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ON JENSEN'S AND RELATED COMBINATORIAL IDENTITIES

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Motivated by the recent work of Chu [Electron. J. Combin. 17 (2010), #N24], we give simple proofs of Jensen's identity

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k}=\sum_{k=0}^{n}\binom{x+y-k}{n-k}z^k,$$

and Chu's and Mohanty-Handa's generalizations of Jensen's identity. We also give a simple proof of an equivalent form of Graham-Knuth-Patashnik's identity

$$\sum_{k\geq 0}\binom{m+r}{m-n-k}\binom{n+k}{n}x^{m-n-k}y^k$$

$$=\sum_{k\geq 0}\binom{-r}{m-n-k}\binom{n+k}{n}(-x)^{m-n-k}(x+y)^k,$$

which was rediscovered, respectively, by Sun in 2003 and Munarini in 2005. Finally we give a multinomial coefficient generalization of this identity.

1. INTRODUCTION

Abel's identity (see, for example, [**8**, §3.1])

$$\sum_{k=0}^{n}\binom{n}{k}x(x+kz)^{k-1}(y-kz)^{n-k}=(x+y)^n$$

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and Rothe's identity [**23**] (or Hagen-Rothe's identity, see, for example, [**9**, §5.4])

$$\sum_{k=0}^{n} \frac{x}{x-kz}\binom{x-kz}{k}\binom{y+kz}{n-k} = \binom{x+y}{n},$$

are famous in literature and play an important role in enumerative combinatorics. Recently, Chu [**6**] gave elementary proofs of Abel's identity and Rothe's identity by using the binomial theorem and the Chu-Vandermonde convolution formula, respectively.

Motivated by Chu's work, we study Jensen's identity [**17**], which is closely related to Rothe's identity, and can be stated as follows:

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k} = \sum_{k=0}^{n}\binom{x+y-k}{n-k}z^k. \tag{1}$$

Jensen's identity (1) has attracted much attention by different authors. Gould [**11**] obtained the following Abel-type analogue:

$$\sum_{k=0}^{n}\frac{(x+kz)^k}{k!}\frac{(y-kz)^{n-k}}{(n-k)!} = \sum_{k=0}^{n}\frac{(x+y)^k}{k!}z^{n-k}. \tag{2}$$

Carlitz [**1**] gave two interesting theorems related to (1) and (2) by mathematical induction. With the help of generating functions, Gould [**12**] derived the following variation of Jensen's identity (1):

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k} = \sum_{k=0}^{n} k\binom{x+y-k}{n-k}\frac{x+y-(n-k)z-k}{x+y-k}z^k.$$

E. G.-Rodeja F. [**10**] deduced Gould's identity (2) from (1) by establishing an identity which includes both. Cohen and Sun [**7**] also gave an expression which unifies (1) and (2). Chu [**4**] generalized Jensen's identity (1) to a multi-sum form:

$$\sum_{k_1+\cdots+k_s=n}\prod_{i=1}^{s}\binom{x_i+k_i z}{k_i} = \sum_{k=0}^{n}\binom{k+s-2}{k}\binom{x_1+\cdots+x_s+nz-k}{n-k}z^k. \tag{3}$$

Moreover, the identities (1) and (3) were respectively generalized by Mohanty and Handa [**19**] and Chu [**5**] to the case of multinomial coefficients (to be stated in Section 4).

The primary purpose of this paper is to give simple proofs of Jensen's identity, Chu's identity (3), Mohanty-Handa's identity, and Chu's generalization of Mohanty-Handa's identity. We use the Chu-Vandermonde convolution formula

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}$$

and the well-known identity

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^r=\begin{cases}0, & \text{if } 0\le r\le n-1,\\ n!, & \text{if } r=n.\end{cases} \tag{4}$$

Equation (4) may be easily deduced from the Stirling numbers of the second kind [**27**, p. 34, (24a)]. The first case of (4) was already utilized by the author [**13**] to give a simple proof of Dixon's identity and by Chu [**6**] in his proofs of Abel's and Rothe's identities.

It is interesting that our proof of Chu's identity (3) also leads to a very short proof of Graham-Knuth-Patashnik's identity, which was rediscovered several times in the past few years. The secondary purpose of this paper is to give a multinomial coefficient generalization of Graham-Knuth-Patashnik's identity.

2. PROOF OF JENSEN'S IDENTITY

By the Chu-Vandermonde convolution formula, we have

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k}=\sum_{k=0}^{n}\binom{x+kz}{k}\sum_{i=k}^{n}\binom{x+y+1}{n-i}\binom{-x-kz-1}{i-k}. \tag{5}$$

Interchanging the summation order in (5) and noticing that

$$\binom{x+kz}{k}\binom{-x-kz-1}{i-k}=(-1)^{i-k}\binom{i}{k}\binom{x+kz+i-k}{i},$$

we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k}&=\sum_{i=0}^{n}\binom{x+y+1}{n-i}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\binom{x+kz+i-k}{i}\\&=\sum_{i=0}^{n}\binom{x+y+1}{n-i}(z-1)^i,\end{aligned} \tag{6}$$

where the second equality holds because $\binom{x+kz+i-k}{i}$ is a polynomial in k of degree i with leading coefficient $(z-1)^i/i!$ and we can apply (4) to simplify. We now substitute $x\to -x-1$, $y\to -y+n-1$ and $z\to -z+1$ in (6) and observe that

$$\binom{-x}{k}=(-1)^k\binom{x+k-1}{k}. \tag{7}$$

Then we obtain

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k}=\sum_{i=0}^{n}\binom{x+y-i}{n-i}z^i,$$

as desired.

Combining (1) and (6), we get the following identity:

$$\sum_{k=0}^{n}\binom{x-k}{n-k}z^k=\sum_{k=0}^{n}\binom{x+1}{n-k}(z-1)^k,$$

which is equivalent to the following identity in Graham et al. [**9**, p. 218]:

$$\sum_{k\le m}\binom{m+r}{k}x^k y^{m-k}=\sum_{k\le m}\binom{-r}{k}(-x)^k(x+y)^{m-k}.$$

3. PROOFS OF CHU'S AND GRAHAM-KNUTH-PATASHNIK'S IDENTITIES

Comparing the coefficients of x^n in both sides of the equation

$$(1+x)^{a_1+\cdots+a_s}=(1+x)^{a_1}\cdots(1+x)^{a_s}$$

by the binomial theorem, we have

$$\binom{a_1+\cdots+a_s}{n}=\sum_{i_1+\cdots+i_s=n}\binom{a_1}{i_1}\cdots\binom{a_s}{i_s}. \tag{8}$$

Letting $a_i=-x_i-k_iz-1$ $(1\le i\le s-1)$ and $a_s=x_1+\cdots+x_s+nz+s-1$ in (8), we have

$$\begin{aligned}\binom{x_s+k_sz}{k_s}&=\binom{x_s+(n-k_1-\cdots-k_{s-1})z}{n-k_1-\cdots-k_{s-1}}\\&=\sum_{j=k_1+\cdots+k_{s-1}}^{n}\sum_{j_1+\cdots+j_{s-1}=j}\binom{x_1+\cdots+x_s+nz+s-1}{n-j}\\&\qquad\times\prod_{i=1}^{s-1}\binom{-x_i-k_iz-1}{j_i-k_i},\end{aligned}$$

where $k_1+\cdots+k_s=n$. It follows that

$$\sum_{k_1+\cdots+k_s=n}\prod_{i=1}^{s}\binom{x_i+k_iz}{k_i}=\sum_{k_1+\cdots+k_{s-1}=0}^{n}\sum_{j=k_1+\cdots+k_{s-1}}^{n}\sum_{j_1+\cdots+j_{s-1}=j}\binom{x_1+\cdots+x_s+nz+s-1}{n-j}\times\prod_{i=1}^{s-1}\binom{x_i+k_iz}{k_i}\binom{-x_i-k_iz-1}{j_i-k_i}. \tag{9}$$

Interchanging the summation order in (9) and observing that

$$\binom{x_i+k_iz}{k_i}\binom{-x_i-k_iz-1}{j_i-k_i}=(-1)^{j_i-k_i}\binom{j_i}{k_i}\binom{x_i+k_iz+j_i-k_i}{j_i}$$

and $\binom{x_i+k_iz+j_i-k_i}{j_i}$ is a polynomial in k_i of degree j_i with leading coefficient $(z-1)^{j_i}/j_i!$, by (4) we get

$$\sum_{k_1+\cdots+k_s=n}\prod_{i=1}^{s}\binom{x_i+k_iz}{k_i}=\sum_{j=0}^{n}\binom{x_1+\cdots+x_s+nz+s-1}{n-j} \tag{10}$$

$$\sum_{j_1+\cdots+j_{s-1}=j}(z-1)^j=\sum_{j=0}^{n}\binom{j+s-2}{j}\binom{x_1+\cdots+x_s+nz+s-1}{n-j}(z-1)^j.$$

Substituting $x_i\to -x_i-1$ $(i=1,\dots,s)$ and $z\to -z+1$ in (10) and using (7), we immediately get Chu's identity (3).

Comparing (3) with (10) and replacing s with $s+2$, we obtain

$$\sum_{k=0}^{n}\binom{k+s}{k}\binom{x-k}{n-k}z^k=\sum_{k=0}^{n}\binom{k+s}{k}\binom{x+s+1}{n-k}(z-1)^k. \tag{11}$$

It is easy to see that the identity (11) is equivalent to each of the following known identities:

- Graham-Knuth-Patashnik's identity [**9**, p. 218]

$$\sum_{k\ge 0}\binom{m+r}{m-n-k}\binom{n+k}{n}x^{m-n-k}y^k \tag{12}$$
$$=\sum_{k\ge 0}\binom{-r}{m-n-k}\binom{n+k}{n}(-x)^{m-n-k}(x+y)^k.$$

- Sun's identity [**29**]

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k}{a}(1+x)^{n+k-a}=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{a}x^{m+k-a}. \tag{13}$$

- Munarini's identity [**20**]

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(1+x)^k=\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k. \tag{14}$$

For example, substituting $n\to m-n$, $s\to n$, $x\to -n-r-1$ and $z\to -y/x$ in (11), we are led to (12). Replacing k by $m-k$ and $n-k$ respectively in both sides of (13), we get

$$\sum_{k=0}^{m+n-a}(-1)^{m-k}\binom{m}{k}\binom{m+n-k}{a}(1+x)^{m+n-k-a}$$
$$=\sum_{k=0}^{m+n-a}\binom{n}{k}\binom{m+n-k}{a}x^{m+n-k-a},$$

which is equivalent to (11) by changing k to $m+n-a-k$.

Moreover, the following special case

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}(1+x)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k \tag{15}$$

was reproved by SIMONS [**26**], HIRSCHHORN [**15**], CHAPMAN [**2**], PRODINGER [**21**], WANG and SUN [**30**].

4. MOHANTY-HANDA'S IDENTITY AND CHU'S GENERALIZATION

Let m be a fixed positive integer. For $\mathbf{a}=(a_1,\dots,a_m)\in\mathbb{N}^m$ and $\mathbf{b}=(b_1,\dots,b_m)\in\mathbb{C}^m$, set $|\mathbf{a}|=a_1+\cdots+a_m$, $\mathbf{a}!=a_1!\cdots a_m!$, $\mathbf{a}+\mathbf{b}=(a_1+b_1,\dots,a_m+b_m)$, $\mathbf{a}\cdot\mathbf{b}=a_1b_1+\cdots+a_mb_m$, and $\mathbf{b}^{\mathbf{a}}=b_1^{a_1}\cdots b_m^{a_m}$. For any variable x and $\mathbf{n}=(n_1,\dots,n_m)\in\mathbb{Z}^m$, the *multinomial coefficient* $\binom{x}{\mathbf{n}}$ is defined by

$$\binom{x}{\mathbf{n}}=\begin{cases}x(x-1)\cdots(x-|\mathbf{n}|+1)/\mathbf{n}!, & \text{if } \mathbf{n}\in\mathbb{N}^m,\\ 0, & \text{otherwise.}\end{cases}$$

Moreover, we let $\mathbf{0}=(0,\dots,0)$ and $\mathbf{1}=(1,\dots,1)$.

Note that the CHU-VANDERMONDE convolution formula has the following trivial generalization

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}\binom{x}{\mathbf{k}}\binom{y}{\mathbf{n}-\mathbf{k}}=\binom{x+y}{\mathbf{n}}, \tag{16}$$

as mentioned by ZENG [**32**], while (4) can be easily generalized as

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}(-1)^{|\mathbf{n}|-|\mathbf{k}|}\binom{\mathbf{n}}{\mathbf{k}}\mathbf{k}^{\mathbf{r}}=\begin{cases}0, & \text{if } r_i<n_i \text{ for some } 1\le i\le m.\\ \mathbf{n}!, & \text{if } \mathbf{r}=\mathbf{n},\end{cases} \tag{17}$$

where

$$\binom{\mathbf{n}}{\mathbf{k}}:=\prod_{i=1}^{m}\binom{n_i}{k_i}.$$

In 1969, MOHANTY and HANDA [**19**] established the following multinomial coefficient generalization of JENSEN's identity

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}\binom{x+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}\binom{y-\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}=\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}\binom{x+y-|\mathbf{k}|}{\mathbf{n}-\mathbf{k}}\binom{|\mathbf{k}|}{\mathbf{k}}\mathbf{z}^{\mathbf{k}}. \tag{18}$$

Here and in what follows, $\mathbf{k} = (k_1, \ldots, k_m)$. Twenty years later, MOHANTY-HANDA's identity was generalized by CHU [**5**] as follows:

(19)
$$\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{x_i + \mathbf{k}_i \cdot \mathbf{z}}{\mathbf{k}_i} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{k}| + s - 2}{\mathbf{k}} \binom{x_1 + \cdots + x_s + \mathbf{n} \cdot \mathbf{z} - |\mathbf{k}|}{\mathbf{n} - \mathbf{k}} \mathbf{z}^{\mathbf{k}},$$

which is also a generalization of (3). Here $\mathbf{k}_i = (k_{i1}, \ldots, k_{im})$, $i = 1, \ldots, m$.

REMARK. Note that the corresponding multinomial coefficient generalization of ROTHE's identity was already obtained by RANEY [**22**] (for a special case) and MOHANTY [**18**]. The reader is referred to STREHL [**28**] for a historical note on RANEY-MOHANTY's identity.

We will give an elementary proof of CHU's identity (19) similar to that of (3).

Lemma 4.1. *For* $\mathbf{n} \in \mathbb{N}^m$ *and* $s \geq 1$, *there holds*

$$\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{|\mathbf{k}_i|}{\mathbf{k}_i} = \binom{|\mathbf{n}| + s - 1}{\mathbf{n}}. \tag{20}$$

Proof. For any nonnegative integers $a_1, \ldots, a_s$ such that $a_1 + \cdots + a_s = |\mathbf{n}|$, by the CHU-VANDERMONDE convolution formula (16), the following identity holds

$$\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{a_i}{\mathbf{k}_i} = \binom{|\mathbf{n}|}{\mathbf{n}}. \tag{21}$$

Moreover, for $\mathbf{k}_1 + \cdots + \mathbf{k}_s = \mathbf{n}$, we have

$$\prod_{i=1}^{s} \binom{a_i}{\mathbf{k}_i} \neq 0 \quad \text{if and only if} \quad |\mathbf{k}_i| = a_i \ (i = 1, \ldots, s).$$

Thus, the identity (21) may be rewritten as

$$\sum_{\substack{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n} \\ |\mathbf{k}_1|=a_1,\ldots,|\mathbf{k}_s|=a_s}} \prod_{i=1}^{s} \binom{a_i}{\mathbf{k}_i} = \binom{|\mathbf{n}|}{\mathbf{n}}.$$

It follows that

$$\begin{aligned} \sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{|\mathbf{k}_i|}{\mathbf{k}_i} &= \sum_{a_1+\cdots+a_s=|\mathbf{n}|} \sum_{\substack{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n} \\ |\mathbf{k}_1|=a_1,\ldots,|\mathbf{k}_s|=a_s}} \prod_{i=1}^{s} \binom{a_i}{\mathbf{k}_i} = \sum_{a_1+\cdots+a_s=|\mathbf{n}|} \binom{|\mathbf{n}|}{\mathbf{n}} \\ &= \binom{|\mathbf{n}| + s - 1}{|\mathbf{n}|} \binom{|\mathbf{n}|}{\mathbf{n}}, \end{aligned}$$

as desired. □

By repeatedly using the convolution formula (16), we may rewrite the left-hand side of (19) as

$$
\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_{s-1}=\mathbf{0}}^{\mathbf{n}} \sum_{\mathbf{j}=\mathbf{k}_1+\cdots+\mathbf{k}_{s-1}}^{\mathbf{n}} \sum_{\mathbf{j}_1+\cdots+\mathbf{j}_{s-1}=\mathbf{j}} \binom{x_1+\cdots+x_s+\mathbf{n}\cdot\mathbf{z}+m-1}{\mathbf{n}-\mathbf{j}} \times \prod_{i=1}^{s-1} \binom{x_i+\mathbf{k}_i\cdot\mathbf{z}}{\mathbf{k}_i} \binom{-x_i-\mathbf{k}_i\cdot\mathbf{z}-1}{\mathbf{j}_i-\mathbf{k}_i}. \tag{22}
$$

Interchanging the summation order in (22), observing that

$$
\binom{x_i+\mathbf{k}_i\cdot\mathbf{z}}{\mathbf{k}_i} \binom{-x_i-\mathbf{k}_i\cdot\mathbf{z}-1}{\mathbf{j}_i-\mathbf{k}_i} = (-1)^{|\mathbf{j}_i|-|\mathbf{k}_i|} \binom{\mathbf{j}_i}{\mathbf{k}_i} \binom{x_i+\mathbf{k}_i\cdot\mathbf{z}+|\mathbf{j}_i|-|\mathbf{k}_i|}{\mathbf{j}_i}
$$

and

$$
\binom{x_i+\mathbf{k}_i\cdot\mathbf{z}+|\mathbf{j}_i|-|\mathbf{k}_i|}{\mathbf{j}_i}
$$

is a polynomial in $k_{i\,1},\dots,k_{i\,m}$ with the coefficient of $\mathbf{k}_i^{\mathbf{j}_i}$ being $\binom{|\mathbf{j}_i|}{\mathbf{j}_i}(\mathbf{z}-\mathbf{1})^{\mathbf{j}_i}/\mathbf{j}_i!$, applying (17), we get

$$
\begin{aligned}
\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{x_i+\mathbf{k}_i\cdot\mathbf{z}}{\mathbf{k}_i}
&= \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{n}} \binom{x_1+\cdots+x_s+\mathbf{n}\cdot\mathbf{z}+s-1}{\mathbf{n}-\mathbf{j}} (\mathbf{z}-\mathbf{1})^{\mathbf{j}} \sum_{\mathbf{j}_1+\cdots+\mathbf{j}_{s-1}=\mathbf{j}} \prod_{i=1}^{m} \binom{|\mathbf{j}_i|}{\mathbf{j}_i} \\
&= \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{j}|+s-2}{\mathbf{j}} \binom{x_1+\cdots+x_s+\mathbf{n}\cdot\mathbf{z}+s-1}{\mathbf{n}-\mathbf{j}} (\mathbf{z}-\mathbf{1})^{\mathbf{j}},
\end{aligned} \tag{23}
$$

where the second equality follows from (20). Substituting $x_i \to -x_i-1$ $(i=1,\dots,s)$ and $\mathbf{z}\to -\mathbf{z}+\mathbf{1}$ in (23) and observing that $\binom{-x}{\mathbf{k}} = (-1)^{|\mathbf{k}|}\binom{x+|\mathbf{k}|-1}{\mathbf{k}}$, we immediately get (19).

Comparing (19) with (23) and replacing s with $s+2$, we obtain the following result.

Theorem 4.2. *For $\mathbf{n}\in\mathbb{N}^m$ and $\mathbf{z}\in\mathbb{C}^m$, it holds that*

$$
\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{k}|+s}{\mathbf{k}} \binom{x-|\mathbf{k}|}{\mathbf{n}-\mathbf{k}} \mathbf{z}^{\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{k}|+s}{\mathbf{k}} \binom{x+s+1}{\mathbf{n}-\mathbf{k}} (\mathbf{z}-\mathbf{1})^{\mathbf{k}}. \tag{24}
$$

It is easy to see that (24) is a multinomial coefficient generalization of (11). Substituting $s\to\beta$, $x\to\alpha-\beta-1$ and $\mathbf{z}\to\mathbf{1}+\mathbf{x}$ in (24), we get

$$
\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} (-1)^{|\mathbf{n}|-|\mathbf{k}|} \binom{\beta-\alpha+|\mathbf{n}|}{\mathbf{n}-\mathbf{k}} \binom{\beta+|\mathbf{k}|}{\mathbf{k}} (\mathbf{1}+\mathbf{x})^{\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{\alpha}{\mathbf{n}-\mathbf{k}} \binom{\beta+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}, \tag{25}
$$

which is a generalization of MUNARINI's identity (14). If $\alpha=\beta=|\mathbf{n}|$, then (25) reduces to

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}(-1)^{|\mathbf{n}|-|\mathbf{k}|}\binom{|\mathbf{n}|}{\mathbf{n}-\mathbf{k}}\binom{|\mathbf{n}|+|\mathbf{k}|}{\mathbf{k}}(\mathbf{1}+\mathbf{x})^{\mathbf{k}}=\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}\binom{|\mathbf{n}|}{\mathbf{n}-\mathbf{k}}\binom{|\mathbf{n}|+|\mathbf{k}|}{\mathbf{k}}\mathbf{x}^{\mathbf{k}},$$

which is a generalization of SIMONS' identity (15). Note that SHATTUCK [**25**] and CHEN and PANG [**3**] have given different combinatorial proofs of (14). It is natural to ask the following problem.

Problem 4.3. Is there a combinatorial interpretation of (25)?

In fact, such a proof was recently found by YANG [**31**].

5. CONCLUDING REMARKS

We know that binomial coefficient identities usually have nice q-analogues. However, there are only curious (not natural) q-analogues of ABEL's and ROTHE's identities (see [**24**] and references therein) up to now. There seems to have no q-analogues of JENSEN's identity in the literature.

It is interesting that HOU and ZENG [**16**] gave a q-analogue of SUN's identity (13):

$$\sum_{k=0}^{m}(-1)^{m-k}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n+k\\a\end{bmatrix}(-xq^a;q)_{n+k-a}q^{\binom{k+1}{2}-mk+\binom{a}{2}} =\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m+k\\a\end{bmatrix}x^{m+k-a}q^{mn+\binom{k}{2}}, \tag{26}$$

where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ and

$$\begin{bmatrix}\alpha\\k\end{bmatrix}=\begin{cases}\dfrac{(q^{\alpha-k+1};q)_k}{(q;q)_k}, & \text{if } k\ge 0,\\ 0, & \text{if } k<0.\end{cases}$$

Clearly, (26) may be written as a q-analogue of MUNARINI's identity (14):

$$\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}\beta-\alpha+n\\n-k\end{bmatrix}\begin{bmatrix}\beta+k\\k\end{bmatrix}q^{\binom{n-k}{2}-\binom{n}{2}}(-x;q)_k =\sum_{k=0}^{n}\begin{bmatrix}\alpha\\n-k\end{bmatrix}\begin{bmatrix}\beta+k\\k\end{bmatrix}q^{\binom{n-k+1}{2}+(\beta-\alpha)(n-k)}x^k, \tag{27}$$

as mentioned by GUO and ZENG [**14**]. We end this paper with the following problem.

Problem 5.1. Is there a q-analogue of (25)? Or equivalently, is there a multi-sum generalization of (27)?

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REFERENCES

1. L. Carlitz: *Some formulas of Jensen and Gould.* Duke Math. J., **27** (1960), 319–321.
2. R. Chapman: *A curious identity revisited.* Math. Gazette, **87** (2003), 139–141.
3. W. Y. C. Chen, S. X. M. Pang: *On the combinatorics of the Pfaff identity.* Discrete Math., **309** (2009), 2190–2196.
4. W. Chu: *On an extension of a partition identity and its Abel analog.* J. Math. Res. Exposition, **6** (4) (1986), 37–39.
5. W. Chu: *Jensen's theorem on multinomial coefficients and its Abel-analog.* Appl. Math. J. Chinese Univ., **4** (1989), 172–178 (in Chinese).
6. W. Chu: *Elementary proofs for convolution identities of Abel and Hagen-Rothe.* Electron. J. Combin., **17** (2010), #N24.
7. M. E. Cohen, H. S. Sun: *A note on the Jensen-Gould convolutions.* Canad. Math. Bull., **23** (1980), 359–361.
8. L. Comtet: *Advanced Combinatorics.* D. Reidel Publishing Company, Dordrecht-Holland, 1974.
9. R. L. Graham, D. E. Knuth, O. Patashnik: *Concrete Mathematics.* 2nd Edition, Addison-Wesley Publishing Company, Reading, MA, 1994.
10. E. G.-Rodeja F.: *On identities of Jensen, Gould and Carlitz.* In: Proc. Fifth Annual Reunion of Spanish Mathematicians (Valencia, 1964), Publ. Inst. "Jorge Juan" Mat., Madrid, 1967, pp. 11–14.
11. H. W. Gould: *Generalization of a theorem of Jensen concerning convolutions.* Duke Math. J., **27** (1960), 71–76.
12. H. W. Gould: *Involving sums of binomial coefficients and a formula of Jensen.* Amer. Math. Monthly, **69** (5) (1962), 400–402.
13. V. J. W. Guo: *A simple proof of Dixon's identity.* Discrete Math., **268** (2003), 309–310.
14. V. J. W. Guo, J. Zeng: *Combinatorial proof of a curious q-binomial coefficient identity.* Electron. J. Combin., **17** (2010), #N13.
15. M. Hirschhorn: *Comment on a curious identity.* Math. Gazette, **87** (2003), 528–530.
16. S. J. X. Hou, J. Zeng: *A q-analog of dual sequences with applications.* European J. Combin., **28** (2007), 214–227.
17. J. L. W. V. Jensen: *Sur une identité d'Abel et sur d'autres formules analogues.* Acta Math., **26** (1902), 307–318.
18. S. G. Mohanty: *Some convolutions with multinomial coefficients and related probability distributions.* SIAM Rev., **8** (1966), 501–509.
19. S. G. Mohanty, B. R. Handa: *Extensions of Vandermonde type convolutions with several summations and their applications, I.* Canad. Math. Bull., **12** (1969), 45–62.
20. E. Munarini: *Generalization of a binomial identity of Simons.* Integers, **5** (2005), #A15.
21. H. Prodinger: *A curious identity proved by Cauchy's integral formula.* Math. Gazette, **89** (2005), 266–267.

22. G. N. Raney: *Functional composition patterns and power series reversion.* Trans. Amer. Math. Soc., **94** (1960), 441–451.
23. H. A. Rothe: *Formulae de serierum reversione demonstratio universalis signis localibus combinatorio-analyticorum vicariis exhibita.* Leipzig, 1793.
24. M. Schlosser: *Abel-Rothe type generalizations of Jacobi's triple product identity.* in: Theory and Applications of Special Functions. Dev. Math., 13, Springer, New York, 2005, pp. 383–400.
25. M. Shattuck: *Combinatorial proofs of some Simons-type binomial coefficient identities.* Integers, **7** (2007), #A27.
26. S. Simons: *A curious identity.* Math. Gazette, **85** (2001), 296–298.
27. R. P. Stanley: *Enumerative Combinatorics.* Vol. 1, Cambridge Studies in Advanced Mathematics, 49, Cambridge University Press, Cambridge, 1997.
28. V. Strehl: *Identities of Rothe-Abel-Schläfli-Hurwitz-type.* Discrete Math., **99** (1992), 321–340.
29. Z.-W. Sun: *Combinatorial identities in dual sequences.* European J. Combin., **24** (2003), 709–718.
30. X. Wang, Y. Sun: *A new proof of a curious identity.* Math. Gazette, **91** (2007), 105–106.
31. D.-M. Yang: *A combinatorial proof of Guo's multi-generalization of Munarini's identity.* Integers, to appear.
32. J. Zeng: *Multinomial convolution polynomials.* Discrete Math., **160** (1996), 219–228.

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