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TRIPLE INTEGRAL IDENTITIES AND ZETA FUNCTIONS

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Some new identities are given for the representation of binomial sums. A master theorem is developed from which integral and closed form results, in terms of Zeta functions and harmonic numbers, are developed for sums of the type $\displaystyle\sum_{n\geq 1}\frac{t^n}{n^4\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+\ell}{\ell}}$.

1. INTRODUCTION

The following definitions and relations will be used throughout this paper. The Gamma function

$$\Gamma(z)=\int_0^\infty w^{z-1}e^{-w}\mathrm{d}w,\ \text{for}\ \operatorname{Re}(z)>0,$$

and Beta function

$$B(s,z)=\int_0^1 w^{s-1}(1-w)^{z-1}\,\mathrm{d}w=\frac{\Gamma(s)\Gamma(w)}{\Gamma(s+w)}$$

for $\operatorname{Re}(s)>0$ and $\operatorname{Re}(z)>0$. The well known Riemann zeta function is defined as:

$$\zeta(z)=\sum_{r=1}^{\infty}\frac{1}{r^z},\ \operatorname{Re}(z)>1.$$

The generalized harmonic numbers of order α are given by

$$H_n^{(\alpha)}=\sum_{r=1}^{n}\frac{1}{r^\alpha}\ \text{for}\ (\alpha,n)\in\mathbb{N}:=\{1,2,3,\ldots\}$$

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and for $\alpha = 1$,

$$H_n^{(1)} = H_n = \int\limits_0^1 \frac{1-t^n}{1-t}\,\mathrm{d}t = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi\left(n+1\right),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r} - \log\left(n\right)\right) = -\psi\left(1\right) \approx 0.5772156649015\ldots$$

and where $\psi\left(z\right)$ denotes the Psi, or digamma function, defined by

$$\psi\left(z\right) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma\left(z\right) = \frac{\Gamma'\left(z\right)}{\Gamma\left(z\right)} = \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right) - \gamma.$$

Similarly

$$\psi\left(z+1\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right) - \gamma,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\psi\left(z\right) = \psi'\left(z\right) = \sum_{n=0}^{\infty}\frac{1}{\left(n+z\right)^2}\,,\ \psi'\left(z+1\right) = \sum_{n=1}^{\infty}\frac{1}{\left(n+z\right)^2} \quad \text{and} \quad \psi''\left(z+1\right) = -\sum_{n=1}^{\infty}\frac{2}{\left(n+z\right)^3}\,, \tag{1.1}$$

also

$$2\psi\left(2z\right) = \psi\left(z\right) + \psi\left(z+\frac{1}{2}\right) + 2\ln 2. \tag{1.2}$$

The aim of this paper is to give a proof of the following theorem. Subsequently a number of interesting corollaries follow which give new closed form expressions of binomial sums in terms of zeta functions and harmonic numbers.

Theorem 1. *Let* $|t| \le 1$, a_i *and* $b_i > 0$ *for* $i = 1, 2, \ldots, s$ *where* $s \in \mathbb{N}$. *Then*

$$\sum_{n\ge1}\frac{t^n}{n^{s+1}\prod\limits_{i=1}^{s}\binom{a_i n + b_i}{b_i}} \tag{1.3}$$

$$= -\prod_{i=1}^{s} a_i \int_{x_i\in(0,1)^s} \prod_{i=1}^{s}\frac{\left(1-x_i\right)^{b_i}}{x_i}\log\left(1 - t\prod_{i=1}^{s}x_i^{a_i}\right)\mathrm{d}x_i \tag{1.4}$$

for $\left|t\prod\limits_{i=1}^{s}x_i^{a_i}\right| < 1$.

Proof. Consider, from (1.3)

$$\sum_{n\geq 1}\frac{t^n}{n^{s+1}\prod\limits_{i=1}^{s}\binom{a_in+b_i}{b_i}}=\sum_{n\geq 1}\frac{t^n}{n^{s+1}}\prod_{i=1}^{s}\frac{\Gamma\left(b_i+1\right)\Gamma\left(a_in+1\right)}{\Gamma\left(a_in+b_i+1\right)}$$

$$=\sum_{n\geq 1}\frac{t^n}{n^{s+1}}\prod_{i=1}^{s}\frac{a_in^s\Gamma\left(b_i+1\right)\Gamma\left(a_in\right)}{\Gamma\left(a_in+b_i+1\right)}=\sum_{n\geq 1}\frac{t^n}{n}\prod_{i=1}^{s}a_iB\left(a_in,b_i+1\right),$$

using the definition of the Beta function produces the $s-$ fold integral representation

$$\sum_{n\geq 1}\frac{t^n}{n^{s+1}\prod\limits_{i=1}^{s}\binom{a_in+b_i}{b_i}}=\prod_{i=1}^{s}a_i\int_{x_i\in(0,1)^s}\prod_{i=1}^{s}\frac{\left(1-x_i\right)^{b_i}}{x_i}\sum_{n\geq 1}\frac{\prod\limits_{i=1}^{s}\left(tx_i^{a_i}\right)^n}{n}\mathrm{d}x_i$$

$$=-\prod_{i=1}^{s}a_i\int_{x_i\in(0,1)^s}\prod_{i=1}^{s}\frac{\left(1-x_i\right)^{b_i}}{x_i}\log\left(1-t\prod_{i=1}^{s}x_i^{a_i}\right)\mathrm{d}x_i$$

for $\left|t\prod\limits_{i=1}^{s}x_i^{a_i}\right|<1.$ □

The representation of sums in terms of integrals is extremely useful because it allows one to estimate bounds on the sums in cases they cannot be written in closed form.

Apéry's [**1**], see also Beukers [**2**], proof of the irrationality of $\zeta(3)$ uses an elementary and quite complicated construction of the approximants $\frac{\alpha_n}{\beta_n}\in Q$ to this number based on a recurrence relation. The integral representation

$$\int_0^1\int_0^1\int_0^1\frac{\left\{x\left(1-x\right)\ y\left(1-y\right)\ z(1-z)\right\}^n}{\left(1-\left(1-x\ y\right)\ z\right)^{n+1}}\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z=\ 2\beta_n\zeta(3)-2\alpha_n$$

for the sequence $\{\alpha_n,\beta_n\}$ was proposed.

It is important to note that other integral representations of $\zeta(3)$ are available in terms of both single and double integrals. Guillera and Sondow [**4**] list a number of them including the classical results

$$\int_0^1\int_0^1\frac{-\ln(xy)}{1-xy}\mathrm{d}x\ \mathrm{d}y=2\zeta(3)$$

and

$$\int_0^1\int_0^1\frac{\ln(2-xy)}{1-xy}\mathrm{d}x\ \mathrm{d}y=\frac{5}{8}\zeta(3).$$

In a recent paper Muzaffar [**7**] also obtained some results of the combinatorial type

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{\binom{2n+1}{1}\binom{2n+k+1}{1}\binom{2n+2k}{n+k}} = \alpha_k \pi^2 + \beta_k$$

by utilizing the power series expansion of $(\sin^{-1} x)^q$ and (α_k, β_k) are constants depending on $k \geq 0$. In this paper we complement and extend some of the results given by Muzaffar. Sofo ([**8**], [**9**], [**10**], [**11**] and [**12**]) also obtained some integral and closed form identities for binomial sums. In the following four corollaries we encounter harmonic numbers at possible rational values of the argument, of the form $H_{r/a}^{(\alpha)}$ where $r = 1, 2, 3, \ldots, k,\ \alpha = 1, 2, 3, \ldots$ and $k \in \mathbb{N}$. The Polygamma function $\psi^{(\alpha)}(z)$ is defined as:

$$\psi^{(\alpha)}(z) = \frac{\mathrm{d}^{\alpha+1}}{\mathrm{d}z^{\alpha+1}} [\log \Gamma(z)] = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}z^{\alpha}} [\psi(z)],\ z \neq \{0, -1, -2, -3, \ldots\}.$$

To evaluate $H_{r/a}^{(\alpha)}$ we have available a relation in terms of the Polygamma function $\psi^{(\alpha)}(z)$, for rational arguments z,

$$H_{r/a}^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^{\alpha}}{\alpha!} \psi^{(\alpha)}\left(\frac{r}{a} + 1\right) \tag{1.5}$$

where $\zeta(z)$ is the Riemann zeta function. We also define

$$H_{r/a}^{(1)} = \gamma + \psi\left(\frac{r}{a} + 1\right), \text{ and } H_0^{(\alpha)} = 0.$$

The evaluation of the Polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [**6**], (see also [**5**]), or Choi and Cvijović [**3**] in terms of the Polylogarithmic or other special functions. Some specific values are given as:

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^n n! \left(2^{n+1} - 1\right) \zeta(n+1)$$

$$H_{1/4}^{(3)} = 64 - \pi^3 - 27\zeta(3),\quad H_{3/4}^{(2)} = \frac{16}{9} + 8G - 5\zeta(2),$$

$$H_{1/3}^{(1)} = \frac{3}{2} + \frac{\pi}{2\sqrt{3}} - \frac{3\ln 3}{2}, \text{ and } H_{1/2}^{(1)} = 2 - 2\ln 2$$

and can be confirmed on a mathematical computer package, such as Mathematica.

2. COROLLARIES

The following lemma will be used in proofs of corollaries in this section.

Lemma 1. *Let a and r be positive real numbers. Then*

$$\sum_{n=1}^{\infty}\frac{1}{n\,(an+r)}=\frac{H^{(1)}_{r/a}}{r}\quad \text{and} \tag{2.1}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,(an+r)}=\frac{1}{r}\left\{H^{(1)}_{r/a}-H^{(1)}_{r/(2a)}\right\}. \tag{2.2}$$

Proof.

$$\sum_{n=1}^{\infty}\frac{1}{n\,(an+r)}=\frac{1}{r}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+r/a}\right)=\frac{1}{r}\left[\gamma+\psi\left(\frac{r}{a}+1\right)\right]=\frac{H^{(1)}_{r/a}}{r},$$

hence (2.1) is attained.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,(an+r)}&=\frac{1}{2r}\left[\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+r/(2a)}\right)\right.\\&\qquad\left.-\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+(r-a)/(2a)}\right)-\sum_{n=1}^{\infty}\frac{1}{n\,(2n-1)}\right]\\&=\frac{1}{2r}\left[-\gamma+\psi\left(\frac{r}{2a}+1\right)+\gamma-\psi\left(\frac{r}{2a}+\frac{1}{2}\right)-2\ln 2\right]\\&=\frac{1}{2r}\left[\psi\left(\frac{r}{2a}+1\right)-\psi\left(\frac{r}{2a}+\frac{1}{2}\right)-2\ln 2\right],\end{aligned}$$

from the definition (1.2)

$$\psi\left(\frac{r}{2a}+\frac{1}{2}\right)=2\psi\left(\frac{r}{a}+1\right)-\psi\left(\frac{r}{2a}+1\right)-2\ln 2,$$

hence

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,(an+r)}&=\frac{1}{2r}\left[2\psi\left(\frac{r}{2a}+1\right)-2\psi\left(\frac{r}{a}+1\right)\right]\\&=\frac{1}{r}\left[H^{(1)}_{r/(2a)}-\gamma-H^{(1)}_{r/a}+\gamma\right],\end{aligned}$$

and (2.2) follows. □

Corollary 1. *From Theorem* 1, *let* $t=1$, $s=3$, $a_2=a>0$, $a_1=a_3=1$, $b_2=k\geq 1$ *and* $b_1=b_3=0$. *Then*

$$\sum_{n\geq 1}\frac{1}{n^4\binom{an+k}{k}}=-a\int_0^1\int_0^1\int_0^1\frac{(1-x_2)^k\log\left(1-x_1^a x_2 x_3\right)}{x_1 x_2 x_3}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \tag{2.3}$$

$$=\zeta(4)-aH_k^{(1)}\zeta(3)+\frac{a^2\zeta(2)}{2}\left[\left(H_k^{(1)}\right)^2+H_k^{(2)}\right]+a^3\sum_{r=1}^{k}\frac{(-1)^r}{r^3}\binom{k}{r}H_{r/a}^{(1)}. \tag{2.4}$$

Proof. By expansion,

$$\sum_{n\geq 1}\frac{1}{n^4\binom{an+k}{k}}=\sum_{n\geq 1}\frac{k!}{n^4\,(an+1)_{k+1}}=\sum_{n\geq 1}\frac{k!}{n^4\prod\limits_{r=1}^{k}(an+r)}$$
$$=\sum_{n\geq 1}\frac{k!}{n^4}\sum_{r=1}^{k}\frac{A_r}{an+r},$$

where

$$A_r=\lim_{n\to(-r/a)}\left\{\frac{an+r}{\prod\limits_{r=1}^{k}(an+r)}\right\}=\frac{(-1)^{r+1}r}{k!}\binom{k}{r},$$

and $(z)_k$ denotes the Pochhammer symbol or the shifted factorial, $z\in\mathbb{C}$, by

$$(z)_k=\frac{\Gamma(z+k)}{\Gamma(k)}=\begin{cases}1, & k=0\\ z(z+1)(z+2)\ldots\ldots(z+k-1), & k\in\mathbb{N}\end{cases}$$

$\mathbb{N}=\{1,2,3,\ldots\}$ denotes the set of natural numbers. Hence, after interchanging the sums, we have

$$\sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\sum_{n\geq 1}\frac{1}{n^4\,(an+r)}$$
$$=\sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\sum_{n\geq 1}\left[\frac{1}{rn^4}-\frac{a}{r^2n^3}+\frac{a^2}{r^3n^2}-\frac{a^3}{r^3n\,(an+r)}\right]$$

and using (2.1) in Lemma 1

$$\sum_{n\geq 1}\frac{1}{n^4\binom{an+k}{k}}=\sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\left[\frac{\zeta(4)}{r}-\frac{a\zeta(3)}{r^2}+\frac{a^2\zeta(2)}{r^3}-\frac{a^3H_{r/a}^{(1)}}{r^4}\right]. \tag{2.5}$$

After some algebraic simplification, (2.5) becomes

$$\sum_{n\geq 1}\frac{1}{n^4\binom{an+k}{k}}=\zeta(4)-aH_k^{(1)}\zeta(3)+\frac{a^2\zeta(2)}{2}\left[\left(H_k^{(1)}\right)^2+H_k^{(2)}\right]$$
$$+a^3\sum_{r=1}^{k}\frac{(-1)^r}{r^3}\binom{k}{r}H_{r/a}^{(1)}.$$

which is the result (2.4). The integral in (2.3) is obtained from (1.4). □

Corollary 2. *From Theorem* 1, *let* $t=-1,\ s=3,\ a_2=a>0,\ a_1=a_3=1,\ b_2=k\geq 1$ *and* $b_1=b_3=0$. *Then*

$$\sum_{n\geq 1}\frac{(-1)^n}{n^4\binom{an+k}{k}}=-a\int_0^1\int_0^1\int_0^1\frac{(1-x_2)^k\log\left(1+x_1^a x_2 x_3\right)}{x_1x_2x_3}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \tag{2.6}$$

$$=-\frac{7}{8}\zeta(4)+\frac{3a}{4}H_k^{(1)}\zeta(3)-\frac{a^2\zeta(2)}{4}\left[\left(H_k^{(1)}\right)^2+H_k^{(2)}\right]+a^3\sum_{r=1}^{k}\frac{(-1)^r}{r^3}\binom{k}{r}\left(H_{r/(2a)}^{(1)}-H_{r/a}^{(1)}\right). \tag{2.7}$$

Proof. The proof will not be detailed, however it follows the same pattern as in Corollary 1 and uses (2.2) in Lemma 1. □

Remark 1. Adding and subtracting (2.3) with (2.6) and (2.4) with (2.7) gives the results:

$$\sum_{n\geq 1}\frac{1}{n^4\binom{2an+k}{k}}=\zeta(4)-2aH_k^{(1)}\zeta(3)+2a^2\zeta(2)\left[\left(H_k^{(1)}\right)^2+H_k^{(2)}\right]+8a^3\sum_{r=1}^{k}\frac{(-1)^r}{r^3}\binom{k}{r}H_{r/(2a)}^{(1)},$$

and

$$\sum_{n\geq 1}\frac{1}{(2n-1)^4\binom{2an-a+k}{k}}=\frac{15}{16}\zeta(4)-\frac{7a}{8}H_k^{(1)}\zeta(3)+\frac{3a^2}{8}\zeta(2)\left[\left(H_k^{(1)}\right)^2+H_k^{(2)}\right]+a^3\sum_{r=1}^{k}\frac{(-1)^r}{r^3}\binom{k}{r}\left(H_{r/a}^{(1)}-\frac{1}{2}H_{r/(2a)}^{(1)}\right).$$

The next corollary also follows from Theorem 1.

Corollary 3. *From Theorem* 1, *let* $t=1,\ s=3,\ a_3=a_2=a>0,\ a_1=1,\ b_3=b_2=k\geq 1$ *and* $b_1=0$. *Then*

$$\sum_{n\geq 1}\frac{1}{n^4\binom{an+k}{k}^2}=-a^2\int_0^1\int_0^1\int_0^1\frac{\left((1-x_2)(1-x_3)\right)^k\log\left(1-x_1^a x_2^a x_3\right)}{x_1x_2x_3}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \tag{2.8}$$

$$(2.9)\qquad = \zeta(4) + \sum_{r=1}^{k} \binom{k}{r}^2 \Bigg[2a \left(H_{k-r}^{(1)} - H_{r-1}^{(1)} - \frac{1}{r} \right) \zeta(3) + a^2 \left(\frac{4}{r^2} - \frac{2}{r} \left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\} \right) \zeta(2) + \left(\frac{2a^3}{r^2} \left\{ H_{k-r}^{(1)} - H_{r-1}^{(1)} \right\} H_{r/a}^{(1)} - \frac{4a^3 H_{r/a}^{(1)}}{r^3} - \frac{a^2 H_{r/a}^{(2)}}{r^2} \right) \Bigg].$$

Proof. Consider the expression

$$(2.10)\qquad \sum_{n\geq 1} \frac{1}{n^4 \binom{an+k}{k}^2} = \sum_{n\geq 1} \frac{(k!)^2}{n^4 \left((an+1)_{k+1} \right)^2} = \sum_{n\geq 1} \frac{(k!)^2}{n^4 \prod_{r=1}^{k} (an+r)^2} = \sum_{n\geq 1} \frac{(k!)^2}{n^4} \sum_{r=1}^{k} \left\{ \frac{A_r}{an+r} + \frac{B_r}{(an+r)^2} \right\},$$

where

$$B_r = \lim_{n\to(-r/a)} \left[\frac{(an+r)^2}{\prod_{r=1}^{k} (an+r)^2} \right] = \left(\frac{r}{k!} \binom{k}{r} \right)^2$$

and

$$A_r = \lim_{n\to(-r/a)} \frac{\mathrm{d}}{\mathrm{d}n} \left[\frac{(an+r)^2}{\prod_{r=1}^{k} (an+r)^2} \right] = -2 \left(\frac{r}{k!} \binom{k}{r} \right)^2 \left[H_{k-r}^{(1)} - H_{r-1}^{(1)} \right].$$

Rearranging (2.10) we have

$$(2.11)\qquad \sum_{r=1}^{k} \left(r \binom{k}{r} \right)^2 \sum_{n\geq 1} \frac{1}{n^4 (an+r)^2} - 2 \sum_{r=1}^{k} \left(r \binom{k}{r} \right)^2 \left[H_{k-r}^{(1)} - H_{r-1}^{(1)} \right] \sum_{n\geq 1} \frac{1}{n^4 (an+r)}.$$

$\sum_{n\geq 1} \frac{1}{n^4 (an+r)}$ can be obtained from (2.5) and

$$\sum_{n\geq 1} \frac{1}{n^4 (an+r)^2} = \sum_{n\geq 1} \left[\frac{1}{r^2 n^4} - \frac{2a}{r^3 n^3} + \frac{3a^2}{r^4 n^2} + \frac{a^4}{r^4 (an+r)^2} - \frac{4a^3}{r^4 n (an+r)} \right]$$

and using (1.1), (1.5) and (2.1),

$$\sum_{n\geq 1}\frac{1}{n^4\left(an+r\right)^2}=\frac{\zeta\left(4\right)}{r^2}-\frac{2a\zeta\left(3\right)}{r^3}+\frac{4a^2\zeta\left(2\right)}{r^4}-\frac{4a^3H^{(1)}_{r/a}}{r^5}-\frac{a^2H^{(2)}_{r/a}}{r^4}. \tag{2.12}$$

Substituting (2.12) into (2.11), we have

$$\begin{aligned}
&\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\sum_{n\geq 1}\left[\frac{\zeta\left(4\right)}{r^2}-\frac{2a\zeta\left(3\right)}{r^3}+\frac{4a^2\zeta\left(2\right)}{r^4}-\frac{4a^3H^{(1)}_{r/a}}{r^5}-\frac{a^2H^{(2)}_{r/a}}{r^4}\right]\\
&\quad-2\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left[H^{(1)}_{k-r}-H^{(1)}_{r-1}\right]\left[\frac{\zeta\left(4\right)}{r}-\frac{a\zeta\left(3\right)}{r^2}+\frac{a^2\zeta\left(2\right)}{r^3}-\frac{a^3H^{(1)}_{r/a}}{r^4}\right]\\
&=\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left[\frac{1}{r^2}-\frac{2}{r}\left\{H^{(1)}_{k-r}-H^{(1)}_{r-1}\right\}\right]\zeta\left(4\right)\\
&\quad+a\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left[-\frac{2}{r^3}+\frac{2}{r^2}\left\{H^{(1)}_{k-r}-H^{(1)}_{r-1}\right\}\right]\zeta\left(3\right)\\
&\quad+a^2\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left[\frac{4}{r^4}-\frac{2}{r^3}\left\{H^{(1)}_{k-r}-H^{(1)}_{r-1}\right\}\right]\zeta\left(2\right)\\
&\quad+\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left[-\frac{4a^3H^{(1)}_{r/a}}{r^5}-\frac{a^2H^{(2)}_{r/a}}{r^4}+\frac{2a^3}{r^4}\left\{H^{(1)}_{k-r}-H^{(1)}_{r-1}\right\}H^{(1)}_{r/a}\right]
\end{aligned}$$

and upon simplification we obtain (2.9) and using the fact that

$$\sum_{r=1}^{k}\binom{k}{r}^2\left[\alpha-2\alpha r\left\{H^{(1)}_{k-r}-H^{(1)}_{r-1}\right\}\right]=\alpha,\qquad \alpha\neq 0. \qquad \square$$

The integral representation (2.8) is obtained from (1.4).

REMARK 2. The following examples can be evaluated from (2.9).

$$\sum_{n\geq 1}\frac{1}{n^4\binom{n/2+4}{4}^2}=\zeta\left(4\right)-\frac{25}{12}\zeta\left(3\right)+\frac{665}{72}\zeta\left(2\right)-\frac{1105724689}{81285120},$$

$$\begin{aligned}
\sum_{n\geq 1}\frac{1}{n^4\binom{4n+5}{5}^2}=\zeta\left(4\right)-\frac{274}{15}\zeta\left(3\right)+\frac{122144}{25}\zeta\left(2\right)-\frac{2028544}{3375}\pi+\frac{401152}{225}G\\
+\frac{3542912}{1125}\ln 2-\frac{20135777}{2025}
\end{aligned}$$

where G is Catalan's constant, defined by

$$G=\frac{1}{2}\int_0^1 K(s)\mathrm{d}s=\sum_{r=1}^{\infty}\frac{\left(-1\right)^r}{\left(2r+1\right)^2}\approx 0.915965\ldots,$$

and $K(s)$ is the complete elliptic integral of the first kind, given by

$$K(s)=\int_0^{\pi/2}\frac{\mathrm{d}t}{\sqrt{1-s^2\sin^2 t}}.$$

The next corollary involves the summation of reciprocals of cubed binomial coefficients.

Corollary 4. *From Theorem* 1, *let* $t=1,\ s=3,\ a_3=a_2=a_1=a>0,\ b_3=b_2=b_1=k\geq 1$ *and* $k\in\mathbb{N}$. *Then*

$$\sum_{n\geq 1}\frac{1}{n^4\binom{an+k}{k}^3} \tag{2.13}$$

$$=-a^3\int_0^1\int_0^1\int_0^1\frac{\left((1-x_1)(1-x_2)(1-x_3)\right)^k\log\left(1-x_1^a x_2^a x_3^a\right)}{x_1x_2x_3}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

$$\begin{aligned}&=\zeta(4)+a\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}^3\left[-\frac{2}{r}+6X(k,r)-\frac{3r}{2}Y(k,r)\right]\zeta(3)\\&\quad+a^2\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}^3\left[\frac{10}{r^2}-\frac{12}{r}X(k,r)+\frac{3}{2}Y(k,r)\right]\zeta(2)\\&\quad+\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}^3\left[-\frac{10a^3H^{(1)}_{r/a}}{r^3}-\frac{4a^2H^{(2)}_{r/a}}{r^2}-\frac{aH^{(3)}_{r/a}}{r}\right.\\&\quad\left.+\frac{3a^2}{r^2}\left(4aH^{(1)}_{r/a}+rH^{(2)}_{r/a}\right)X(k,r)-\frac{3a^3}{2r}H^{(1)}_{r/a}Y(k,r)\right],\end{aligned} \tag{2.14}$$

where

$$X(k,r)=H^{(1)}_{k-r}-H^{(1)}_{r-1}\quad \textit{and} \tag{2.15}$$

$$Y(k,r)=3X^2(k,r)+H^{(2)}_{k-r}+H^{(2)}_{r-1}. \tag{2.16}$$

Proof. Consider the expansion

$$\sum_{n\geq 1}\frac{1}{n^4\binom{an+k}{k}^3}=\sum_{n\geq 1}\frac{(k!)^3}{n^4\left((an+1)_{k+1}\right)^3}=\sum_{n\geq 1}\frac{(k!)^3}{n^4\prod\limits_{r=1}^{k}(an+r)^3}$$

$$=\sum_{n\geq 1}\frac{(k!)^3}{n^4}\sum_{r=1}^{k}\left\{\frac{A_r}{an+r}+\frac{B_r}{(an+r)^2}+\frac{C_r}{(an+r)^3}\right\},$$

where

$$C_r = \lim_{n\to(-r/a)} \left\{ \frac{(an+r)^3}{\prod_{r=1}^{k}(an+r)^3} \right\} = (-1)^{r+1}\left(\frac{r}{k!}\binom{k}{r}\right)^3,$$

$$B_r = \lim_{n\to(-r/a)} \frac{\mathrm{d}}{\mathrm{d}n}\left\{ \frac{(an+r)^3}{\prod_{r=1}^{k}(an+r)^3} \right\} = 3(-1)^r\left(\frac{r}{k!}\binom{k}{r}\right)^3\left[H_{k-r}^{(1)} - H_{r-1}^{(1)}\right]$$

and

$$\begin{aligned} A_r &= \frac{1}{2}\lim_{n\to(-r/a)} \frac{\mathrm{d}^2}{\mathrm{d}n^2}\left[\frac{(an+r)^3}{\prod_{r=1}^{k}(an+r)^3} \right] \\ &= \frac{3}{2}(-1)^{r+1}\left(\frac{r}{k!}\binom{k}{r}\right)^3\left[3\left(H_{k-r}^{(1)} - H_{r-1}^{(1)}\right)^2 + H_{k-r}^{(2)} + H_{r-1}^{(2)}\right]. \end{aligned}$$

Now we can write

$$\begin{aligned} (2.17)\qquad & \sum_{r=1}^{k}(-1)^{r+1}\left(r\binom{k}{r}\right)^3\sum_{n\geq1}\frac{1}{n^4(an+r)^3} \\ &+\sum_{r=1}^{k}3(-1)^r\left(\frac{r}{k!}\binom{k}{r}\right)^3 X(k,r)\sum_{n\geq1}\frac{1}{n^4(an+r)^2} \\ &+\sum_{r=1}^{k}\frac{3}{2}(-1)^{r+1}\left(\frac{r}{k!}\binom{k}{r}\right)^3 Y(k,r)\sum_{n\geq1}\frac{1}{n^4(an+r)}. \end{aligned}$$

From (2.5) and (2.10) respectively we have results for $\sum_{n\geq1}\frac{1}{n^4(an+r)}$ and $\sum_{n\geq1}\frac{1}{n^4(an+r)^2}$. Similarly with the aid of (1.1), (1.5) and (2.1) in Lemma 1, we can evaluate

$$\begin{aligned} &\sum_{n\geq1}\frac{1}{n^4(an+r)^3} \\ &=\sum_{n\geq1}\left[\frac{1}{r^3n^4} - \frac{3a}{r^4n^3} + \frac{6a^2}{r^5n^2} + \frac{a^4}{r^4(an+r)^3} + \frac{4a^4}{r^5(an+r)^2} - \frac{10a^3}{r^5n(an+r)}\right] \\ &= \frac{\zeta(4)}{r^3} - \frac{2a\zeta(3)}{r^4} + \frac{10a^2\zeta(2)}{r^5} - \frac{10a^3H_{r/a}^{(1)}}{r^6} - \frac{4a^2H_{r/a}^{(2)}}{r^5} - \frac{aH_{r/a}^{(3)}}{r^4}, \end{aligned}$$

and hence we have, from (2.17)

$$\sum_{r=1}^{k}(-1)^{r+1}\left(r\binom{k}{r}\right)^{3}\left[\begin{array}{c}\frac{\zeta(4)}{r^{3}}-\frac{2a\zeta(3)}{r^{4}}+\frac{10a^{2}\zeta(2)}{r^{5}}-\frac{10a^{3}H_{r/a}^{(1)}}{r^{6}}-\frac{4a^{2}H_{r/a}^{(2)}}{r^{5}}\\-\frac{aH_{r/a}^{(3)}}{r^{4}}\end{array}\right]$$

$$+\sum_{r=1}^{k}3(-1)^{r}\left(r\binom{k}{r}\right)^{3}X(k,r)\left[\frac{\zeta(4)}{r^{2}}-\frac{2a\zeta(3)}{r^{3}}+\frac{4a^{2}\zeta(2)}{r^{4}}-\frac{4a^{3}H_{r/a}^{(1)}}{r^{5}}-\frac{a^{2}H_{r/a}^{(2)}}{r^{4}}\right]$$

$$+\sum_{r=1}^{k}\frac{3}{2}(-1)^{r+1}\left(r\binom{k}{r}\right)^{3}Y(k,r)\left[\frac{\zeta(4)}{r}-\frac{a\zeta(3)}{r^{2}}+\frac{a^{2}\zeta(2)}{r^{3}}-\frac{a^{3}H_{r/a}^{(1)}}{r^{4}}\right].$$

Collecting the $\zeta(4)$, $\zeta(3)$, $\zeta(2)$ and constant terms,

$$\begin{aligned}&\sum_{r=1}^{k}(-1)^{r+1}\left(r\binom{k}{r}\right)^{3}\left[\frac{1}{r^{3}}-\frac{3}{r^{2}}X(k,r)+\frac{3}{2r}Y(k,r)\right]\zeta(4)\\&\quad+a\sum_{r=1}^{k}(-1)^{r+1}\left(r\binom{k}{r}\right)^{3}\left[-\frac{2}{r^{4}}+\frac{6}{r^{3}}X(k,r)-\frac{3}{2r^{2}}Y(k,r)\right]\zeta(3)\\&\quad+a^{2}\sum_{r=1}^{k}(-1)^{r+1}\left(r\binom{k}{r}\right)^{3}\left[\frac{10}{r^{5}}-\frac{12}{r^{4}}X(k,r)+\frac{3}{2r^{3}}Y(k,r)\right]\zeta(2)\\&\quad+\sum_{r=1}^{k}(-1)^{r+1}\left(r\binom{k}{r}\right)^{3}\left[-\frac{10a^{3}H_{r/a}^{(1)}}{r^{6}}-\frac{4a^{2}H_{r/a}^{(2)}}{r^{5}}-\frac{aH_{r/a}^{(3)}}{r^{4}}\right.\\&\qquad\left.+3\,\frac{X(k,r)}{r^{5}}\left(4aH_{r/a}^{(1)}+rH_{r/a}^{(2)}\right)-\frac{3a^{2}H_{r/a}^{(1)}}{2r^{4}}Y(k,r)\right],\end{aligned}$$

upon simplification (2.14) is attained since we use the fact that

$$\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}^{3}\left[1-3rX(k,r)+\frac{3r^{2}}{2}Y(k,r)\right]=1,$$

where $X(k,r)$ and $Y(k,r)$ are defined by (2.15) and (2.16) respectively. The integral representation (2.13) is obtained from (1.4). □

Remark 3. The following examples can be evaluated from (2.14).

$$\sum_{n\geq 1}\frac{1}{n^4\binom{n/3+4}{4}^3}=\zeta(4)-\frac{175}{18}\zeta(3)-\frac{5005}{648}\zeta(2)+\frac{5997355904916157}{255600043776000},$$

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{n^4\binom{4n+2}{2}^3}&=\zeta(4)+864\zeta(3)+32\pi^3+840\zeta(2)+1024\pi+1024G\\&\quad+5216\ln 2-11184\\&=\frac{1}{3375}\,{}_{11}F_{10}\left[\begin{matrix}1,1,1,1,1,\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{5}{4},\frac{5}{4},\frac{5}{4}\\2,2,2,2,\frac{5}{2},\frac{5}{2},\frac{5}{2},\frac{9}{4},\frac{9}{4},\frac{9}{4}\end{matrix}\;\middle|\;1\right]\end{aligned}$$

where the standard hypergeometric notation

$${}_kF_s\left[\begin{matrix}\alpha_1,\ldots,\alpha_k\\\beta_1,\ldots,\beta_s\end{matrix}\;\middle|\;t\right]=\sum_{r=0}^{\infty}\frac{(\alpha_1)_r\cdots(\alpha_k)_r}{r!\,(\beta_1)_r\cdots(\beta_s)_r}t^r,$$

$$\left(\begin{matrix}k,s\in\{0,1,2,3,\ldots\};\ k\leq s+1;\ k\leq s \text{ and } |t|<\infty;\\ k=s+1 \text{ and } |t|<1;\quad k=s+1,\quad |t|=1 \text{ and}\\ \operatorname{Re}\left\{\sum_{m=1}^{s}\beta_m-\sum_{m=1}^{k}\alpha_m\right\}>0,\ \beta_m,\gamma\notin\{0,-1,-2,-3,\ldots\}\end{matrix}\right).$$

REFERENCES

1. R. Apéry: *Irrationalitè* $\zeta(2)$ *and* $\zeta(3)$. Journees Arithmètiques de Luminy, Astérisque, **61** (1979), 11–13.
2. F. Beukers: *A note on the irrationality of* $\zeta(2)$ *and* $\zeta(3)$. Bull. London Math. Soc., **11** (1979), 268–272.
3. J. Choi, D. Cvijović: *Values of the polygamma functions at rational arguments.* J. Phys. A: Math. Theor., **40** (2007), 15019–15028.
4. J. Guillera, J. Sondow: *Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent.* Ramanujan J., **16** (2008), 247–270.
5. K. Kölbig: *The polygamma function* $\psi(x)$ *for* $x=1/4$ *and* $x=3/4$. J. Comput. Appl. Math., **75** (1996), 43–46.
6. K. Kölbig: *The polygamma function and the derivatives of the cotangent function for rational arguments.* CERN-IT-Reports CERN-CN., (1996), 96-005.
7. H. Muzaffar: *Some Interesting Series Arising From the Power Series Expansion of* $(\sin^{-1}x)^q$. Int. J. Math. Math. Sci., **14** (2005), 2329–2336.
8. A. Sofo: *On sums of binomial coefficients.* Proyecciones, **28** (2009), 35–45.
9. A. Sofo: *Sums of reciprocals of triple binomial coefficients.* Int. J. Math. Math. Sci, (2008), Art. ID 794181, 11pp.
10. A. Sofo: *Computational Techniques for the Summation of Series.* Kluwer Academic/ Plenum Publishers, New York, 2003.

11. A. Sofo: *Integral forms of sums associated with Harmonic numbers.* Appl. Math. Comput., **207** (2009), 365–372.

12. A. Sofo: *General properties involving reciprocals of binomial coefficients.* J. Integer Seq., **9** (2006), article 06.4.5.

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