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BESSELIAN *G*-FRAMES AND NEAR *G*-RIESZ BASES

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In this paper we introduce and study near g-Riesz basis, Besselian g-frames and unconditional g-frames. We show that a near g-Riesz basis is a Besselian g-frame and we conclude that under some conditions the kernel of associated synthesis operator for a near g-Riesz basis is finite dimensional. Finally, we show that a g-frame is a g-Riesz basis for a Hilbert space $\mathcal{H}$ if and only if there is an equivalent inner product on $\mathcal{H}$, with respect to which it becomes an g-orthonormal basis for $\mathcal{H}$.

1. INTRODUCTION

The concept of frame was introduced by Duffin and Schaeffer [**4**] in 1952. Afterwards, several generalizations of frames in Hilbert spaces have been proposed [**1, 7, 5, 3**]. *G*-frames, the most recent generalization of frames, introduced by W. Sun [**9**].

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space and $\{\mathcal{H}_i\}_{i\in I}$ is a sequence of separable Hilbert spaces, where I is a subset of $\mathbb{N}$. $\mathcal{B}(\mathcal{H},\mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_i$.

Definition 1.1. *A sequence* $\{\Lambda_i \in \mathcal{B}(\mathcal{H},\mathcal{H}_i) : i \in I\}$ *is called a g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I}$ *if there exist two positive constants* A *and* B *such that*

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \tag{1.1}$$

for all $f \in \mathcal{H}$. *We call* A *and* B *the lower and upper* $g-$*frame bounds, respectively. We call* $\{\Lambda_i\}_{i\in I}$ *a tight g-frame if* $A = B$ *and Parseval g-frame if* $A = B = 1$. *The sequence* $\{\Lambda_i \in \mathcal{B}(\mathcal{H},\mathcal{H}_i) : i \in I\}$ *is called the g-Bessel sequence if the right hand inequality in* (1.1) *holds for all* $f \in \mathcal{H}$.

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Let us define the set

$$\left(\sum_{i\in I}\oplus\mathcal{H}_i\right)_{l_2}=\left\{\{f_i\}:f_i\in\mathcal{H}_i,\sum_{i\in I}\|f_i\|^2<\infty\right\}$$

with the inner product given by $\langle\{f_i\},\{g_i\}\rangle=\sum_{i\in I}\langle f_i,g_i\rangle$. It is clear that $\left(\sum_{i\in I}\oplus\mathcal{H}_i\right)_{l_2}$ is a Hilbert space with respect to the pointwise operations. If $\{\Lambda_i\in\mathcal{B}(\mathcal{H},\mathcal{H}_i): i\in I\}$ is a g-Bessel sequence for $\mathcal{H}$, then the operator

$$T:\left(\sum_{i\in I}\oplus\mathcal{H}_i\right)_{l_2}\to\mathcal{H}$$

defined by

$$T(\{f_i\})=\sum_{i\in I}\Lambda_i^*(f_i) \tag{1.2}$$

is well defined, bounded and its adjoint is $T^*f=\{\Lambda_i f\}_{i\in I}$. A sequence $\{\Lambda_i\in\mathcal{B}(\mathcal{H},\mathcal{H}_i): i\in I\}$ is a g-frame if and only if the operator T is defined as (1.2) is bounded and onto (see [**8**]). We call the operators T and T^*, synthesis and analysis operators, respectively.

Proposition 1.2. [**9**] *Let $\{\Lambda_i\in\mathcal{B}(\mathcal{H},\mathcal{H}_i): i\in I\}$ be a g-Bessel sequence for $\mathcal{H}$. The operator*

$$S:\mathcal{H}\to\mathcal{H},\quad Sf=\sum_{i\in I}\Lambda_i^*\Lambda_i f$$

is a positive and bounded operator.

A simple computation shows that $\langle Sf,f\rangle=\sum_{i\in I}\|\Lambda_i f\|^2$ for all $f\in\mathcal{H}$. This implies that S is an invertible operator if and only if $\{\Lambda_i\in\mathcal{B}(\mathcal{H},\mathcal{H}_i): i\in I\}$ is a g-frame for $\mathcal{H}$. If $\{\Lambda_i\in\mathcal{B}(\mathcal{H},\mathcal{H}_i): i\in I\}$ is a g-frame for $\mathcal{H}$, then every $f\in\mathcal{H}$ has an expansion

$$f=\sum_{i\in I}S^{-1}\Lambda_i^*\Lambda_i f=\sum_{i\in I}\Lambda_i^*\Lambda_i S^{-1}f.$$

The operator S is called the g-frame operator of $\{\Lambda_i\}_{i\in I}$. If $\{\Lambda_i\in\mathcal{B}(\mathcal{H},\mathcal{H}_i): i\in I\}$ is a g-Bessel sequence, then $S=TT^*$.

Definition 1.3. [**9**] *A sequence $\{\Lambda_i\in\mathcal{B}(\mathcal{H},\mathcal{H}_i): i\in I\}$ is called*

(1) *g-complete, if $\{f:\ \Lambda_i f=0, i\in I\}=0$;*

(2) *a g-Riesz basis for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I}$, *if* $\{\Lambda_i \in \mathcal{B}(\mathcal{H},\mathcal{H}_i) : i \in I\}$ *is g-complete and there exist two positive constants* A *and* B *such that for any finite subset* $F \subseteq I$ *and* $g_i \in \mathcal{H}_i$

$$A\sum_{i\in F}\|g_i\|^2 \leq \left\|\sum_{i\in F}\Lambda_i^* g_i\right\|^2 \leq B\sum_{i\in F}\|g_i\|^2;$$

(3) *a g-orthonormal basis for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I}$, *if* $\sum_{i\in I}\|\Lambda_i f\|^2 = \|f\|^2$ *for all* $f \in \mathcal{H}$ *and* $\langle\Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle, \quad g_i \in \mathcal{H}_i,\ g_j \in \mathcal{H}_j, \quad i,j \in I.$

2. NEAR g-RIESZ BASES

As usual, we denote by $\ell^2(I)$ the Hilbert space of all square-summable sequences of scalars $\{c_i\}_{i\in I}$. If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then $\sum_{i\in I} c_i f_i$ converges if $\{c_i\}_{i\in I} \in \ell^2(I)$. But the converse is not true in general (see [**6**]). A frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$ is called

- *Besselian*, if whenever $\sum_{i\in I} c_i f_i$ converges, then $\{c_i\}_{i\in I} \in \ell^2(I)$;
- a *near-Riesz basis*, if there is a finite set σ for which $\{f_i\}_{i\in I\setminus\sigma}$ is a Riesz basis for $\mathcal{H}$.

We recall the following characterization of frames which are near-Riesz bases.

Theorem 2.1. [**6**] *If* $\{f_i\}_{i\in I}$ *is a frame for* $\mathcal{H}$, *the following are equivalent:*

(i) $\{f_i\}_{i\in I}$ *is a near-Riesz basis for* $\mathcal{H}$;

(ii) $\{f_i\}_{i\in I}$ *is Besselian;*

(iii) $\sum_{i\in I} c_i f_i$ *converges if and only if* $\{c_i\}_{i\in I} \in l^2(I)$.

For the rest of the paper we need the following proposition.

Proposition 2.2. [**8**] *Let* $\{e_{ij}\}_{i\in J_i}$ *be an orthonormal basis for the Hilbert space* $\mathcal{H}_i$, *where* J_i *is a subset of* $\mathbb{N}$ *and* $i \in I$. *If*

$$(E_{ij})_k = \begin{cases} e_{ij}, & i = k, \\ 0, & i \neq k, \end{cases} \tag{2.1}$$

then $\{E_{ij}\}_{i\in I, j\in J_i}$ *is an orthonormal basis for* $\left(\sum_{i\in I}\oplus\mathcal{H}_i\right)_{l_2}$.

Theorem 2.3. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and let T be the associated synthesis operator for Λ. If T has the property that $\dim(\operatorname{Ker} T) < \infty$, then there is a g-Riesz basis $\{\Theta_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, where W_i is a closed subspace of $\mathcal{H}_i$, such that $\Theta_i = \Lambda_i$ and $W_i = \mathcal{H}_i$ for all $i \in I$ except finitely many i.*

Proof. Let $\{g_{ij}\}_{i\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$, $i \in I$. Then $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ [**9**]. Let Q be the associated synthesis operator for $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i}$. We define

$$\Psi : \operatorname{Ker} Q \to \operatorname{Ker} T, \quad \Psi(\{c_{ij}\}_{i\in I, j\in J_i}) = \sum_{i\in I} \sum_{j\in J_i} c_{ij} E_{ij},$$

where E_{ij} was defined by (2.1). It is clear that Ψ is well defined, linear and injective since $\{E_{ij}\}$ is an orthonormal basis for $\Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{l_2}$.

Let $f = \{f_i\}_{i\in I} \in \operatorname{Ker} T$, then $f_i = \sum_{j\in J_i} \lambda_{ij} g_{ij}$ for all $i \in I$. Since $\sum_{i\in I, j\in J_i} |\lambda_{ij}|^2 = \sum_{i\in I} \|f_i\|^2 < \infty$, we get $\{\lambda_{ij}\}_{i\in I, j\in J_i} \in l^2$ and

$$\Psi(\{\lambda_{ij}\}_{i\in I, j\in J_i}) = \sum_{i\in I, j\in J_i} \lambda_{ij} E_{ij} = f.$$

Therefore Ψ is surjective and we conclude that $\dim(\operatorname{Ker} Q) = \dim(\operatorname{Ker} T) < \infty$. So $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i}$ is a near-Riesz basis for $\mathcal{H}$ [**6**]. Therefore there exists a finite subsequence $\mathcal{M}$ of $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i}$ such that $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i} \setminus \mathcal{M}$ is a Riesz basis for $\mathcal{H}$. Let us consider

$$K_i = \{j \in J_i : \Lambda_i^* g_{ij} \notin \mathcal{M}\},$$

and define

$$\Theta_i : \mathcal{H} \to \mathcal{H}_i, \quad \Theta_i f = \sum_{j\in K_i} \langle f, \Lambda_i^* g_{ij}\rangle g_{ij}$$

for all $i \in I$. By Theorem 3.1 in [**9**], $\{\Theta_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, where $W_i = \overline{\operatorname{span}}\{g_{ij}\}_{j\in K_i} \subseteq \mathcal{H}_i$. Since $K_i = J_i$ for all $i \in I$ except finitely many, we have

$$\Lambda_i f = \sum_{j\in J_i} \langle \Lambda_i f, g_{ij}\rangle g_{ij} = \sum_{j\in K_i} \langle f, \Lambda_i^* g_{ij}\rangle g_{ij} = \Theta_i f$$

for all $f \in \mathcal{H}$ and all $i \in I$ except finitely many i. This completes the proof.

Definition 2.4. *We say that a g-frame $\{\Lambda_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ is*

(1) *a Besselian g-frame, if $\sum_{i\in I} \Lambda_i^* g_i$ converges, then $\{g_i\}_{i\in I} \in \Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{\ell_2}$;*

(2) *a near g-Riesz basis, if there exists a finite subset* σ *of* I *for which* $\{\Lambda_i\}_{i\in I\setminus\sigma}$ *is a g-Riesz basis for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I\setminus\sigma}$.

EXAMPLE 2.5. Let $A = [0, +\infty)$ with the Lebesque measure μ and $A_1 = [0, 5)$, $A_2 = [5, 10)$ and $A_n = [n-3, n-2)$ for all integers $n \geq 3$. Let $\mathcal{H} = L^2(A)$, $\mathcal{H}_i = L^2(A_i)$ and Λ_i be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_i$. Then $\{\Lambda_i\}_{i\in\mathbb{N}}$ is near g-Riesz basis for $\mathcal{H} = L^2(A)$, because $\{\Lambda_i\}_{i\geq 3}$ is g-Riesz basis for $\mathcal{H} = L^2(A)$.

Theorem 2.6. *Let* $\Lambda = \{\Lambda_i\}_{i\in I}$ *be a g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I}$ *and let* T *be the associated synthesis operator for* Λ*. If* $\Lambda = \{\Lambda_i\}_{i\in I}$ *is a near g-Riesz basis, then* Λ *is a Besselian g-frame.*

Proof. Since $\Lambda = \{\Lambda_i\}_{i\in I}$ is a near g-Riesz basis, there exists a finite subset σ of I such that $\{\Lambda_i\}_{i\in I\setminus\sigma}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I\setminus\sigma}$. Suppose that $\sum_{i\in I}\Lambda_i^* g_i$ converges, where $g_i \in \mathcal{H}_i$ for all $i \in I$. So $\sum_{i\in I\setminus\sigma}\Lambda_i^* g_i$ converges. Since $\{\Lambda_i\}_{i\in I\setminus\sigma}$ is a g-Riesz basis, there exists a bounded invertible operator U and a g-orthonormal basis $\{\Theta_i\}_{i\in I\setminus\sigma}$ such that $\Lambda_i = \Theta_i U$ for $i \in I \setminus \sigma$ (see [**9**], Corollary 3.4). So

$$\sum_{i\in I\setminus\sigma}\Lambda_i^* g_i = \sum_{i\in I\setminus\sigma} U^*\Theta_i^* g_i = U^*\left(\sum_{i\in I\setminus\sigma}\Theta_i^* g_i\right).$$

Since $\{\Theta_i\}_{i\in I\setminus\sigma}$ is a g-orthonormal basis, we have

$$\sum_{i\in I\setminus\sigma}\|g_i\|^2 = \left\|\sum_{i\in I\setminus\sigma}\Theta_i^* g_i\right\|^2 < \infty.$$

Then $\{g_i\}_{i\in I\setminus\sigma} \in \left(\sum_{i\in I\setminus\sigma}\oplus\,\mathcal{H}_i\right)_{l_2}$ and this implies that $\{g_i\}_{i\in I} \in \left(\sum_{i\in I}\oplus\,\mathcal{H}_i\right)_{l_2}$. Hence $\Lambda = \{\Lambda_i\}_{i\in I}$ is a Besselian g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$.

Corollary 2.7. *Suppose that* $\dim\mathcal{H}_i < \infty$ *for each* $i \in I$*. Let* $\Lambda = \{\Lambda_i\}_{i\in I}$ *be a g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I}$ *and let* T *be the associated synthesis operator for* Λ*. If* $\Lambda = \{\Lambda_i\}_{i\in I}$ *is a near g-Riesz basis, then* $\dim(\operatorname{Ker} T) < \infty$.

Proof. It follows from Theorem 2.6 that $\Lambda = \{\Lambda_i\}_{i\in I}$ is a Besselian g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$. Then $\{\Lambda_i^* e_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$. Suppose that $\sum_{i\in I}\sum_{j\in J_i} c_{ij}\Lambda_i^* e_{ij}$ converges. Since Λ is a Besselian g-frame, we get $\left\{\sum_{j\in J_i} c_{ij}e_{ij}\right\}_{i\in I} \in \left(\sum_{i\in I}\oplus\,\mathcal{H}_i\right)_{l_2}$. So

$$\sum_{i\in I}\sum_{j\in J_i}|c_{ij}|^2 = \sum_{i\in I}\left\|\sum_{j\in J_i} c_{ij}e_{ij}\right\|^2 < \infty.$$

Hence $\{\Lambda_i^* e_{ij}\}_{i\in I, j\in J_i}$ is Besselian. Let Q be the associated synthesis operator for $\{\Lambda_i^* e_{ij}\}_{i\in I, j\in J_i}$, then $\dim(\operatorname{Ker} Q) < \infty$ [**6**, Theorem 2.3]. Let us define $E_{ij} \in \Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{l_2}$ by

$$(E_{ij})_k = \begin{cases} e_{ij}, & i = k, \\ 0, & i \neq k, \end{cases}$$

for all $i, j, k \in I$. By Proposition 2.2, $\{E_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{l_2}$. By the definition of Q and T, it is clear that

$$Q(\{c_{ij}\}_{i\in I, j\in J_i}) = \sum_{i\in I}\sum_{j\in J_i} c_{ij}\Lambda_i^* e_{ij} = T\Bigg(\sum_{i\in I}\sum_{j\in J_i} c_{ij}E_{ij}\Bigg).$$

Now we consider the mapping

$$\varphi : \operatorname{Ker} Q \to \operatorname{Ker} T, \quad \varphi(\{c_{ij}\}_{i\in I, j\in J_i}) = \sum_{i\in I}\sum_{j\in J_i} c_{ij}E_{ij}.$$

It is obvious that φ is linear and injective. We claim that φ is surjective. Let $\{g_i\}_{i\in I} \in \operatorname{Ker} T$. Then $g_i \in \mathcal{H}_i$ and $g_i = \sum_{j\in J_i} \lambda_{ij} e_{ij}$ for each $i \in I$. Since $\|g_i\|^2 = \sum_{j\in J_i} |\lambda_{ij}|^2$, we have $\sum_{i\in I}\sum_{j\in J_i} |\lambda_{ij}|^2 = \sum_{i\in I} \|g_i\|^2 < \infty$. Therefore $\{\lambda_{ij}\}_{i\in I, j\in J_i} \in l^2$ and

$$\begin{aligned} Q\big(\{\lambda_{ij}\}_{i\in I, j\in J_i}\big) &= T\Bigg(\sum_{i\in I}\sum_{j\in J_i} \lambda_{ij}E_{ij}\Bigg) = T\big(\{g_i\}_{i\in I}\big) = 0, \\ \varphi\big(\{\lambda_{ij}\}_{i\in I, j\in J_i}\big) &= \sum_{i\in I}\sum_{j\in J_i} \lambda_{ij}E_{ij} = \{g_i\}_{i\in I}. \end{aligned}$$

Hence $\dim(\operatorname{Ker} T) = \dim(\operatorname{Ker} Q) < \infty$.

Definition 2.8. *A g-frame $\{\Lambda_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ is called an unconditional g-frame if it satisfies that, if $\sum_{i\in I} \Lambda_i^* g_i$ converges, then $\sum_{i\in I} \Lambda_i^* g_i$ converges unconditionally, where $g_i \in \mathcal{H}_i$ for each $i \in I$.*

Proposition 2.9. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a Besselian g-frame (near g-Riesz basis) for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with the upper bound B. Then Λ is an unconditional g-frame.*

Proof. By Theorem 2.6 every near g-Riesz basis is a Besselian g-frame. Suppose that $\sum_{i\in I} \Lambda_i^* g_i$ converges, where $g_i \in \mathcal{H}_i$ for all $i \in I$. Since $\Lambda = \{\Lambda_i\}_{i\in I}$ is Besselian, we get $\{g_i\}_{i\in I} \in \Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{l_2}$. We show that $\sum_{i\in I} \Lambda_i^* g_i$ converges unconditionally. Let J be an arbitrary finite subset of I. Then

$$\Big\|\sum_{i\in J}\Lambda_i^* g_i\Big\| = \sup_{\|g\|=1}\Big|\Big\langle \sum_{i\in J}\Lambda_i^* g_i, g\Big\rangle\Big| = \sup_{\|g\|=1}\Big|\sum_{i\in J}\langle g_i, \Lambda_i g\rangle\Big|$$
$$\leq \sup_{\|g\|=1}\Big(\sum_{i\in J}\|\Lambda_i g\|^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in J}\|g_i\|^2\Big)^{\frac{1}{2}} \leq \sqrt{B}\Big(\sum_{i\in I}\|g_i\|^2\Big)^{\frac{1}{2}}.$$

Since $\sum_{i\in I}\|g_i\|^2$ converges unconditionally, $\sum_{i\in I}\Lambda_i^* g_i$ converges unconditionally.

Theorem 2.10. *Let $\{\Lambda_i\}_{i\in I}$ be a Besselian g-frame for $\mathcal{H}$ with bounds A, B and $\{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a sequence of bounded operators such that for any finite subset $J \subseteq I$ and for each $f \in \mathcal{H}$,*

$$\Big\|\sum_{i\in J}(\Lambda_i^* f_i - \Theta_i^* f_i)\Big\| \leq \lambda\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| + \mu\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\|, \tag{2.2}$$

where $0 \leq \lambda, \mu < 1$ and $f_i \in \mathcal{H}_i$ for all $i \in J$. Then $\{\Theta_i\}_{i\in I}$ is a Besselian g-frame for $\mathcal{H}$ with the bounds

$$\Big[\frac{(1-\lambda)\sqrt{A}}{1+\mu}\Big]^2 \quad \text{and} \quad \Big[\frac{(1+\lambda)\sqrt{B}}{1-\mu}\Big]^2. \tag{2.3}$$

Proof. It follows from (2.2) that $\{\Theta_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with the required bounds (see [**8**]). Assume that $J \subseteq I$ with $|J| < +\infty$ and $f_i \in \mathcal{H}_i$ for all $i \in J$. We have

$$\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| \leq \Big\|\sum_{i\in J}(\Lambda_i^* f_i - \Theta_i^* f_i)\Big\| + \Big\|\sum_{i\in J}\Theta_i^* f_i\Big\|$$
$$\leq (1+\mu)\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| + \lambda\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\|.$$

Hence

$$\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| \leq \frac{1+\mu}{1-\lambda}\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\|.$$

This implies that $\sum_{i\in I}\Lambda_i^* f_i$ converges if $\sum_{i\in I}\Theta_i^* f_i$ converges. Therefore $\{\Theta_i\}_{i\in I}$ is Besselian.

Definition 2.11. *A g-frame $\{\Lambda_i\}_{i\in I}$ for $\mathcal{H}$ is called a g-Riesz frame if every subfamily $\{\Lambda_i\}_{i\in J}$ of $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in J}$ with uniform g-frame bounds A, B.*

Theorem 2.12. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ such that*

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{ij}\langle g_i, g_j\rangle, \quad g_i \in \mathcal{H}_i,\, g_j \in \mathcal{H}_j, \quad i, j \in I. \tag{2.4}$$

Then there exist $I_1 \subseteq I$ and a g-Riesz basis $\{\Theta_i\}_{i\in I_1}$ for $\mathcal{H}$ with respect to $\{K_i\}_{i\in I_1}$, where K_i is a closed subspace of $\mathcal{H}_i$ for all $i \in I_1$.

Proof. Let A, B be the g-frame bounds for $\{\Lambda_i\}_{i\in I}$ and let $E \subseteq I$. Since $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$, we get $\sum_{i\in E}\Lambda_i^*\Lambda_i f$ converges for all $f \in \mathcal{H}$. We show that $f = \sum_{i\in E}\Lambda_i^*\Lambda_i f$ for all $f \in \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$. Let $f \in \text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$, then $f = \sum_{i\in E}\Lambda_i^* g_i$ where $g_i \in \mathcal{H}_i$ and the set $\{\, i \in E : \Lambda_i^* g_i \neq 0\,\}$ is finite. We show that $g_i = \Lambda_i f$ for $i \in E$. Let $h \in \mathcal{H}_i$, then

$$\langle \Lambda_i f, h\rangle = \Big\langle \sum_{k\in E}\Lambda_i\Lambda_k^* g_k, h\Big\rangle = \sum_{k\in E}\langle \Lambda_k^* g_k, \Lambda_i^* h\rangle = \langle \Lambda_i^* g_i, \Lambda_i^* h\rangle = \langle g_i, h\rangle.$$

So $g_i = \Lambda_i f$ for $i \in E$ and $f = \sum_{i\in E}\Lambda_i^*\Lambda_i f$.

For the case $f \in \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$, there exists a sequence $\{f_n\}$ in $\text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$ such that $f_n \to f$ as $n \to \infty$. We have

$$\begin{aligned}\Big\|\sum_{i\in E}\Lambda_i^*\Lambda_i f_n - \sum_{i\in E}\Lambda_i^*\Lambda_i f\Big\|^2 &= \Big\|\sum_{i\in E}\Lambda_i^*\Lambda_i(f_n - f)\Big\|^2 \\ &= \sum_{i\in E}\|\Lambda_i(f_n - f)\|^2 \leq B\|f_n - f\|^2 \to 0.\end{aligned}$$

Hence $f = \sum_{i\in E}\Lambda_i^*\Lambda_i f$. Therefore it follows from (2.4) that

$$\|f\|^2 = \Big\|\sum_{i\in E}\Lambda_i^*\Lambda_i f\Big\|^2 = \sum_{i\in E}\|\Lambda_i f\|^2 \leq \sum_{i\in I}\|\Lambda_i f\|^2 \leq B\|f\|^2$$

for all $f \in \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$. This means that $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-Riesz frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with the uniform g-frame bounds 1 and B. Assume that $\{g_{ij}\}_{j\in J_i}$ is an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$. Then $\{g_{ij}\}_{j\in J_i}$ is a Riesz frame for $\mathcal{H}_i$ with bounds equal to 1. We show that $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i}$ is a Riesz frame for $\mathcal{H}$. Let $I_0 \subseteq I$, $J_i^0 \subseteq J_i$ and $f \in \text{span}\{\Lambda_i^* g_{ij}\}_{i\in I_0, J_i^0}$. Then $\Lambda_i f \in \text{span}\{g_{ij}\}_{j\in J_i^0}$ for all $i \in I_0$. So

$$\sum_{i\in I_0}\|\Lambda_i f\|^2 = \sum_{i\in I_0}\sum_{j\in J_i^0}|\langle \Lambda_i f, g_{ij}\rangle|^2 = \sum_{i\in I_0}\sum_{j\in J_i^0}|\langle f, \Lambda_i^* g_{ij}\rangle|^2. \tag{2.5}$$

Since $\{\Lambda_i\}_{i\in I}$ is g-Riesz frame, we have

$$\|f\|^2 \leq \sum_{i\in I_0} \|\Lambda_i f\|^2 \leq B\|f\|^2. \tag{2.6}$$

Hence (2.6) and (2.5) imply

$$\|f\|^2 \leq \sum_{i\in I_0} \sum_{j\in J_i^0} |\langle f, \Lambda_i^* g_{ij}\rangle|^2 \leq B\|f\|^2.$$

Therefore $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i}$ is a Riesz frame for $\mathcal{H}$. By Theorem 6.3.3 in [**2**], it follows that $\{\Lambda_i^* g_{ij}\}_{i\in I, J_i}$ contains a Riesz basis. Let $I_1 \subseteq I$ and $J_i^1 \subseteq J_i$ such that $\{\Lambda_i^* g_{ij}\}_{i\in I_1, j\in J_i^1}$ is a Riesz basis for $\mathcal{H}$. Consider $K_i = \overline{\operatorname{span}}\{g_{ij}\}_{j\in J_i^1}$ for $i \in I_1$ and define

$$\Theta_i : \mathcal{H} \to K_i, \quad \Theta_i f = \sum_{j\in J_i^1} \langle f, \Lambda_i^* g_{ij}\rangle, \qquad i \in I_1.$$

By Theorem 3.1 of [**9**], we obtain that $\{\Theta_i\}_{i\in I_1}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{K_i\}_{i\in I_1}$.

Theorem 2.13. *Let $\{\Lambda_i\}_{i\in I}$ be a g-Riesz basis for $\mathcal{H}$ with bounds A, B and $\{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a sequence of bounded operators. Assume that there exist $\lambda, \gamma, \mu \geq 0$ such that $\max\left\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\right\} < 1$. Suppose that for any finite subset $J \subseteq I$ and for each $f \in \mathcal{H}$,*

$$\Big\|\sum_{i\in J}(\Lambda_i^* f_i - \Theta_i^* f_i)\Big\| \leq \lambda \Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| + \mu\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| + \gamma\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}, \tag{2.7}$$

where $f_i \in \mathcal{H}_i$ for all $i \in J$. Then $\{\Theta_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H}$ with the bounds

$$\left[\frac{(1-\lambda)\sqrt{A} - \gamma}{1+\mu}\right]^2 \quad \text{and} \quad \left[\frac{(1+\lambda)\sqrt{B} + \gamma}{1-\mu}\right]^2. \tag{2.8}$$

Especially, if $\{\Lambda_i\}_{i\in I}$ is a near g-Riesz basis for $\mathcal{H}$, then $\{\Theta_i\}_{i\in I}$ is a near g-Riesz basis for $\mathcal{H}$.

Proof. It follows from (2.7) that $\{\Theta_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ and therefore $\{\Theta_i\}_{i\in I}$ is g-complete (see [**8**]). Assume that $J \subseteq I$ with $|J| < +\infty$ and $f_i \in \mathcal{H}_i$ for all $i \in J$. We have

$$\begin{aligned}\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| &\leq \Big\|\sum_{i\in J}(\Lambda_i^* f_i - \Theta_i^* f_i)\Big\| + \Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| \\ &\leq (1+\lambda)\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| + \mu\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| + \gamma\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Then

$$\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| \leq \frac{1+\lambda}{1-\mu}\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| + \frac{\gamma}{1-\mu}\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}.$$

Since $\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\|^2 \leq B\sum_{i\in J}\|f_i\|^2$, we get

$$\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\|^2 \leq \Big[\frac{(1+\lambda)\sqrt{B}+\gamma}{1-\mu}\Big]^2\sum_{i\in J}\|f_i\|^2.$$

Similarly, we have

$$\begin{aligned}\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| &\leq \Big\|\sum_{i\in J}(\Lambda_i^* f_i - \Theta_i^* f_i)\Big\| + \Big\|\sum_{i\in J}\Theta_i^* f_i\Big\|\\ &\leq (1+\mu)\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| + \lambda\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| + \gamma\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\| \leq \frac{1+\mu}{1-\lambda}\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\| + \frac{\gamma}{1-\lambda}\Big(\sum_{i\in J}\|f_i\|^2\Big)^{\frac{1}{2}}.$$

Since $\Big\|\sum_{i\in J}\Lambda_i^* f_i\Big\|^2 \geq A\sum_{i\in J}\|f_i\|^2$, we get

$$\Big\|\sum_{i\in J}\Theta_i^* f_i\Big\|^2 \geq \Big[\frac{(1-\lambda)\sqrt{A}-\gamma}{1+\mu}\Big]^2\sum_{i\in J}\|f_i\|^2.$$

This completes the proof. □

Let V be a normed space with norm $\|.\|$. If $\|.\|_1$ is another norm on V, $\|.\|$ and $\|.\|_1$ are said to be equivalent if there are positive constants m and M such that $m\|f\| \leq \|f\|_1 \leq M\|f\|$ for all $f \in V$. Two inner products on a vector space are said to be equivalent if they generate equivalent norms. A sequence $\{f_n\}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis if and only if there exists an equivalent inner product on $\mathcal{H}$, with respect to which the sequence $\{f_n\}$ becomes an orthonormal basis for $\mathcal{H}$ [**10**]. In the next theorem we show that every g-Riesz basis for $\mathcal{H}$ can be considered as a g-orthonormal basis for $\mathcal{H}$ with respect to equivalent inner product on $\mathcal{H}$.

Theorem 2.14. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then Λ is a g-Riesz basis for $\mathcal{H}$ if and only if there is an equivalent inner product on $\mathcal{H}$, with respect to which $\Lambda = \{\Lambda_i\}_{i\in I}$ becomes an g-orthonormal basis for $\mathcal{H}$.*

Proof. Let $\langle .,. \rangle$ be the usual inner product of $\mathcal{H}$. Assume that $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H}$ and $\{g_{ij}\}_{j\in J_i}$ is an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$.

Then by [**9**] Theorem 3.1, $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i}$ is a Riesz basis for $\mathcal{H}$. By Theorem 9 in [**10**, page 32] there exists an equivalent inner product $\langle .,,.\rangle_1$ on $\mathcal{H}$ such that $\{\Lambda_i^* g_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\mathcal{H}$ with respect to $\langle .,,.\rangle_1$. We show that $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis with respect to $\langle .,,.\rangle_1$. Let $\|.\|_1$ be the induced norm by $\langle .,,.\rangle_1$ and $f \in \mathcal{H}$. Then

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle g_{ij}, \Lambda_i f\rangle|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle \Lambda_i^* g_{ij}, f\rangle_1|^2 = \|f\|_1^2.$$

If $g \in \mathcal{H}_i$, $h \in \mathcal{H}_j$ and $i \neq j$, we have

$$\langle \Lambda_i^* g, \Lambda_j^* h\rangle_1 = \left\langle \sum_{k\in J_i} \langle g, g_{ik}\rangle \Lambda_i^* g_{ik}, \sum_{l\in J_j} \langle h, g_{jl}\rangle \Lambda_j^* g_{jl} \right\rangle_1 = 0.$$

If $g, h \in \mathcal{H}_i$, then

$$\langle \Lambda_i^* g, \Lambda_i^* h\rangle_1 = \left\langle \sum_{k\in J_i} \langle g, g_{ik}\rangle \Lambda_i^* g_{ik}, \sum_{l\in J_i} \langle h, g_{il}\rangle \Lambda_i^* g_{il} \right\rangle_1 = \sum_{k\in J_i} \langle g, g_{ik}\rangle \langle g_{ik}, h\rangle = \langle g, h\rangle.$$

Conversely, let $\langle .,,.\rangle_1$ be an equivalent inner product on $\mathcal{H}$ such that $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$ w.r. to $\langle .,,.\rangle_1$. Since $\langle .,,.\rangle$ and $\langle .,,.\rangle_1$ are equivalent, there are positive numbers m, M so that

$$m\|f\| \le \|f\|_1 \le M\|f\|, \quad f \in \mathcal{H}.$$

If $g_i \in \mathcal{H}_i$ and J is a finite subset of I then

$$\begin{aligned} \frac{1}{M^2}\sum_{i\in J} \|g_i\|^2 &= \frac{1}{M^2}\Big\|\sum_{i\in J} \Lambda_i^* g_i\Big\|_1^2 \le \Big\|\sum_{i\in J} \Lambda_i^* g_i\Big\|^2 \\ &\le \frac{1}{m^2}\Big\|\sum_{i\in J} \Lambda_i^* g_i\Big\|_1^2 = \frac{1}{m^2}\sum_{i\in J} \|g_i\|^2. \end{aligned}$$

Since for all $f \in \mathcal{H}$, we have $\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|_1^2$ so $\{f | \Lambda_i f = 0, i \in I\} = \{0\}$. Therefore, $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis with respect to the original inner product on $\mathcal{H}$.

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