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EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS FOR A CLASS OF FIRST-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

Chengjun Guo, Donal O'Regan, Ravi P. Agarwal

In this paper we establish three different existence results for periodic solutions for a class of first-order neutral differential equations. The first one is based on a generalized version of the Poincaré-Birkhoff fixed point theorem where we establish conditions on f which guarantee that a first-order neutral differential equations has infinitely many periodic solutions. The second one is based on Mawhin's continuation theorem and the third one is based on Krasnoselskii fixed point theorem.

1. INTRODUCTION

The purpose of this paper is to discuss the existence of periodic solutions for the equation

$$x^{'}(t) + cx^{'}(t-\tau) = f(x(t), x(t-\tau)), \tag{1.1}$$

where $\tau > 0$ is a constant, c is a real number and $f \in C(\mathbb{R}^2, \mathbb{R})$.

Periodic solutions for differential equations were studied in [**3-5**, **7**, **9-14**, **16**, **17**, **21**] and we note that most of the results in the literature concern delay problems. For neutral problems we refer the reader to [**8**, **11**, **13**, **18**, **20**] and we note that only a few results are available in the literature on the existence of infinitely many periodic solutions for neutral differential equations [**2**, **19**]. In

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this paper, by using a generalized version of the Poincaré-Birkhoff fixed point theorem [**1**, **5**, **15**], we show that there are infinitely many periodic solutions of (1.1). Also we establish the existence of periodic solutions of (1.1) by Mawhin's continuation theorem [**6**] and Krasnoselskii fixed point theorem respectively.

In Section 2 some preliminary results will be given. In Section 3 by employing a generalized version of the Poincaré-Birkhoff fixed point theorem, we establish conditions on f which guarantee that (1.1) has infinitely many periodic solutions. In Section 4 and 5 we prove a second and a third existence result for (1.1) by Mawhin's continuation theorem and Krasnoselskii fixed point theorem respectively. In Section 6 we provide some examples to illustrate our results.

2. PRELIMINARIES

For the sake of completeness, three lemmas will be stated here which will be used in the proof of our main results.

Let (r, θ) be a polar coordinate expression and O be the corresponding polar point. Let $\mathbf{A}$ be an annular region in $\mathbb{R}^2 : R_1 \leq r \leq R_2, (0 < R_1 < R_2)$.

Definition 2.1. *A map $T : \mathbf{A} \to \mathbb{R}^2 \backslash \{0\}$ is called torsional if*

(i) $$r^* = g(r, \theta), \quad \theta^* = \theta + h(r, \theta),$$

where (r^*, θ^*) denotes the image of (r, θ) under T, and g and h are continuous and $2\pi-$periodic in θ.

(ii) *the twist condition*: $h(R_1, \theta) \cdot h(R_2, \theta) < 0$ *is satisfied.*

Let $\mathbf{A}$ be an annular region in $\mathbb{R}^2$ bounded by two disjoint simple closed curves Γ_1 and Γ_2. Let D_i denote the open set bounded by Γ_i, $(i = 1, 2)$. Assume that $0 \in D_1 \subset \overline{D}_1 \subset D_2$.

Lemma 2.1. (Generalized version of the Poincaré-Birkhoff fixed point theorem) *Suppose $T : \mathbf{A} \to \mathbf{T(A)} \subset \mathbb{R}^{\mathbf{2}} \backslash \{\mathbf{0}\}$ is an area-preserving homeomorphism. Suppose*

(1) *T is torsional*;

(2) *there exists an area-preserving homeomorphism $T_1 : \overline{D}_2 \to \mathbb{R}^2$, such that $T_1|_{\mathbf{A}} = T$ and $0 \in T_1(D_1)$.*

Then T has at least two fixed points in $\mathbf{A}$.

Let X and Y be two Banach space, $L : \operatorname{Dom} L \subset X \to Y$ a linear mapping and $N : X \to Y$ a continuous mapping.

Definition 2.2. *A mapping L will be called a Fredholm mapping of index zero if* $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$, *and* $\operatorname{Im} L$ *is closed in* Y.

If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$.

It follows that $L|_{\mathrm{Dom}\, L\cap \mathrm{Ker}\, P} : (I-P)X \to \mathrm{Im}\, L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, we have

Definition 2.3. *A mapping N will be called $L-$compact on Ω if $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact.*

Since $\mathrm{Im}\, Q$ is isomorphic to $\mathrm{Ker}\, L$, there exists an isomorphism $J : \mathrm{Im}\, Q \to \mathrm{Ker}\, L$.

Lemma 2.2. (Mawhin's continuation theorem) *Let L be a Fredholm mapping of index zero, and let N be $L-$compact on $\overline{\Omega}$. Suppose*

(1) *for each $\lambda \in (0,1)$ and $x \in \partial\Omega, Lx \neq \lambda Nx$ and*

(2) *for each $x \in \partial\Omega \cap \mathrm{Ker}(L), QNx \neq 0$ and $\deg(QN, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.3. (Krasnoselskii fixed point theorem) *Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Let $U, S : X \to \Omega$ satisfy the following conditions:*

(1) *$Ux + Sy \in X$ for any $x, y \in X$;*

(2) *U is a contraction mapping;*

(3) *S is completely continuous.*

Then $U + S$ has a fixed point in X.

3. EXISTENCE RESULT (I)

In this section we prove that (1.1) has infinitely many nonconstant periodic solutions. The proof is based on the generalized version of the Poincaré-Birkhoff fixed point theorem.

Now we make the following assumptions:

(H_1) There exists a continuous function $g \in C(\mathbb{R}^2, \mathbb{R})$ such that

$$f(x,y) = g(x,y) - cg(y,x);$$

(H_2) g is odd, i.e. $g(-x,y) = g(x,y)$, $g(x,-y) = -g(x,y)$;

(H_3) For any given $k > 0$, we have $\lim\limits_{|x|\to\infty} \dfrac{g(x,kx)}{x} = +\infty$;

(H_4) there exist two positive constants α_0 and R_0, such that

$$yg(x,y) + xg(y,x) \geq \alpha_0(x^2+y^2), \quad x^2+y^2 \geq R_0^2.$$

Now let us state our the first existence result.

Theorem 3.1. *Suppose that conditions $(H_1) - (H_4)$ are satisfied and also assume that*

(H_5) *Every solution of* (1.1) *exists on the whole real line.*

Then (1.1) *has infinitely many nonconstant* 4τ*−periodic solutions.*

REMARK 3.1. In Theorem 3.1, c is any real number. In Section 4 and 5, we will need $|c| < 1$.

In order to prove Theorem 3.1, set

$$E := \{x | x(t-\tau) = -x(t+\tau), \quad \forall t \in \mathbb{R}\}.$$

This implies that $x(t) = x(t + 4\tau)$.

Let $x(t-\tau) = y(t), x \in E$, and we have from (1.1), (H_1) and (H_2) that

$$\begin{cases} x^{'}(t) = g(x(t), y(t)), \\ y^{'}(t) = -g(y(t), x(t)), \quad x \in E. \end{cases} \tag{3.1}$$

From (H_5), let $x = \varphi(t, x_0, y_0), y = \psi(t, x_0, y_0)$ be the unique solution of (3.1) which satisfies $x(0) = x_0, y(0) = y_0$.

Under the assumption of (H_5), define a continuous mapping $P_t : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$P_t(x, y) = (\varphi(t, x, y), \psi(t, x, y)).$$

Obviously we have

(i) $P_{4\tau}$ is an area-preserving mapping (for details see [**12**, **14**]);

(ii) For each fixed t, from (H_5), P_t is a homeomorphism.

Lemma 3.1. ([**12**]) *Suppose that* (1.1) *satisfies condition* (H_5). *Then there exists a constant* $\mu > 0$ *for arbitrary* $\nu \geq 0$, *such that*

$$|P_t(v)| > \nu, \quad |v| > \mu,$$

where $|\cdot|$ *denotes the the Euclidean norm of* $\mathbb{R}^2$ *and* $v \in \mathbb{R}^2$.

Proof of Theorem 3.1. From Lemma 3.1, these exists a $R_1 > 0$, such that

$$|P_t(v)| > 0, \quad t \in [0, 4\tau] \tag{3.2}$$

with $|v| \geq R_1$ and $v \in \mathbb{R}^2$.

Suppose $\mathbf{A}$ is an annular region in $\mathbb{R}^2$ with $R_1 \leq r \leq R_2 (0 < R_1 < R_2)$ such that $0 \notin P_t(\mathbf{A})$ for $t \in [0, 4\tau]$, where R_2 will be defined later.

Let

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \tag{3.3}$$

where x and y are defined in (3.1).

From (3.1) and (3.3), we have

$$\begin{cases} x^{'} = r^{'}\cos\theta - \theta^{'} r\sin\theta = g(r\cos\theta, r\sin\theta), \\ y^{'} = r^{'}\sin\theta + \theta^{'} r\cos\theta = -g(r\sin\theta, r\cos\theta). \end{cases} \tag{3.4}$$

From (3.4), we have

$$\text{(3.5)}\qquad \begin{cases} r^{'} = g(r\cos\theta, r\sin\theta)\cos\theta - g(r\sin\theta, r\cos\theta)\sin\theta, \\ \theta^{'} = -\dfrac{g(r\sin\theta, r\cos\theta)\cos\theta}{r} - \dfrac{g(r\cos\theta, r\sin\theta)\sin\theta}{r}. \end{cases}$$

Note (3.5) shows that r and θ depend on t. Let $r = \overline{r}(t, r_0, \theta_0)$, $\theta = \overline{\theta}(t, r_0, \theta_0)$ be a solution of (3.5) which satisfies $r(0) = r_0$, $\theta(0) = \theta_0$. Obviously, if $r \neq 0$ and $\overline{r}(t, r, \theta) \neq 0$, $\overline{r}$ and $\overline{\theta}$ are continuous in (t, r, θ).

For $r \geq R_1$, we have

$$\overline{r}(t, r, \theta) = |P_t(Q)| > 0, \quad t \in [0, 4\tau],$$

where $Q = (r, \theta)$.

Obviously, for $(t, r, \theta) \in [0, 4\tau] \times [R_1, \infty) \times \mathbb{R}$, we have $\overline{r}(t, r, \theta + 2\pi) = \overline{r}(t, r, \theta)$ and $\overline{\theta}(t, r, \theta + 2\pi) - \overline{\theta}(t, r, \theta) = 2k\pi$, where k is a integer. Furthermore, using these two functions we obtain a polar coordinate expression for $P_{4\tau}$ as follows

$$\text{(3.6)}\qquad r^* = \overline{r}(4\tau, r, \theta), \quad \theta^* = \theta + [\overline{\theta}(4\tau, r, \theta) - \theta + 2m\pi],$$

where m is a arbitrary integer.

Let m be large enough such that

$$\text{(3.7)}\qquad h(R_1, \theta) \equiv \overline{\theta}(4\tau, R_1, \theta) - \theta + 2m\pi > 0 \quad \forall\theta.$$

Following the argument in [**12**] we note that for any positive integer m, there exists a constant $R_2 > 0$ such that

$$\text{(3.8)}\qquad \overline{\theta}(4\tau, R_2, \theta) - \theta < -2m\pi, \quad r \geq R_2,$$

where r is given by (3.3). Now (3.7) and (3.8) show that $P_{4\tau}$ is torsional on $\mathbf{A}$ (for details see [**12**]). By applying Lemma 2.1, $P_{4\tau}$ has at least two fixed points in $\overline{D}_2$.

On the other hand, by the above construction of the annular region $\mathbf{A}$, R_1 is arbitrary, so as a result we can construct infinitely many disjoint annular region in $\mathbb{R}^2$ such that $P_{4\tau}$ has at least two fixed points in every one annular region. Obviously, the infinitely many $4\tau-$periodic solutions $x(t)$ for (1.1) satisfying $x \in E$, is equivalent to the infinitely many fixed points of $P_{4\tau}$ in $\mathbb{R}^2$. □

4. EXISTENCE RESULT (II)

The proof in this section is based on Mawhin's continuation theorem. We now assume that following hypothesis is satisfied:

(V_1) $\quad f(x(t), x(t-\tau)) = -a(t)x(t) - f_1(x(t-\tau))$,
where $a(t) > 0$ is a real continuous function defined on $\mathbb{R}$ with $\tau-$periodic in t and $f_1(x)$ is a real continuous function defined on $\mathbb{R}$.

Now we make the following assumptions on $a(t)$:

(V_2) $M = \max_{t\in[0,T]} a(t) \geq a(t) \geq m_0 = \min_{t\in[0,T]} a(t) > 0$;

(V_3) $|c| < \dfrac{1 - T\sigma(M - m + r)}{2 + 2T\sigma m}$, where $\sigma = \dfrac{e^{MT}}{e^{MT} - 1}$ and r will be defined in later.

Our second existence results are the following theorems.

Theorem 4.1. *Suppose* $(V_1) - (V_3)$ *hold and assume*

$$\lim_{|x|\to\infty} \sup \left|\frac{f_1(x)}{x}\right| \leq r \tag{4.1}$$

and

$$\lim_{|x|\to\infty} \operatorname{sgn}(x) f_1(x) = +\infty. \tag{4.2}$$

Then (1.1) *has at least one* $T-$*periodic solution.*

Theorem 4.2. *Suppose* $(V_1) - (V_3)$ *hold and assume*

$$\lim_{x\to 0} \sup \left|\frac{f_1(x)}{x}\right| \leq r \tag{4.3}$$

and

$$\lim_{x\to 0} \operatorname{sgn}(x) f_1(x) = 0. \tag{4.4}$$

Then (1.1) *has at least one* $T-$*periodic solution.*

Set $X := \{x | x \in C^1(R, R), x(t + T) = x(t), \forall t \in R\}$ and $Y := \{y | y \in C(R, R), y(t + T) = y(t), \forall t \in R\}$. We define the norms on X and Y by $||x|| = \max\{\max_{t\in[0,T]} |x(t)|, \max_{t\in[0,T]} |x^{'}(t)|\}$ and $||y||_0 = \max_{t\in[0,T]} |y(t)|$ respectively.

Define the operators $L : X \longrightarrow Y$ and $N : X \longrightarrow Y$ respectively by

$$Lx(t) = x^{'}(t), \quad t \in \mathbb{R}, \tag{4.5}$$

and

$$Nx(t) = -cx^{'}(t - \tau) - a(t)x(t) - f_1(x(t - \tau)). \tag{4.6}$$

For $x(t) \in X$, $y(t) \in Y$, let us define $P : X \to X$ and $Q : Y \to Y/\operatorname{Im}(L)$ respectively by $Px(t) = x(0)$ and $Qy(t) = \dfrac{1}{T}\int_0^T y(t)\mathrm{d}t$. Let Ω be an open and bounded subset of X. Note L is a Fredholm mapping of index zero and N is $L-$compact on $\overline{\Omega}$.

Now, we consider the following auxiliary equation

$$x^{'}(t) + \lambda\big[cx^{'}(t - \tau) + a(t)x(t) + f_1(x(t - \tau))\big] = 0, \quad 0 < \lambda < 1. \tag{4.7}$$

Lemma 4.1. *Suppose* (4.1) *and* (4.2) *hold. If* x *is a* $T-$*periodic solution of* (4.7), *then there are positive constants* D_0 *and* D_1, *which are independent of* λ, *such that*

$$||x^{(i)}||_0 \leq D_i, \quad i = 0, 1, \tag{4.8}$$

where $x^{(0)}(t) = x(t)$.

Proof. Let $x(t)$ be a $T-$ periodic solution of (4.7) and $\varepsilon = \frac{1}{2}\left[\frac{1 - T\sigma(M - m)}{T\sigma} - r\right]$. By (4.1), we know that exists a $\widetilde{M} > 0$, such that

$$|f_1(x(t-\tau))| \leq (r + \varepsilon)|x(t-\tau)|, \quad |x(t))| > \widetilde{M}, \quad t \in \mathbb{R}. \tag{4.9}$$

Set $E_1 = \{t | t \in [0, T], |x(t)| > \widetilde{M}\}$, $E_2 = [0, T] \backslash E_1$ and $\rho = \max_{|x| \leq \widetilde{M}} |f_1(x)|$.

The definitions of E_1, E_2, ρ, (4.7) and (4.10), yield

$$\begin{aligned} ||x^{'}||_0 &\leq \max_{0 \leq t \leq T} \lambda\big\{|cx^{'}(t-\tau) + a(t)x(t)| + f_1(x(t-\tau))|\big\} \\ &\leq \max_{x \in E_1} |f_1(x(t-\tau))| + \max_{x \in E_2} |f_1(x(t-\tau))| \\ &\qquad + \max_{0 \leq t \leq T} |c||x^{'}(t)| + M||x||_0 \\ &\leq \max_{0 \leq t \leq T} |c||x^{'}(t)| + M||x||_0 + (r + \varepsilon)||x||_0 + \rho. \end{aligned} \tag{4.10}$$

From (4.7) we get

$$x(t) = \int_0^T \Phi(t, s)\lambda[-cx^{'}(s-\tau) + (M - a(s))x(s) - f_1(x(s-\tau))]\mathrm{d}t_1 \tag{4.11}$$

and

$$\begin{aligned} ||x||_0 &\leq \max_{t \in [0,T]} \int_0^T |\lambda\Phi(t, s)[|f_1(x(s-\tau))| \\ &\qquad + (M - a(s))|x(s)| + |c||x^{'}(s-\tau)|]\mathrm{d}s \\ &\leq T\sigma\{(M - m)||x||_0 + |c| \max_{t \in [0,T]} |x^{'}(t)|\} \\ &\qquad + \sigma \max_{0 \leq s \leq T} \int_{E_1} |f_1(x(s-\tau))|\mathrm{d}s + \sigma \max_{0 \leq s \leq T} \int_{E_2} |f_1(x(s-\tau))|\mathrm{d}s \\ &\leq T\sigma\{(M - m)||x||_0 + |c| \max_{t \in [0,T]} |x^{'}(t)| + (r + \varepsilon)||x||_0 + \rho\}, \end{aligned} \tag{4.12}$$

where $\Phi(t, s) = \dfrac{e^{\lambda M(s-t)}}{e^{\lambda MT} - 1}$ and $\Phi(t, s) \leq \Phi(t, t + T) = \dfrac{e^{\lambda MT}}{e^{\lambda MT} - 1} \leq \sigma$.

Now the definition of ε, (4.10) and (4.12), yield

$$\begin{aligned} ||x^{'}||_0 &= \max_{t \in [0,T]} |x^{'}(t)| \\ &\leq \frac{2\rho(1 + T\sigma m)}{1 - T\sigma(M_0 - m_0 + r)}\left[1 - |c| - |c|\frac{1 + T\sigma(M + m + r)}{1 - T\sigma(M_{\_}m + r)}\right]^{-1} = D_1 \end{aligned} \tag{4.13}$$

and

$$||x||_0 \le \frac{2T\sigma[|c|D_1 + \rho]}{1 - T\sigma(M - m + r)} = D_0. \tag{4.14}$$

Now Lemma 4.1 follows from (4.13) and (4.14).

Lemma 4.2. *Suppose* (4.3) *and* (4.4) *hold. If x is a $T-$periodic solution of* (4.7), *then there are positive constants $\widetilde{D}_0$ and $\widetilde{D}_1$, which are independent of λ, such that*

$$||x^{(i)}||_0 \le \widetilde{D}_i, \quad i = 0, 1. \tag{4.15}$$

Proof of Theorem 4.1. Suppose that $x(t)$ is a $T-$ periodic solution of (4.7). By Lemma 4.1, there exist positive constants D_0 and D_1 which are independent of λ such that (4.8) holds. By (4.2), we know that exists a $\widehat{M} > 0$ for any $\epsilon > 0$, such that

$$\operatorname{sgn}(x) f_1(x) > \epsilon, \quad |x| > \widehat{M}. \tag{4.16}$$

For any positive constant $\overline{D} > \max\{D_0, D_1\} + \widehat{M}$. Set $\Omega := \{x \in X | ||x|| < \overline{D}\}$.

We have $Lx \neq Nx$ for any $x \in \partial\Omega \cap \operatorname{Dom}(L)$ and $\lambda \in (0, 1)$. Since for any $x \in \partial\Omega \cap \operatorname{Ker}(L)$, $x = \overline{D}(> D)$ or $x = -\overline{D}$. Hence, it is easy to see that

$$(QNx) = \frac{1}{T}\int_0^T [-cx^{'}(t-\tau) - a(t)x(t) - f_1(x(t-\tau))]\mathrm{d}t \neq 0.$$

Then, for any $x \in \operatorname{Ker} L \cap \partial\Omega$ and $\eta \in [0, 1]$, we obtain

$$xH(x,\eta) = -\eta x^2 - \frac{x}{T}(1-\eta)\int_0^T [cx^{'}(t-\tau) + a(t)x(t) + f_1(x(t-\tau))]\mathrm{d}t \neq 0,$$

hence, $H(x, \eta)$ is a homotopy. This shows that

$$\begin{aligned}\deg\{QN,\Omega \cap \operatorname{Ker}(L), 0\} &= \deg\{-\frac{1}{T}\int_0^T [cx^{'}(t-\tau) + a(t)x(t)\\ &\quad + f_1(x(t-\tau))]\mathrm{d}t, \Omega \cap \operatorname{Ker}(L), 0\} = \deg\{-x, \Omega \cap \operatorname{Ker}(L), 0\} \neq 0.\end{aligned}$$

By Lemma 2.2, the equation $Lx = Nx$ has at least a solution in $\operatorname{Dom}(L) \cap \overline{\Omega}$, so there exists a T-periodic solution of (1.1). □

Similarly, we can prove Theorem 4.2. Details are left to the readers.

5. EXISTENCE RESULT (III)

In this section the proof is based on the KRASNOSELSKII fixed point theorem.

Theorem 5.1. *Suppose* $(V_1) - (V_2)$ *hold and also assume there exists a constant* $K_0 > 0$ *such that*

$$0 < \frac{\|f_1\|_0}{1 - (2 + T\sigma M)|c| - (M - m)T\sigma} < K_0, \tag{5.1}$$

where $\|f_1\|_0 = \max_{\{t\in[0,T],|x|\le K_0\}} |f_1(x)|$ *and* $\sigma = \dfrac{e^{MT}}{e^{MT}-1}$. *Then* (1.1) *possesses a nontrivial* $T-$*periodic solution.*

In order to prove the main theorem we need some preliminaries. Set $X := \{x|x \in C^1(\mathbb{R},\mathbb{R}), x(t+T) = x(t), \forall t \in \mathbb{R}\}$ and define the norm on X by $||x|| = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |x^{'}(t)|$.

Proof of Theorem 5.1. For $\forall x \in X$, define the operators $U : X \longrightarrow X$ and $S : X \longrightarrow X$ respectively by

$$(Ux)(t) = -cx(t-\tau) \tag{5.2}$$

and

$$(Sx)(t) = cx(t-\tau)+\int_0^T \Psi(t,s)[-cx^{'}(s-\tau)+(M-a(s))x(s)-f_1(x(s-\tau))]\mathrm{d}s, \tag{5.3}$$

where $\Psi(t,s) = \dfrac{e^{M(s-t)}}{e^{MT}-1}$ and $\Psi(t,s) \le \Psi(t,t+T) = \dfrac{e^{MT}}{e^{MT}-1} = \sigma$. It is clear that a fixed point of $U+S$ is a $T-$periodic solution of (1.1).
We are going to demonstrate that U and S satisfy the conditions of Lemma 2.3.

Let $x, y \in X$ and $|x| \le K_0$, $|y| \le K_0$ (here K_0 is as in the statement of Theorem 5.1). Now we prove that $|Ux+Sy| \le K_0$ holds.

From (V_1), (V_2) and (5.1)–(5.3), we have

$$\begin{aligned}
|(Uy)(t)+(Sx)(t)| &\le |(Uy)(t)| + |(Sx)(t)| \\
&\le 2|c|K_0 + \left|\int_0^T \Psi(t,s)[-cx^{'}(s-\tau)\right. \\
&\qquad \left. + (M-a(s))x(s) - f_1(x(s-\tau))]\mathrm{d}s\right| \\
&\le 2|c|K_0 + T\sigma[(M-m)K_0 + \|f_1\|_0] \\
&\qquad + |c|M\left|\int_0^T \Psi(t,s)x(s-\tau)\mathrm{d}s\right| \\
&\le 2|c|K_0 + T\sigma[(M-m)K_0 + \|f_1\|_0] + |c|MT\sigma K_0 \\
&\le K_0, \quad x, y \in X,
\end{aligned} \tag{5.4}$$

where $\|f_1\|_0$ is given in (5.1).

Set $K_1 = \dfrac{mK_0 + \|f_1\|_0}{1-|c|}$ and $G = \{x \in X : |x(t)| \le K_0, |x^{'}(t)| \le K_1\}$. It is clear that G is a bounded, convex and closed subset of X.

(1) $Ux+Sy \in G$ for $\forall x, y \in G$. For $\forall x, y \in G$, from (5.2) and (5.3), we have

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}[(Uy)(t)+(Sx)(t)]\right| &= |cy^{'}(t-\tau) - a(t))x(t) - f_1(x(t-\tau))| \\
&\le |c|K_1 + mK_0 + \|f_1\|_0 \le K_1.
\end{aligned} \tag{5.5}$$

From (5.4) and (5.5), we have $Ux + Sy \in G$ for $\forall x, y \in G$.

(2) U is a contraction mapping. Let $x, y \in G$ and we from (5.2) that

$$\begin{aligned}\|Ux - Uy\| &= \max_{t\in[0,T]} |cx(t-\tau) - cy(t-\tau)| + \max_{t\in[0,T]} |cx^{'}(t-\tau) - cy^{'}(t-\tau)| \\ &= |c|\{\max_{t\in[0,T]} |x(t-\tau) - y(t-\tau)| + \max_{t\in[0,T]} |x^{'}(t-\tau) - y^{'}(t-\tau)|\} \\ &= |c| \ \|x - y\|.\end{aligned}$$

Since $|c| < 1$, U is a contraction mapping.

(3) S is completely continuous. To see this, suppose that $x_k \in G$ and $\|x_k - x\| \to 0$ as $k \to +\infty$. Since G is closed convex subset of X, we have $x \in G$. Then

$$\begin{aligned}(5.6)\quad |Sx_k - Sx| = |c[x_k(t-\tau) - x(t-\tau)] + &\int_0^T \Psi(t,s)\{(M - a(s))(x_k(s) - x(s)) \\ &- c[x_k^{'}(s-\tau) - x^{'}(s-\tau)] - [f_1(x_k(s-\tau)) - f_1(x(s-\tau))]\}\mathrm{d}s|.\end{aligned}$$

From the continuity of $a(t)$ and $f_1(x(t-\tau))$ for $t \in [0,T], x \in G$, we have from (5.3) and (5.6) that

$$\lim_{k\to+\infty} \|Sx_k - Sx\| = 0.$$

Then S is continuous.

It is easy to check that Sx is relatively compact. Since S is continuous and is relatively compact, S is completely continuous. By Lemma 2.3 (Krasnoselskii fixed point theorem), we have a fixed point x of $U + S$. That means that x is a $T-$periodic solution of (1.1).

6. SOME EXAMPLES

Example 1. Consider

$$(6.1)\qquad x^{'}(t) + cx^{'}(t-\tau) = a[1 + x^2(t) + x^2(t-\tau)][x(t-\tau) - cx(t)],$$

where $a > 0$, $\tau > 0$ and c is a constant. Then $f(x,y) = a(1 + x^2 + y^2)(y - cx)$. Choose $g(x,y) = a(1+x^2+y^2)y$. It is easy to check that $g(x,y)$ satisfies the conditions of Theorem 3.1. Theorem 3.1 guarantees that (6.1) has infinitely many nonconstant $4\tau-$periodic solutions.

Example 2. Consider

$$(6.2)\qquad x^{'}(t) + \frac{1}{64}x^{'}\left(t - \frac{\pi}{2}\right) + \frac{2 + \cos 8t}{96}x(t) + \frac{1}{50}x\left(t - \frac{\pi}{2}\right) = 0.$$

Let $T = \frac{\pi}{4}$, $r = \frac{1}{50}$ and note all the conditions of Theorem 4.1 and Theorem 4.2 are satisfied. Theorem 4.1 or Theorem 4.2 guarantee that (6.2) has a $\frac{\pi}{4}-$ periodic solution.

EXAMPLE 3. Consider

$$x^{'}(t)+\frac{1}{16}x^{'}\left(t-\frac{1}{2}\right)+\frac{2+\sin 2\pi t}{12}x(t)+\frac{\sin 2\pi t}{4}\exp^{-x^2(t-1/2)}=0. \tag{6.3}$$

Let $T=1$, $K_0=4$ and note all the conditions of Theorem 5.1 are satisfied. Theorem 5.1 guarantees that (6.3) has a $1-$ periodic solution.

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School of Applied Mathematics,
Guangdong University of Technology,
510006 Guangzhou,
P. R. China
E-mail: guochj817@163.com

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Department of Mathematics,
National University of Ireland, Galway,
Ireland
E-mail: donal.oregan@nuigalway.ie

Department of Mathematics/Department of Mathematics and Statistics,
Florida institute of Technology/KFUPM
Melbourne/Dhahran
32901 Florida, USA/Saudi Arabia 31261
E-mail: agarwal@fit.edu