, we have

$$Q_2(G) = \sum_{1\le i<j\le 4}\left[s_{ij}\sum_{k\notin\{i,j\}}\binom{p_k}{2}\right].$$

It is obvious that $Q_3(G) = s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23}$. Therefore,

$$\begin{aligned}Q(G) &= Q(F) - \sum_{1\le i<j\le 4} (p_i-1)(p_j-1)s_{ij} + \binom{s}{2} - s_{12}(s_{13}+s_{14}+s_{23} \\ &\quad + s_{24}) - s_{13}(s_{14}+s_{23}+s_{34}) - s_{14}(s_{24}+s_{34}) - s_{23}(s_{24}+s_{34}) \\ &\quad - s_{24}s_{34} + \sum_{1\le i<j\le 4}\left[s_{ij}\sum_{k\notin\{i,j\}}\binom{p_k}{2}\right].\end{aligned}$$

Hence,

$$Q(G) \le Q(F) - \sum_{1\le i<j\le 4} (p_i-1)(p_j-1)s_{ij} + \binom{s}{2} + \sum_{1\le i<j\le 4}\left[s_{ij}\sum_{k\notin\{i,j\}}\binom{p_k}{2}\right]$$

with the equality holds if and only if $S = S_{ij}\cup S_{k\ell}$ for $i<j$, $k<\ell$ and $\{i,j,k,\ell\} = \{1,2,3,4\}$. Now, observe that $(p_1-1)(p_2-1)s \le (p_i-1)(p_j-1)s_{ij} + (p_k-1)(p_\ell-1)s_{k\ell}$ and the equality holds if and only if $S = S_{ij}\cup S_{k\ell}$ for $i<j$, $k<\ell$ and $\{i,j,k,\ell\} = \{1,2,3,4\}$ when $p_1 = p_2 = p_3 = p_4$, or $S = S_{12}$ otherwise. Hence, $\max\{Q(G)\}$ is attained if and only if S is a set of a possibility discussed above.

We now find $K(G)$. Observe that each K_4 subgraph in F has at most two edges in S. Let $K_m(G)$ be the number of K_4 subgraphs in F that contains m edges in S for $m = 1, 2$. Hence, $K(G) = K(F) - K_1(G) + K_2(G)$. Let v_iv_j denote an edge in S such that $v_i \in V_i$ and $v_j \in V_j$. Then, the number of K_4 subgraphs in F that contains v_iv_j is p_kp_ℓ where $i<j, k<\ell$ and $\{i,j,k,\ell\} = \{1,2,3,4\}$. Hence,

$$K_1(G) = \sum_{\substack{i<j,\,k<\ell \\ \{i,j,k,\ell\}=\{1,2,3,4\}}} s_{ij}p_kp_\ell.$$

Observe that there is a one-to-one correspondence between the set of Type 3 induced C_4 in G and the set of K_4 subgraphs in F that contain two edges in S. Hence, $K_2(G) = Q_3(G)$. It follows that

$$K(G) = K(F) - \sum_{\substack{i<j,\,k<\ell \\ \{i,j,k,\ell\}=\{1,2,3,4\}}} s_{ij}p_kp_\ell + (s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23}).$$

Therefore,

$$K(G) \geq K(F) - \sum_{\substack{i<j,\,k<\ell \\ \{i,j,k,\ell\}=\{1,2,3,4\}}} s_{ij}p_kp_\ell$$

with the equality holds if and only if $s' = s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23} = 0$. Now, observe that $sp_3p_4 \geq s_{12}p_3p_4 + s_{13}p_2p_4 + s_{14}p_2p_3 + s_{23}p_1p_4 + s_{24}p_1p_3 + s_{34}p_1p_2$. Hence, when $s' = 0$, the equality holds if and only if $S = S_{ij} \cup S_{k\ell} \cup S_{mn}$ where $(i,j) \in \{(1,2),(3,4)\}$, $(k,\ell) \in \{(1,3),(2,4)\}$ and $(m,n) \in \{(1,4),(2,3)\}$ when $p_1 = p_2 = p_3 = p_4$, or $s = s_{12}$ otherwise. Hence, $\min\{K(G)\}$ is attained if and only if S is a set of a possibility discussed above. Consequently, $\max\{Q(G) - 2K(G)\}$ is attained if and only if each edge in S joins vertices in the same two partite sets of the smallest size in F. This completes the proof. □

3. CHARACTERIZATION

In this section, we shall characterize certain complete 4-partite graphs $G = K(p_1,p_2,p_3,p_4)$ according to the number of 5-independent partitions of G where $p_4 - p_1 \leq 5$.

Lemma 3.1. *Let $G = K(p_1,p_2,p_3,p_4)$ be a complete 4-partite graph such that $p_1+p_2+p_3+p_4 = 4p$. Define $\theta(G) = (\alpha(G,5) - 2^{p+1} + 4)/2^{p-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(p,p,p,p)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(p-1,p,p,p+1)$;

(iii) $\theta(G) = 2$ *if and only if* $G = K(p-1,p-1,p+1,p+1)$;

(iv) $\theta(G) = 2\,\frac{1}{2}$ *if and only if* $G = K(p-2,p,p+1,p+1)$;

(v) $\theta(G) = 4$ *if and only if* $G = K(p-1,p-1,p,p+2)$;

(vi) $\theta(G) = 4\,\frac{1}{4}$ *if and only if* $G = K(p-3,p+1,p+1,p+1)$;

(vii) $\theta(G) = 4\,\frac{1}{2}$ *if and only if* $G = K(p-2,p,p,p+2)$;

(viii) $\theta(G) = 5\,\frac{1}{2}$ *if and only if* $G = K(p-2,p-1,p+1,p+2)$;

(ix) $\theta(G) = 6\frac{1}{4}$ *if and only if* $G = K(p-3, p, p+1, p+2)$;

(x) $\theta(G) = 9$ *if and only if* $G = K(p-2, p-2, p+2, p+2)$;

(xi) $\theta(G) = 9\frac{1}{4}$ *if and only if* $G = K(p-3, p-1, p+2, p+2)$;

(xii) $\theta(G) = 11$ *if and only if* $G = K(p-1, p-1, p-1, p+3)$;

(xiii) $\theta(G) = 11\frac{1}{2}$ *if and only if* $G = K(p-2, p-1, p, p+3)$;

(xiv) $\theta(G) = 13$ *if and only if* $G = K(p-2, p-2, p+1, p+3)$.

Proof. In order to complete the proof of the theorem, we first give a table about the θ-value of various complete 4-partite graphs with $4p$ vertices as shown in Table 1.

Table 1: Some complete 4-partite graphs with $4p$ vertices and their θ-values

G_i $(1 \le i \le 16)$	$\theta(G_i)$	G_i $(17 \le i \le 31)$	$\theta(G_i)$
$G_1 = K(p,p,p,p)$	0	$G_{17} = K(p-4, p+1, p+1, p+2)$	$8\frac{1}{8}$
$G_2 = K(p-1, p, p, p+1)$	1	$G_{18} = K(p-4, p, p+2, p+2)$	$10\frac{1}{8}$
$G_3 = K(p-1, p-1, p+1, p+1)$	2	$G_{19} = K(p-4, p, p+1, p+3)$	$14\frac{1}{8}$
$G_4 = K(p-2, p, p+1, p+1)$	$2\frac{1}{2}$	$G_{20} = K(p-2, p-1, p-1, p+4)$	$26\frac{1}{2}$
$G_5 = K(p-1, p-1, p, p+2)$	4	$G_{21} = K(p-2, p-2, p, p+4)$	27
$G_6 = K(p-2, p, p, p+2)$	$4\frac{1}{2}$	$G_{22} = K(p-3, p-1, p, p+4)$	$27\frac{1}{4}$
$G_7 = K(p-2, p-1, p+1, p+2)$	$5\frac{1}{2}$	$G_{23} = K(p-4, p, p, p+4)$	$28\frac{1}{8}$
$G_8 = K(p-3, p+1, p+1, p+1)$	$4\frac{1}{4}$	$G_{24} = K(p-3, p-2, p+2, p+3)$	$16\frac{3}{4}$
$G_9 = K(p-3, p, p+1, p+2)$	$6\frac{1}{4}$	$G_{25} = K(p-4, p-1, p+2, p+3)$	$17\frac{1}{8}$
$G_{10} = K(p-1, p-1, p-1, p+3)$	11	$G_{26} = K(p-3, p-2, p+1, p+4)$	$28\frac{3}{4}$
$G_{11} = K(p-2, p-1, p, p+3)$	$11\frac{1}{2}$	$G_{27} = K(p-5, p+1, p+2, p+2)$	$12\frac{1}{16}$
$G_{12} = K(p-3, p, p, p+3)$	$12\frac{1}{4}$	$G_{28} = K(p-5, p, p+2, p+3)$	$18\frac{1}{16}$
$G_{13} = K(p-2, p-2, p+2, p+2)$	9	$G_{29} = K(p-6, p+2, p+2, p+2)$	$16\frac{1}{32}$
$G_{14} = K(p-3, p-1, p+2, p+2)$	$9\frac{1}{4}$	$G_{30} = K(p-5, p+1, p+1, p+3)$	$16\frac{1}{16}$
$G_{15} = K(p-2, p-2, p+1, p+3)$	13	$G_{31} = K(p-6, p+1, p+2, p+3)$	$20\frac{1}{32}$
$G_{16} = K(p-3, p-1, p+1, p+3)$	$13\frac{1}{4}$		

By the definition of improvement, we have the following.

(i) G_1 is the improvement of G_2 with $\theta(G_2) = 1$;

(ii) G_2 is the improvement of G_3, G_4, G_5 and G_6 with $\theta(G_3) = 2$, $\theta(G_4) = 2\,\frac{1}{2}$, $\theta(G_5) = 4$ and $\theta(G_6) = 4\,\frac{1}{2}$;

(iii) G_3 is the improvement of G_4, G_5 and G_7 with $\theta(G_4) = 2\,\frac{1}{2}$, $\theta(G_5) = 4$ and $\theta(G_7) = 5\,\frac{1}{2}$;

(iv) G_4 is the improvement of G_6, G_7, G_8 and G_9 with $\theta(G_6) = 4\,\frac{1}{2}$, $\theta(G_7) = 5\,\frac{1}{2}$, $\theta(G_8) = 4\,\frac{1}{4}$ and $\theta(G_9) = 6\,\frac{1}{4}$;

(v) G_5 is the improvement of G_6, G_7, G_{10} and G_{11} with $\theta(G_6) = 4\,\frac{1}{2}$, $\theta(G_7) = 5\,\frac{1}{2}$, $\theta(G_{10}) = 11$ and $\theta(G_{11}) = 11\,\frac{1}{2}$;

(vi) G_6 is the improvement of G_7, G_9, G_{11} and G_{12} with $\theta(G_7) = 5\,\frac{1}{2}$, $\theta(G_9) = 6\,\frac{1}{4}$, $\theta(G_{11}) = 11\,\frac{1}{2}$ and $\theta(G_{12}) = 12\,\frac{1}{4}$;

(vii) G_7 is the improvement of G_9, G_{11}, G_{13}, G_{14}, G_{15} and G_{16} with $\theta(G_9) = 6\,\frac{1}{4}$, $\theta(G_{11}) = 11\,\frac{1}{2}$, $\theta(G_{13}) = 9$, $\theta(G_{14}) = 9\,\frac{1}{4}$, $\theta(G_{15}) = 13$ and $\theta(G_{16}) = 13\,\frac{1}{4}$;

(viii) G_8 is the improvement of G_9 and G_{17} with $\theta(G_9) = 6\,\frac{1}{4}$ and $\theta(G_{17}) = 8\,\frac{1}{8}$;

(ix) G_9 is the improvement of G_{12}, G_{14}, G_{16}, G_{17}, G_{18} and G_{19} with $\theta(G_{12}) = 12\,\frac{1}{4}$, $\theta(G_{14}) = 9\,\frac{1}{4}$, $\theta(G_{16}) = 13\,\frac{1}{4}$, $\theta(G_{17}) = 8\,\frac{1}{8}$, $\theta(G_{18}) = 10\,\frac{1}{8}$ and $\theta(G_{19}) = 14\,\frac{1}{8}$;

(x) G_{10} is the improvement of G_{11} and G_{20} with $\theta(G_{11}) = 11\,\frac{1}{2}$ and $\theta(G_{20}) = 26\,\frac{1}{2}$;

(xi) G_{11} is the improvement of G_{12}, G_{15}, G_{16}, G_{20}, G_{21} and G_{22} with $\theta(G_{12}) = 12\,\frac{1}{4}$, $\theta(G_{15}) = 13$, $\theta(G_{16}) = 13\,\frac{1}{4}$, $\theta(G_{20}) = 26\,\frac{1}{2}$, $\theta(G_{21}) = 27$ and $\theta(G_{22}) = 27\,\frac{1}{4}$;

(xii) G_{12} is the improvement of G_{16}, G_{19}, G_{22} and G_{23} with $\theta(G_{16}) = 13\,\frac{1}{4}$, $\theta(G_{19}) = 14\frac{1}{8}$, $\theta(G_{22}) = 27\,\frac{1}{4}$ and $\theta(G_{23}) = 28\,\frac{1}{8}$;

(xiii) G_{13} is the improvement of G_{14}, G_{15} and G_{24} with $\theta(G_{14}) = 9\,\frac{1}{4}$, $\theta(G_{15}) = 13$ and $\theta(G_{24}) = 16\,\frac{3}{4}$;

(xiv) G_{14} is the improvement of G_{16}, G_{18}, G_{24} and G_{25} with $\theta(G_{16}) = 13\,\frac{1}{4}$, $\theta(G_{18}) = 10\,\frac{1}{8}$, $\theta(G_{24}) = 16\,\frac{3}{4}$ and $\theta(G_{25}) = 17\,\frac{1}{8}$;

(xv) G_{15} is the improvement of G_{16}, G_{21}, G_{24} and G_{26} with $\theta(G_{16}) = 13\,\frac{1}{4}$, $\theta(G_{21}) = 27$, $\theta(G_{24}) = 16\,\frac{3}{4}$ and $\theta(G_{26}) = 28\,\frac{3}{4}$;

(xvi) G_{18} is the improvement of G_{19}, G_{25}, G_{27} and G_{28} with $\theta(G_{19}) = 14\,\frac{1}{8}$, $\theta(G_{25}) = 17\,\frac{1}{8}$, $\theta(G_{27}) = 12\,\frac{1}{16}$ and $\theta(G_{28}) = 18\,\frac{1}{16}$;

(xvii) G_{27} is the improvement of G_{28}, G_{29}, G_{30} and G_{31} with $\theta(G_{28}) = 18\,\frac{1}{16}$, $\theta(G_{29}) = 16\,\frac{1}{32}$, $\theta(G_{30}) = 16\,\frac{1}{16}$ and $\theta(G_{31}) = 20\,\frac{1}{32}$.

Hence, by Lemma 2.4 and the above arguments, we know that (i) to (xiv) hold. The proof is thus complete. □

Similar to the proof of Lemma 3.1, we obtain Lemmas 3.2 to 3.4.

Lemma 3.2. *Let G be a complete 4-partite graph with $4p+1$ vertices. Define $\theta(G) = (\alpha(G,5) - 2^{p-1} - 2^{p+1} + 4)/2^{p-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(p,p,p,p+1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(p-1,p,p+1,p+1)$;

(iii) $\theta(G) = 2\,\frac{1}{2}$ *if and only if* $G = K(p-2,p+1,p+1,p+1)$;

(iv) $\theta(G) = 3$ *if and only if* $G = K(p-1,p,p,p+2)$;

(v) $\theta(G) = 4$ *if and only if* $G = K(p-1,p-1,p+1,p+2)$;

(vi) $\theta(G) = 4\,\frac{1}{2}$ *if and only if* $G = K(p-2,p,p+1,p+2)$;

(vii) $\theta(G) = 6\,\frac{1}{4}$ *if and only if* $G = K(p-3,p+1,p+1,p+2)$;

(viii) $\theta(G) = 7\,\frac{1}{2}$ *if and only if* $G = K(p-2,p-1,p+2,p+2)$;

(ix) $\theta(G) = 8\,\frac{1}{4}$ *if and only if* $G = K(p-3,p,p+2,p+2)$;

(x) $\theta(G) = 10$ *if and only if* $G = K(p-1,p-1,p,p+3)$;

(xi) $\theta(G) = 10\,\frac{1}{2}$ *if and only if* $G = K(p-2, p, p, p+3)$;

(xii) $\theta(G) = 11\,\frac{1}{2}$ *if and only if* $G = K(p-2, p-1, p+1, p+3)$;

(xiii) $\theta(G) = 15$ *if and only if* $G = K(p-2, p-2, p+2, p+3)$;

(xiv) $\theta(G) = 25$ *if and only if* $G = K(p-1, p-1, p-1, p+4)$.

Lemma 3.3. *Let G be a complete 4-partite graph with $4p+2$ vertices. Define $\theta(G) = (\alpha(G,5) - 2^p - 2^{p+1} + 4)/2^{p-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(p, p, p+1, p+1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(p-1, p+1, p+1, p+1)$;

(iii) $\theta(G) = 2$ *if and only if* $G = K(p, p, p, p+2)$;

(iv) $\theta(G) = 3$ *if and only if* $G = K(p-1, p, p+1, p+2)$;

(v) $\theta(G) = 4\,\frac{1}{2}$ *if and only if* $G = K(p-2, p+1, p+1, p+2)$;

(vi) $\theta(G) = 6$ *if and only if* $G = K(p-1, p-1, p+2, p+2)$;

(vii) $\theta(G) = 6\,\frac{1}{2}$ *if and only if* $G = K(p-2, p, p+2, p+2)$;

(viii) $\theta(G) = 8\,\frac{1}{4}$ *if and only if* $G = K(p-3, p+1, p+2, p+2)$;

(ix) $\theta(G) = 9$ *if and only if* $G = K(p-1, p, p, p+3)$;

(x) $\theta(G) = 10$ *if and only if* $G = K(p-1, p-1, p+1, p+3)$;

(xi) $\theta(G) = 10\,\frac{1}{2}$ *if and only if* $G = K(p-2, p, p+1, p+3)$;

(xii) $\theta(G) = 13\,\frac{1}{2}$ *if and only if* $G = K(p-2, p-1, p+2, p+3)$;

(xiii) $\theta(G) = 21$ *if and only if* $G = K(p-2, p-2, p+3, p+3)$;

(xiv) $\theta(G) = 24$ *if and only if* $G = K(p-1, p-1, p, p+4)$.

Lemma 3.4. *Let G be a complete 4-partite graph with $4p+3$ vertices. Define $\theta(G) = (\alpha(G,5) - 2^{p-1} - 2^p - 2^{p+1} + 4)/2^{p-1}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(p, p+1, p+1, p+1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(p, p, p+1, p+2)$;

(iii) $\theta(G) = 1\,\frac{1}{2}$ *if and only if* $G = K(p-1, p+1, p+1, p+2)$;

(iv) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(p-1, p, p+2, p+2)$;

(v) $\theta(G) = 3\frac{1}{4}$ *if and only if* $G = K(p-2, p+1, p+2, p+2)$;

(vi) $\theta(G) = 4$ *if and only if* $G = K(p, p, p, p+3)$;

(vii) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(p-1, p, p+1, p+3)$;

(viii) $\theta(G) = 5\frac{1}{8}$ *if and only if* $G = K(p-3, p+2, p+2, p+2)$;

(ix) $\theta(G) = 5\frac{1}{4}$ *if and only if* $G = K(p-2, p+1, p+1, p+3)$;

(x) $\theta(G) = 6$ *if and only if* $G = K(p-1, p-1, p+2, p+3)$;

(xi) $\theta(G) = 6\frac{1}{4}$ *if and only if* $G = K(p-2, p, p+2, p+3)$;

(xii) $\theta(G) = 9\frac{3}{4}$ *if and only if* $G = K(p-2, p-1, p+3, p+3)$;

(xiii) $\theta(G) = 11\frac{1}{2}$ *if and only if* $G = K(p-1, p, p, p+4)$;

(xiv) $\theta(G) = 12$ *if and only if* $G = K(p-1, p-1, p+1, p+4)$.

4. CHROMATICALLY CLOSED 4-PARTITE GRAPHS

In this section, we deduce the χ-closed families of graphs obtained from the graphs in Lemma 3.1 to Lemma 3.4 with a set S of s edges deleted.

Lemma 4.1. *The family of graphs* $\mathcal{K}^{-s}(p_1, p_2, p_3, p_4)$ *where* $p_1 + p_2 + p_3 + p_4 = 4p$, $p_4 - p_1 \leq 5$ *and* $p_1 \geq s+3$ *is* χ*-closed.*

Proof. By Lemma 3.1, there are 14 cases to consider. Denote each graph in Lemma 3.1 (i), (ii), ..., (xiv) by $G_1, G_2, \ldots, G_{14}$, respectively. Suppose $H \sim G_i - S$. It suffices to show that $H \in \{G_i - S\}$. By Lemma 2.1, we know there exists a complete 4-partite graph $F = K(w, x, y, z)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s \geq 0$.

Case i. Let $G = G_1$ with $p \geq s+2$. In this case, $H \sim G - S \in \mathcal{K}^{-s}(p, p, p, p)$. By Lemma 2.5,

$$\alpha(G-S, 5) = \alpha(G, 5) + \alpha_5(G-S) \text{ with } s \leq \alpha_5(G-S) \leq 2^s - 1,$$

$$\alpha(F-S', 5) = \alpha(F, 5) + \alpha_5(F-S') \text{ with } 0 \leq s' \leq \alpha_5(F-S').$$

Hence,

$$\alpha(F-S', 5) - \alpha(G-S, 5) = \alpha(F, 5) - \alpha(G, 5) + \alpha_5(F-S') - \alpha_5(G-S).$$

By definition, $\alpha(F,5)-\alpha(G,5)=2^{p-2}(\theta(F)-\theta(G))$. By Lemma 3.1, $\theta(F)\geq 0$. Suppose $\theta(F)>0$, then

$$\begin{aligned}\alpha(F-S',5)-\alpha(G-S,5) &\geq 2^{p-2}+\alpha_5(F-S')-\alpha_5(G-S)\\ &\geq 2^s+\alpha_5(F-S')-2^s+1\geq 1,\end{aligned}$$

contradicting $\alpha(F-S',5)=\alpha(G-S,5)$. Hence, $\theta(F)=0$ and so $F\cong G$ and $s=s'$. Therefore, $H\in\mathcal{K}^{-s}(p,p,p,p)$.

Case ii. Let $G=G_2$ with $p\geq s+2$. In this case, $H\sim G-S\in\mathcal{K}^{-s}(p-1,p,p,p+1)$. By Lemma 2.5,

$$\alpha(G-S,5)=\alpha(G,5)+\alpha_5(G-S) \text{ with } s\leq\alpha_5(G-S)\leq 2^s-1,$$

$$\alpha(F-S',5)=\alpha(F,5)+\alpha_5(F-S') \text{ with } 0\leq s'\leq\alpha_5(F-S').$$

Hence,

$$\alpha(F-S',5)-\alpha(G-S,5)=\alpha(F,5)-\alpha(G,5)+\alpha_5(F-S')-\alpha_5(G-S).$$

By definition, $\alpha(F,5)-\alpha(G,5)=2^{p-2}(\theta(F)-\theta(G))$. Suppose $\theta(F)\neq\theta(G)$. We consider two subcases.

Subcase a. $\theta(F)<\theta(G)$. By Lemma 3.1, $F=G_1$ and so $H=G_1-S'\in\{G_1-S'\}$. However, $G-S\notin\{G_1-S'\}$ since $\{G_1-S'\}$ is χ-closed, a contradiction.

Subcase b. $\theta(F)>\theta(G)$. By Lemma 3.1, $\alpha(F,5)-\alpha(G,5)\geq 2^{p-2}$. So,

$$\begin{aligned}\alpha(F-S',5)-\alpha(G-S,5) &\geq 2^{p-2}+\alpha_5(F-S')-\alpha_5(G-S)\\ &\geq 2^s+\alpha_5(F-S')-2^s+1\geq 1,\end{aligned}$$

contradicting $\alpha(F-S',5)=\alpha(G-S,5)$. Hence, $\theta(F)-\theta(G)=0$ and so $F=G$ and $s=s'$. Therefore, $H\in\mathcal{K}^{-s}(p-1,p,p,p+1)$.

Using Table 1, we can prove (iii) to (xiv) in a similar way. This completes the proof. □

Similarly, we can prove Lemmas 4.2 to 4.4.

Lemma 4.2. *The family of graphs $\mathcal{K}^{-s}(p_1,p_2,p_3,p_4)$ where $p_1+p_2+p_3+p_4=4p+1$, $p_4-p_1\leq 5$ and $p_1\geq s+4$ is χ-closed.*

Lemma 4.3. *The family of graphs $\mathcal{K}^{-s}(p_1,p_2,p_3,p_4)$ where $p_1+p_2+p_3+p_4=4p+2$, $p_4-p_1\leq 5$ and $p_1\geq s+5$ is χ-closed.*

Lemma 4.4. *The family of graphs $\mathcal{K}^{-s}(p_1,p_2,p_3,p_4)$ where $p_1+p_2+p_3+p_4=4p+3$, $p_4-p_1\leq 5$ and $p_1\geq s+2$ is χ-closed.*

5. CHROMATICALLY UNIQUE 4-PARTITE GRAPHS

The following two Lemmas give several families of chromatically unique complete 4-partite graphs having $4p$ vertices with a set S of s edges deleted where the deleted edges induce a star $K(1,s)$ and a matching sK_2, respectively.

Lemma 5.1. *The graphs* $K_{i,j}^{-K(1,s)}(p_1,p_2,p_3,p_4)$ *where* $p_1+p_2+p_3+p_4=4p$, $p_4-p_1\le 5$ *and* $p_1\ge s+3$ *are* χ*-unique for* $1\le i\ne j\le 4$.

Proof. By Lemma 3.1, there are 14 cases to consider. Denote each graph in Lemma 3.1 (i), (ii), ..., (xiv) by $G_1, G_2, \dots, G_{14}$, respectively. The proofs for each graph obtained from G_i $(i=1,2,\dots,14)$ are similar, so we only give the detailed proof for the graphs obtained from G_2 below.

By Lemma 2.5 and 4.1, we know that $\mathcal{K}_{i,j}^{-K(1,s)}(p-1,p,p,p+1)=\{K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\mid(i,j)\in\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}\}$ is χ-closed for $p\ge s+2$. Note that

$$\begin{aligned}
&t\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)=t(G_2)-2p-1 \text{ for } (i,j)\in\{(1,2),(2,1)\},\\
&t\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)=t(G_2)-2p \text{ for } (i,j)\in\{(1,4),(4,1)\},\\
&t\big(K_{2,3}^{-K(1,s)}(p-1,p,p,p+1)\big)=t(G_2)-2p,\\
&t\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)=t(G_2)-2p+1 \text{ for } (i,j)\in\{(2,4),(4,2)\}.
\end{aligned}$$

By Lemmas 2.2 and 2.6, we conclude that $\sigma\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)\ne\sigma\big(K_{j,i}^{-K(1,s)}(p-1,p,p,p+1)\big)$ for each $(i,j)\in\{(1,2),(1,4),(2,4)\}$. We now show that $K_{2,3}^{-K(1,s)}(p-1,p,p,p+1)\not\sim K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)$ for $(i,j)\in\{(1,4),(4,1)\}$. We have

$$\begin{aligned}
&Q\big(K_{2,3}^{-K(1,s)}(p-1,p,p,p+1)\big)=Q(G_2)-(p-1)^2+\binom{s}{2}+\binom{p-1}{2}+\binom{p+1}{2},\\
&Q\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)=Q(G_2)-p(p-2)+\binom{s}{2}+2\binom{p}{2}\\
&\qquad\qquad \text{for } (i,j)\in\{(1,4),(4,1)\},
\end{aligned}$$

with

$$Q\big(K_{2,3}^{-K(1,s)}(p-1,p,p,p+1)\big)-Q\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)=0,$$

and that

$$\begin{aligned}
&K\big(K_{2,3}^{-K(1,s)}(p-1,p,p,p+1)\big)=K(G_2)-s(p-1)(p+1),\\
&K\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)=K(G_2)-sp^2 \text{ for } (i,j)\in\{(1,4),(4,1)\},
\end{aligned}$$

with

$$K\big(K_{2,3}^{-K(1,s)}(p-1,p,p,p+1)\big)-K\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)=s.$$

This means $2K\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big) - Q\big(K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)\big)$ $\neq 2K\big(K_{2,3}^{-K(1,s)}(p-1,p,p,p+1)\big) - Q\big(K_{2,3}^{-K(1,s)}(p-1,p,p,p+1)\big)$, contradicting Lemma 2.1. Hence, $K_{i,j}^{-K(1,s)}(p-1,p,p,p+1)$ where $p \geq s+2$ is χ-unique for $1 \leq i \neq j \leq 4$. The proof is thus complete. □

Lemma 5.2. *The graphs $K_{1,2}^{-sK_2}(p_1,p_2,p_3,p_4)$ where $p_1+p_2+p_3+p_4 = 4p$, $p_4-p_1 \leq 5$ and $p_1 \geq s+3$ are χ-unique.*

Proof. By Lemma 3.1, there are 14 cases to consider. Denote each graph in Lemma 3.1 (i), (ii), ..., (xiv) by $G_1, G_2, \ldots, G_{14}$, respectively. For a graph $K(w,x,y,z)$, let $S = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_s\}$ be a set of s edges in $E(K(w,x,y,z))$ and let $t(\epsilon_i)$ denote the number of triangles containing ϵ_i in $K(w,x,y,z)$. The proofs for each graph obtained from G_i $(i = 1,2,\ldots,14)$ are similar, so we only give the proofs for the graphs obtained from G_2 and G_3 as follows.

Suppose $H \sim G = K_{1,2}^{-sK_2}(p-1,p,p,p+1)$ for $p \geq s+2$. By Lemma 4.1 and Lemma 2.1, $H \in \mathcal{K}^{-s}(p-1,p,p,p+1)$ and $\alpha_5(H) = \alpha_5(G) = s$. Let $H = F - S$ where $F = K(p-1,p,p,p+1)$. Clearly, $t(\epsilon_i) \leq 2p+1$ for each $\epsilon_i \in S$. So,

$$t(H) \geq t(F) - s(2p+1) \tag{1}$$

with equality holds only if $t(\epsilon_i) = 2p+1$ for all $\epsilon_i \in S$. Since $t(H) = t(G) = t(F) - s(2p+1)$, equality in (1) holds with $t(\epsilon_i) = 2p+1$ for all $\epsilon_i \in S$. Therefore, each edge in S has an end-vertex in V_1 and another end-vertex in V_2 or in V_3. Moreover, S must induce a matching in F. Otherwise, equality in (1) does not hold or $\alpha_5(H) > s$. By Lemma 2.8, $Q(G) - 2K(G) = Q(F) - s(p-2)(p-1) + \binom{s}{2}$ $+s\left[\binom{p}{2} + \binom{p+1}{2}\right] - 2[K(F) - sp(p+1)] \geq Q(H) - 2K(H)$ and the equality holds if and only if $s = s_{1j}$ $(2 \leq j \leq 3)$. Hence, $\langle S\rangle \cong sK_2$ and $H \cong G$.

Now, suppose $H \sim G = K_{1,2}^{-sK_2}(p-1,p-1,p+1,p+1)$ for $p \geq s+3$. By Lemma 4.1 and Lemma 2.1, $H \in \mathcal{K}^{-s}(p-1,p-1,p+1,p+1)$ and $\alpha_5(H)$ $= \alpha_5(G) = s$. Let $H = F - S$ where $F = K(p-1,p-1,p+1,p+1)$. Clearly, $t(\epsilon_i) \leq 2p+2$ for each $\epsilon_i \in S$. So,

$$t(H) \geq t(F) - s(2p+2) \tag{2}$$

with equality holds only if $t(\epsilon_i) = 2p+2$ for all $\epsilon_i \in S$. Since $t(H) = t(G) = t(F) - s(2p+2)$, equality in (2) holds with $t(\epsilon_i) = 2p+2$ for all $\epsilon_i \in S$. Therefore, each edge in S has an end-vertex in V_1, and another end-vertex in V_2. Moreover, S must induce a matching in F. Otherwise, $\alpha_5(H) > s$. Hence, $\langle S\rangle \cong sK_2$ and $H \cong G$. The proof is thus complete. □

Similarly to the proofs of Lemmas 5.1 and 5.2, we can prove the following six lemmas.

Lemma 5.3. *The graphs $K_{i,j}^{-K(1,s)}(p_1,p_2,p_3,p_4)$ where $p_1+p_2+p_3+p_4 = 4p+1$, $p_4-p_1 \leq 5$ and $p_1 \geq s+4$ are χ-unique for $1 \leq i \neq j \leq 4$.*

Lemma 5.4. *The graphs* $K_{1,2}^{-sK_2}(p_1, p_2, p_3, p_4)$ *where* $p_1 + p_2 + p_3 + p_4 = 4p + 1$, $p_4 - p_1 \leq 5$ *and* $p_1 \geq s + 4$ *are* χ*-unique.*

Lemma 5.5. *The graphs* $K_{i,j}^{-K(1,s)}(p_1, p_2, p_3, p_4)$ *where* $p_1 + p_2 + p_3 + p_4 = 4p + 2$, $p_4 - p_1 \leq 5$ *and* $p_1 \geq s + 5$ *are* χ*-unique for* $1 \leq i \neq j \leq 4$.

Lemma 5.6. *The graphs* $K_{1,2}^{-sK_2}(p_1, p_2, p_3, p_4)$ *where* $p_1 + p_2 + p_3 + p_4 = 4p + 2$, $p_4 - p_1 \leq 5$ *and* $p_1 \geq s + 5$ *are* χ*-unique.*

Lemma 5.7. *The graphs* $K_{i,j}^{-K(1,s)}(p_1, p_2, p_3, p_4)$ *where* $p_1 + p_2 + p_3 + p_4 = 4p + 3$, $p_4 - p_1 \leq 5$ *and* $p_1 \geq s + 2$ *are* χ*-unique for* $1 \leq i \neq j \leq 4$.

Lemma 5.8. *The graphs* $K_{1,2}^{-sK_2}(p_1, p_2, p_3, p_4)$ *where* $p_1 + p_2 + p_3 + p_4 = 4p + 3$, $p_4 - p_1 \leq 5$ *and* $p_1 \geq s + 2$ *are* χ*-unique.*

We thus have our main theorem as follows.

Theorem 5.1. *The graphs* $K_{i,j}^{-K(1,s)}(p_1, p_2, p_3, p_4)$ *where* $1 \leq i \neq j \leq 4$, *and* $K_{1,2}^{-sK_2}(p_1, p_2, p_3, p_4)$ *are* χ*-unique for integers* $p_4 - p_1 \leq 5$ *and* $p_1 \geq s + 5$.

Note that our results significantly improve the condition of Theorems 6.5.2 to 6.5.4 in [**9**] especially when s is “sufficiently” large.

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