Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **6** (2012), 140–155. doi:10.2298/AADM120319008P

EXPONENTIAL DICHOTOMIES FOR LINEAR DISCRETE-TIME SYSTEMS IN BANACH SPACES

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In this paper we investigate some dichotomy concepts for linear difference equations in Banach spaces. Characterizations of these concepts are given. Some illustrating examples clarify the relations between these concepts.

1. INTRODUCTION

In the mathematical literature of the last decades the asymptotic behavior of difference equations is one of the most important subjects due to large application area (see [**1**],[**4**],[**6**],[**7**],[**12**]).

The classical paper of Perron [**15**] served as a starting point for numerous works on the stability theory. A discrete variant of Perron's results was given by Ta Li in [**18**]. Several results about exponential dichotomy were obtained by C. V. Coffman and J. J. Schäffer [**5**], D. Henry [**8**], J. Kurzweil and G. Papaschinopoulos [**11**], M. Pinto [**17**], P. H. Ngoc and T. Naito [**13**], N. T. Huy and V. T. Ngoc [**10**], K. J. Palmer [**14**].

In their notable contribution [**4**], L. Barreira and C. Valls obtain results in the case of a notion of nonuniform exponential dichotomy, which is motivated by ergodic theory. A principal motivation for weakening the assumption of uniform exponential behavior is that from the point of view of ergodic theory, almost all linear variational equations in a finite dimensional space admit a nonuniform exponential dichotomy.

In this paper we study some dichotomy concepts for non-autonomous linear discrete-time systems in Banach spaces. The most common classes of exponential dichotomy used in the qualitative theory of difference equations are the uniform and nonuniform exponential dichotomy.

2010 Mathematics Subject Classification. 34D05, 34D09.
Keywords and Phrases. Uniform exponential dichotomy, nonuniform exponential dichotomy, exponential dichotomy, strong exponential dichotomy.

The present paper considers three concepts of nonuniform exponential dichotomy and the classical property of uniform exponential dichotomy for difference equations in Banach spaces. Two of these concepts are inspired from the papers of L. Barreira and C. Valls ([**2**], [**3**]). In the particular case of reversible systems we prove the equivalence of exponential dichotomy concept studied in this paper with the concept used by L. Barreira and C. Valls. Some illustrating examples clarify the relations between these concepts.

The obtained results extend the framework to the study of dichotomy of difference equations, hold without any requirement on the coefficients and are applicable to all systems of difference equations.

2. NOTATIONS AND DEFINITIONS

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X into itself. The norms of both these spaces will be denoted by $\|.\|$. Let $\mathbb{N}$ be the set of all positive integers and Δ be the set of all pairs (m, n) of positive integers satisfying the inequality $m \geq n$. We also denote by T the set of all triplets (m, n, p) of positive integers with (m, n) and $(n, p) \in \Delta$.

In this paper we consider linear discrete-time systems of the form

$$x_{n+1} = A(n)x_n \tag{$\mathfrak{A}$}$$

where $A : \mathbb{N} \longrightarrow \mathcal{B}(X)$ is a sequence in $\mathcal{B}(X)$. Then every solution $x = (x_n)$ of the system ($\mathfrak{A}$) is given by

$$x_m = \mathcal{A}(m, n)x_n, \quad \text{for all} \quad (m, n) \in \Delta,$$

where $A : \Delta \longrightarrow \mathcal{B}(X)$ is defined by

$$\mathcal{A}(m, n) = \begin{cases} A(m-1) \cdots A(n), & m \geq n+1 \\ I, & m = n. \end{cases} \tag{2.1}$$

where I is the identity operator on X.

For the particular case when $A(n) = A \in \mathcal{B}(X)$ we have that $\mathcal{A}(m, n) = A^{m-n}$ for all $(m, n) \in \Delta$.

It is obvious that

$$\mathcal{A}(m, n)\mathcal{A}(n, p) = \mathcal{A}(m, p), \quad \text{for all} \quad (m, n, p) \in T.$$

If for each $n \in \mathbb{N}$ the operator $A(n)$ is invertible, then the linear discrete-time system ($\mathfrak{A}$) is called *reversible.* If ($\mathfrak{A}$) is reversible, then there exists $\mathcal{A}(m, n)^{-1} = \mathcal{A}(n, m)$, for all $(m, n) \in \mathbb{N}^2$.

Definition 2.1. *An application $P : \mathbb{N} \to \mathcal{B}(X)$ is said to be a family of projections on X if*

$$P^2(n) = P(n),$$

for every $n \in \mathbb{N}$.

Remark 2.1. If $P:\mathbb{N}\to\mathcal{B}(X)$ is a family of projections, then $Q:\mathbb{N}\to\mathcal{B}(X)$, with $Q(n)=I-P(n)$ is also a family of projections on X, and it is called *the complementary projection* of P. It is obvious that

$$P(n)Q(n)=Q(n)P(n)=0,$$

for each $n\in\mathbb{N}$.

Definition 2.2. *A family of projections $P:\mathbb{N}\to\mathcal{B}(X)$ is said to be compatible with the system* $(\mathfrak{A})$, *if*

$$A(n)P(n)=P(n+1)A(n),$$

for every $n\in\mathbb{N}$.

Remark 2.2. For the particular case when $(\mathfrak{A})$ is autonomous, i.e. $A(n)=A\in\mathcal{B}(X)$ for all $n\in\mathbb{N}$, and $P(n)=P$ then P is compatible with system $(\mathfrak{A})$ if and only if $AP=PA$.

Remark 2.3. If $P:\mathbb{N}\to\mathcal{B}(X)$ is a family of projections compatible with system $(\mathfrak{A})$ then

$$\mathcal{A}(m,n)P(n)=P(m)\mathcal{A}(m,n),\ \text{ and }\ \mathcal{A}(m,n)Q(n)=Q(m)\mathcal{A}(m,n)$$

for all $(m,n)\in\Delta$.

Remark 2.4. If $P:\mathbb{N}\to\mathcal{B}(X)$ is a family of projections compatible with the reversible system $(\mathfrak{A})$ then $\mathcal{A}(m,n)$ is invertible for every $(m,n)\in\mathbb{N}^2$ and

$$\mathcal{A}(m,n)^{-1}P(m)=P(n)\mathcal{A}(m,n)^{-1}$$

and

$$\mathcal{A}(m,n)^{-1}Q(m)=Q(n)\mathcal{A}(m,n)^{-1}$$

for all $(m,n)\in\Delta$, where $Q(n)=I-P(n)$.

In what follows, we will denote by $\mathcal{A}_P:\Delta\to\mathcal{B}(X)$ and $\mathcal{A}_Q:\Delta\to\mathcal{B}(X)$ the mappings defined by

$$\mathcal{A}_P(m,n)=\begin{cases} A(m-1)\cdots A(n)P(n), & m>n\\ P(n), & m=n.\end{cases}$$

and

$$\mathcal{A}_Q(m,n)=\begin{cases} A(m-1)\cdots A(n)Q(n), & m>n\\ Q(n), & m=n.\end{cases}$$

for all $(m,n)\in\Delta$.

3. UNIFORM EXPONENTIAL DICHOTOMY

Let $P:\mathbb{N}\to\mathcal{B}(X)$ be a family of projections compatible with system $(\mathfrak{A})$, thus

Definition 3.1. *The linear discrete-time system* $(\mathfrak{A})$ *is said to be P-uniformly exponentially dichotomic* (*and denote P-u.e.d.*) *if there exist two constants* $N \geq 1$ *and* $\alpha > 0$ *such that*

$$e^{\alpha(m-n)} (\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|) \leq N (\|P(n)x\| + \|\mathcal{A}_Q(m,n)x\|) \tag{3.1}$$

for all $(m,n,x) \in \Delta \times X$.

Remark 3.1. The linear discrete-time system $(\mathfrak{A})$ is P-uniformly exponentially dichotomic if and only if there exist two constants $N \geq 1$ and $\alpha > 0$ such that

$$e^{\alpha(m-n)} (\|\mathcal{A}_P(m,p)x\| + \|\mathcal{A}_Q(n,p)x\|) \leq N (\|\mathcal{A}_P(n,p)x\| + \|\mathcal{A}_Q(m,p)x\|)$$

for all $(m,n,p,x) \in T \times X$.

The notion of uniform exponential dichotomy for linear discrete-time systems is defined in different ways according to different authors, both in finite and infinite-dimensional time. Our notion of uniform exponential dichotomy is closely related by the notion introduced by C. V. Coffman and J. J. Schäffer ([**5**]) and developing work done by P. Preda and M. Megan [**16**].

An example of P-uniform exponential dichotomic system is given in

Example 3.1. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$A(n)(x_1, x_2) = \left(\frac{x_1}{a_n}, a_n x_2\right)$$

for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$, where $a_n = e^{n+\frac{1}{2}}$. Then for $P : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$P(n)(x_1, x_2) = (x_1, 0)$$

for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$, we have that

$$\mathcal{A}_P(m,n)(x_1,x_2) = \begin{cases} \left(e^{\frac{n^2-m^2}{2}} x_1, 0\right) & m > n \\ (x_1, 0) & m = n, \end{cases}$$

$$\mathcal{A}_Q(m,n)(x_1,x_2) = \begin{cases} \left(0, e^{\frac{m^2-n^2}{2}} x_2\right) & m > n \\ (0, x_2) & m = n, \end{cases}$$

Hence, for $N = 1$ and $\alpha = \dfrac{1}{2}$ we have

$$e^{\frac{1}{2}(m-n)} (\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|) \leq \|P(n)x\| + \|\mathcal{A}_Q(m,n)x\|,$$

and thus the system $(\mathfrak{A})$ is P-u.e.d.

We remark that the system $(\mathfrak{A})$ is reversible with

$$\sup_{n\in\mathbb{N}} \|A(n)\| = \sup_{n\in\mathbb{N}} \|A(n)^{-1}\| = \infty.$$

The following example presents a reversible P-uniform exponential dichotomic system $(\mathfrak{A})$ with

$$\sup_{n\in\mathbb{N}}\|A(n)\| < \infty \ \text{ and } \ \sup_{n\in\mathbb{N}}\|A(n)^{-1}\| < \infty.$$

Example 3.2. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$A(n)(x_1, x_2) = \left(c\, e^{a_n} x_1, \frac{x_2}{c}\right)$$

where $c \in \left(0, \frac{1}{e^4}\right)$ and $a_n = (n+3)\sin\ln(n+2) - (n+2)\sin\ln(n+1)$, for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$.

If we denote by $f(x) = (x+1)\sin\ln x$, then by the mean value theorem it results that for every $n \in \mathbb{N}$ there exists $c_n \in (n+1, n+2)$ such that

$$a_n = f(n+2) - f(n+1) = \sin\ln c_n + \frac{1+c_n}{c_n}\cos\ln c_n$$

and hence

$$|a_n| \leq 3, \ \text{ for every } \ n \in \mathbb{N}.$$

This implies that

$$\sup_{n\in\mathbb{N}}\|A(n)\| \leq ce^3 + \frac{1}{c} \ \text{ and } \ \sup_{n\in\mathbb{N}}\|A(n)^{-1}\| \leq c + \frac{e^3}{c}.$$

Then $P : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$, defined by

$$P(n)(x_1, x_2) = (x_1, 0)$$

is a family of projections compatible with $(\mathfrak{A})$ with the complementary projection

$$Q(n)(x_1, x_2) = (0, x_2)$$

for every $n \in \mathbb{N}$.

A simple calculus shows that

$$\mathcal{A}_P(m,n)(x_1,x_2) = \begin{cases} \left(c^{m-n}e^{a_{mn}}x_1, 0\right) & m > n \\ (x_1, 0) & m = n, \end{cases}$$

and

$$\mathcal{A}_Q(m,n)(x_1,x_2) = \begin{cases} \left(0, x_2 c^{-(m-n)}\right) & m > n \\ (0, x_2) & m = n, \end{cases}$$

where $a_{mn} = (m+2)\sin\ln(m+1) - (n+2)\sin\ln(n+1)$.

Then

$$\begin{aligned} &e^{(m-n)}\left(\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|\right) = e^{\alpha(m-n)}\left(c^{m-n}e^{a_{mn}}|x_1| + |x_2|\right) \\ &\leq e^3\left(|x_1| + c^{-(m-n)}|x_2|\right) = e^3\left(\|P(n)x\| + \|\mathcal{A}_Q(m,n)x\|\right) \end{aligned}$$

for all $(m,n,x) \in \Delta \times \mathbb{R}^2$, where $\alpha = -\ln(ce^3) > -\ln\frac{1}{e} = 1 > 0$. Finally, by Definition 3.1 we obtain that system $(\mathfrak{A})$ is P-u.e.d.

4. NONUNIFORM EXPONENTIAL DICHOTOMY

A natural generalization of the concept of P-uniform exponential dichotomy is given by

Definition 4.1. *The linear discrete-time system* $(\mathfrak{A})$ *is said to be P-nonuniformly exponentially dichotomic* (*and denote P-n.e.d.*) *if there exists a constant* $\alpha > 0$ *and a nondecreasing sequence of real numbers* $N : \mathbb{N} \longrightarrow \mathbb{R}_+^*$ *such that*

$$(4.1)\quad e^{\alpha(m-n)}\left(\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|\right) \le N(n)\|P(n)x\| + N(m)\|\mathcal{A}_Q(m,n)x\|$$

for all $(m,n,x) \in \Delta \times X$.

Remark 4.1. The linear discrete-time system $(\mathfrak{A})$ is P-nonuniformly exponentially dichotomic if and only if there exists a constant $\alpha > 0$ and a nondecreasing sequence of real numbers $N : \mathbb{N} \longrightarrow \mathbb{R}_+^*$ such that

$$e^{\alpha(m-n)}\left(\|\mathcal{A}_P(m,p)x\| + \|\mathcal{A}_Q(n,p)x\|\right) \le N(n)\|\mathcal{A}_P(n,p)x\| + N(m)\|\mathcal{A}_Q(m,p)x\|$$

for all $(m,n,p,x) \in T \times X$.

Remark 4.2. A P-uniformly exponentially dichotomic system $(\mathfrak{A})$ is P-nonuniformly exponentially dichotomic. Now, we present an example which shows that the converse implication is not valid.

Example 4.1. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$A(n)(x_1,x_2) = \left(ba_n x_1, \frac{x_2}{b}\right)$$

where $b \in (0,1)$, $c > 0$ and

$$a_n = \begin{cases} \dfrac{1}{(n+2)^c} & \text{if } n = 2k \\ (n+1)^c & \text{if } n = 2k+1 \end{cases} \quad \text{for all} \quad (n,x_1,x_2) \in \mathbb{N} \times \mathbb{R}^2.$$

Then for $P : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$P(n)(x_1,x_2) = (x_1,0)$$

for all $(n,x_1,x_2) \in \mathbb{N} \times \mathbb{R}^2$, we have that

$$\mathcal{A}_P(m,n)(x_1,x_2) = \begin{cases} \left(b^{m-n} a_{mn} x_1, 0\right) & m > n \\ (x_1,0) & m = n, \end{cases}$$

and

$$\mathcal{A}_Q(m,n)(x_1,x_2) = \begin{cases} \left(0, \dfrac{x_2}{b^{m-n}}\right) & m > n \\ (0,x_2) & m = n, \end{cases}$$

where

$$a_{mn} = \begin{cases} 1 & \text{if } m = 2q \text{ and } n = 2p \\ (n+1)^c & \text{if } m = 2q \text{ and } n = 2p+1 \\ \left(\dfrac{n+1}{m+1}\right)^c & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ \left(\dfrac{1}{m+1}\right)^c & \text{if } m = 2q+1 \text{ and } n = 2p. \end{cases}$$

As we have

$$e^{\alpha(m-n)}\left(b^{m-n}a_{mn}|x_1|+|x_2|\right)\leq N(n)|x_1|+\frac{N(m)}{b^{m-n}}|x_2|,$$

for all $(m,n,x_1,x_2)\in\Delta\times\mathbb{R}^2$, where $N(n)=(n+1)^c$ and $\alpha=-\ln b$. By Remark 4.1 we obtain that system $(\mathfrak{A})$ is P-n.e.d.

If we suppose that system $(\mathfrak{A})$ is P-u.e.d. then there are two constants $N\geq 1$ and $\alpha>0$ such that

$$e^{\alpha(m-n)}\left(b^{m-n}a_{mn}|x_1|+|x_2|\right)\leq N\left(|x_1|+\frac{|x_2|}{b^{m-n}}\right),$$

for all $(m,n,x_1,x_2)\in\Delta\times\mathbb{R}^2$. In particular, for $m=2q+2$ and $n=2q+1$ we have that

$$e^{\alpha}\left(b(2q+2)^c|x_1|+|x_2|\right)\leq N\left(|x_1|+\frac{|x_2|}{b}\right),$$

for all $q\in\mathbb{N}$, which is a contradiction. Hence, system $(\mathfrak{A})$ is not P-u.e.d. □

Theorem 4.1. *The linear discrete-time system* $(\mathfrak{A})$ *is P-nonuniformly exponentially dichotomic if and only if there exists a constant* $d>0$ *and a sequence of real numbers* $S:\mathbb{N}\longrightarrow\mathbb{R}_+^*$ *such that*

$$\sum_{m=n}^{\infty}e^{d(m-n)}\|\mathcal{A}_P(m,p)x\|+\sum_{k=n}^{m}e^{d(m-k)}\|\mathcal{A}_Q(k,n)x\| \leq S(n)\|\mathcal{A}_P(n,p)x\|+S(m)\|\mathcal{A}_Q(m,n)x\| \tag{4.2}$$

for all $(m,n,p,x)\in T\times X$.

Proof. *Necessity.* For $d\in(0,\alpha)$ we have that

$$\sum_{m=n}^{\infty}e^{d(m-n)}\|\mathcal{A}_P(m,p)x\|+\sum_{k=n}^{m}e^{d(m-k)}\|\mathcal{A}_Q(k,n)x\| \leq \frac{e^{\alpha}N(n)}{e^{\alpha}-e^{d}}\|\mathcal{A}_P(n,p)x\|+\frac{e^{\alpha}N(m)}{e^{\alpha}-e^{d}}\|\mathcal{A}_Q(m,n)x\|.$$

Hence, for $S(n)=\dfrac{e^{\alpha}N(n)}{e^{\alpha}-e^{d}}$ we obtain (4.2), with $\alpha>0$ and sequence $N(n)$ given by Definition 4.1.

Sufficiency. Firstly, we have that

$$\sum_{m=n}^{\infty}e^{d(m-n)}\|\mathcal{A}_P(m,p)x\|\leq S(n)\|\mathcal{A}_P(n,p)x\|$$

and thus

$$e^{d(m-n)}\|\mathcal{A}_P(m,n)x\|\leq S(n)\|P(n)x\|, \tag{4.3}$$

for all $(m,n,x)\in\Delta\times X$. Similarly, it follows that

$$e^{d(m-n)}\|Q(n)x\|\leq S(m)\|\mathcal{A}_Q(m,n)x\|. \tag{4.4}$$

Finally, using inequalities (4.3) and (4.4) and Remark 4.1 we conclude that the system $(\mathfrak{A})$ is P-n.e.d. □

As a particular case, we obtain a characterization of P-uniform exponential dichotomy given by

Corollary 4.1. *The linear discrete-time system* $(\mathfrak{A})$ *is P- uniformly exponentially dichotomic if and only if there exist two constants* $D, d > 0$ *such that*

$$\sum_{j=m}^{\infty} e^{d(j-n)} \|\mathcal{A}_P(j,n)x\| + \sum_{k=n}^{m} e^{d(m-k)} \|\mathcal{A}_Q(k,n)x\| \leq D\left(\|\mathcal{A}_P(m,n)x\| + \|\mathcal{A}_Q(m,n)x\|\right),$$

for all $(m, n, x) \in \Delta \times X$.

Another characterization of the P-uniform exponential dichotomy property is given by

Corollary 4.2. *The linear discrete-time system* $(\mathfrak{A})$ *is P- uniformly exponentially dichotomic if and only if there exists* $D > 0$ *such that*

$$\sum_{j=m}^{\infty} \|\mathcal{A}_P(j,n)x\| + \sum_{k=n}^{m} \|\mathcal{A}_Q(k,n)x\| \leq D\left(\|\mathcal{A}_P(m,n)x\| + \|\mathcal{A}_Q(m,n)x\|\right),$$

for all $(m, n, x) \in \Delta \times X$.

Proof. *Necessity.* It is a simple verification as in the proof ot Theorem 4.1.

Sufficiency. It is immediate from Corollary 4.1.

Remark 4.3. An equivalent variant of the preceding Corollary was proved by P. Preda and M. Megan in [**16**].

5. EXPONENTIAL DICHOTOMY

A particular case of nonuniform exponential dichotomy is introduced by

Definition 5.1. *The linear discrete-time system* $(\mathfrak{A})$ *is said to be P-exponentially dichotomic (and denote P-e.d.) if there exist the constants* $N \geq 1$, $\alpha > 0$ *and* $\beta \geq 0$ *such that*

$$e^{\alpha(m-n)}\left(\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|\right) \leq N\left(e^{\beta n}\|P(n)x\| + e^{\beta m}\|\mathcal{A}_Q(m,n)x\|\right) \tag{5.1}$$

for all $(m, n, x) \in \Delta \times X$.

Remark 5.1. The linear discrete-time system $(\mathfrak{A})$ is P-exponentially dichotomic if and only if there exist the constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$e^{\alpha(m-n)}\left(\|\mathcal{A}_P(m,p)x\| + \|\mathcal{A}_Q(n,p)x\|\right) \leq N\left(e^{\beta n}\|\mathcal{A}_P(n,p)x\| + e^{\beta m}\|\mathcal{A}_Q(m,p)x\|\right)$$

for all $(m, n, p, x) \in T \times X$.

Remark 5.2. If the system $(\mathfrak{A})$ is P-uniformly exponentially dichotomic then it is P-exponentially dichotomic. Now, we present an example which shows that the converse is not valid. Moreover, in this example $A(n) = A$ is a bounded linear invertible operator. Thus even for autonomous systems, the concepts of P-uniform exponential dichotomy and P-exponential dichotomy are distinct.

Example 5.1. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$A(n)(x_1, x_2) = (cx_1, x_2)$$

for $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$, where $c \in \left(0, \frac{1}{e}\right]$. Then for $P, Q : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$P(n)(x_1, x_2) = (x_1, 0) \text{ respectively } Q(n)(x_1, x_2) = (0, x_2)$$

we have that

$$\mathcal{A}_P(m, n)(x_1, x_2) = \left(c^{m-n}x_1, 0\right) \text{ and } \mathcal{A}_Q(x_1, x_2) = (0, x_2).$$

Then

$$\begin{aligned} & e^{m-n}\left(\|\mathcal{A}_P(m, n)x\| + \|Q(n)x\|\right) = (ec)^{m-n}|x_1| + e^{m-n}|x_2| \\ & \leq |x_1| + e^m|x_2| \leq e^n|x_1| + e^m|x_2| = e^n\|P(n)x\| + e^m\|\mathcal{A}_Q(m, n)x\| \end{aligned}$$

for all $(m, n, x) \in \Delta \times \mathbb{R}^2$. Thus Definition 5.1 is satisfied for $N = \alpha = \beta = 1$, hence $(\mathfrak{A})$ is P-e.d.

If we suppose that $(\mathfrak{A})$ is P-u.e.d. then there are $N \geq 1$ and $\alpha > 0$ with

$$e^{\alpha(m-n)}\left(c^{m-n}|x_1| + |x_2|\right) \leq N\left(|x_1| + |x_2|\right)$$

for all $(m, n) \in \Delta$ and all $x = (x_1, x_2) \in \mathbb{R}^2$. This implies that

$$e^{\alpha(m-n)} \leq N \text{ for all } (m, n) \in \Delta,$$

which is a contradiction.

Definition 5.2. *A family of projections $P : \mathbb{N} \to \mathcal{B}(X)$ is called exponentially bounded if there exist $N \geq 1$ and $\beta \geq 0$ such that*

$$\|P(n)\| \leq Ne^{\beta n},$$

for every $n \in \mathbb{N}$.

A characterization of exponential dichotomy for reversible systems with respect to a family of exponentially bounded projections is given by

Theorem 5.1. *Let $(\mathfrak{A})$ be a reversible linear discrete-time system and let $P : \mathbb{N} \to \mathcal{B}(X)$ be a family of exponentially bounded projections compatible with $(\mathfrak{A})$.*

Then $(\mathfrak{A})$ is P-exponentially dichotomic if and only if there are $M \geq 1$, $\alpha > 0$ and $\gamma > 0$ such that

$$(5.2) \qquad e^{\alpha(m-n)}\left(\|\mathcal{A}(m, n)P(n)\| + \|\mathcal{A}(m, n)^{-1}Q(m)\|\right) \leq M\left(e^{\gamma n} + e^{\gamma m}\right)$$

for all $(m, n) \in \Delta$.

Proof. *Necessity.* If $(\mathfrak{A})$ is P-e.d. where P is exponentially bounded then there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$
\begin{aligned}
& e^{\alpha(m-n)} \left(\|\mathcal{A}(m,n)P(n)x\| + \|\mathcal{A}(m,n)^{-1}Q(m)x\| \right) \\
& = e^{\alpha(m-n)} \left(\|\mathcal{A}(m,n)P(n)x\| + \|Q(n)\mathcal{A}(m,n)^{-1}x\| \right) \\
& \leq N \left(e^{\beta n}\|P(n)x\| + e^{\beta m}\|\mathcal{A}(m,n)Q(n)\mathcal{A}(m,n)^{-1}x\| \right) \\
& \leq N^2 \left(e^{2\beta n} + e^{2\beta m} \right) \|x\|,
\end{aligned}
$$

for all $x \in X$, which implies (5.2).

Sufficiency. From (5.2) it follows that

$$\|P(n)\|,\ \|Q(n)\| \leq Me^{\gamma n}$$

for every $n \in \mathbb{N}$. Moreover,

$$
\begin{aligned}
& e^{\alpha(m-n)} \left(\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\| \right) \\
& = e^{\alpha(m-n)} \left(\|\mathcal{A}(m,n)P(n)x\| + \|\mathcal{A}(m,n)^{-1}Q(m)\mathcal{A}(m,n)Q(n)x\| \right) \\
& \leq M^2 \left(e^{\gamma n}\|P(n)x\| + e^{\gamma m}\|\mathcal{A}_Q(m,n)x\| \right),
\end{aligned}
$$

for all $(m,n,x) \in \Delta \times X$. Finally, we obtain that system $(\mathfrak{A})$ is P-e.d.

Remark 5.3. In the papers [**2**] and [**3**] L. Barreira and C. Valls define a concept of nonuniform exponential dichotomy for reversible linear discrete-time systems by the inequality (5.2). Thus, we consider that the concept introduced by Definition 5.1 is a generalization of exponential dichotomy studied by L. Barreira and C. Valls.

In the particular case $\beta = \gamma = 0$ we obtain a characterization of P-uniform exponential dichotomy given by

Corollary 5.1. *Let $(\mathfrak{A})$ be a reversible linear discrete-time system and let $P : \mathbb{N} \to \mathcal{B}(X)$ be a family of exponentially bounded projections compatible with $(\mathfrak{A})$.*

Then $(\mathfrak{A})$ is P-uniformly exponentially dichotomic if and only if there exist $M \geq 1$ and $\alpha > 0$ such that

$$e^{\alpha(m-n)} \left(\|\mathcal{A}(m,n)P(n)\| + \|\mathcal{A}(m,n)^{-1}Q(m)\| \right) \leq M$$

for all $(m,n) \in \Delta$.

A characterization of P-exponential dichotomy is given by

Theorem 5.2. *The linear discrete-time system $(\mathfrak{A})$ is P-exponentially dichotomic if and only if there exist some constants $D \geq 1$, $d > 0$ and $c \geq 0$ such that*

$$
\begin{aligned}
& \sum_{m=n}^{\infty} e^{d(m-n)}\|\mathcal{A}_P(m,p)x\| + \sum_{k=n}^{m} e^{d(m-k)}\|\mathcal{A}_Q(k,n)x\| \\
& \leq D \left(e^{cn}\|\mathcal{A}_P(n,p)x\| + e^{cm}\|\mathcal{A}_Q(m,n)x\| \right)
\end{aligned}
\tag{5.3}
$$

for all $(m,n,p,x) \in T \times X$.

Proof. *Necessity.* For $d \in (0, \alpha)$ we have that

$$\sum_{m=n}^{\infty} e^{d(m-n)} \|\mathcal{A}_P(m,p)x\| + \sum_{k=n}^{m} e^{d(m-k)} \|\mathcal{A}_Q(k,n)x\|$$
$$\leq N\|\mathcal{A}_P(n,p)x\| \sum_{m=n}^{\infty} e^{d(m-n)} e^{-\alpha(m-n)} e^{\beta n} +$$
$$+ N e^{(\beta+d-\alpha)m} \|\mathcal{A}_Q(m,n)x\| \sum_{k=n}^{m} e^{(\alpha-d)k}$$
$$\leq \frac{Ne^{\alpha}}{e^{\alpha}-e^{d}} e^{\beta n} \|\mathcal{A}_P(n,p)x\| + \frac{Ne^{\alpha}}{e^{\alpha}-e^{d}} e^{\beta m} \|\mathcal{A}_Q(m,n)x\|.$$

Hence, for $D = 1 + \dfrac{Ne^{\alpha}}{e^{\alpha}-e^{d}}$ and $c = \beta$ we obtain relation (5.3), with constants N, α and β offered by Definition 5.1.

Sufficiency. According to relation (5.3) we have that

$$\sum_{m=n}^{\infty} e^{d(m-n)} \|\mathcal{A}_P(m,n)x\| + \sum_{k=n}^{m} e^{d(m-k)} \|\mathcal{A}_Q(k,n)x\|$$
$$\geq e^{d(m-n)} \|\mathcal{A}_P(m,n)x\| + e^{d(m-n)} \|Q(n)x\|.$$

Hence,

$$e^{d(m-n)} \left(\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|\right) \leq D\left(e^{cn}\|P(n)x\| + e^{cm}\|\mathcal{A}_Q(m,n)x\|\right)$$

for all $(m,n,x) \in \Delta \times X$. Now, using Remark 5.1 we obtain that system $(\mathfrak{A})$ is P-e.d., which completes the proof.

Remark 5.4. If the system $(\mathfrak{A})$ is P-exponentially dichotomic then it is P-nonuniformly exponentially dichotomic. The following example shows that the converse is not true.

Example 5.2. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$A(n)(x_1,x_2) = \left(ca_n x_1, \frac{x_2}{c}\right)$$

for all $(n,x_1,x_2) \in \mathbb{N} \times \mathbb{R}^2$, where c is a positive constant and

$$a_n = \begin{cases} e^{n(1+2^n)} & \text{if } n = 2k \\ e^{-(n+1)(1+2^{n+1})} & \text{if } n = 2k+1 \end{cases}$$

Then for $P : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$P(n)(x_1,x_2) = (x_1,0)$$

for all $(n,x_1,x_2) \in \mathbb{N} \times \mathbb{R}^2$, we have that

$$\mathcal{A}_P(m,n)(x_1,x_2) = \begin{cases} \left(c^{m-n} a_{mn} x_1, 0\right) & m > n \\ (x_1,0) & m = n \end{cases},$$

and

$$\mathcal{A}_Q(m,n)(x_1,x_2) = \begin{cases} \left(0, \dfrac{x_2}{c^{m-n}}\right) & m > n \\ (0, x_2) & m = n \end{cases},$$

where

$$a_{mn} = \begin{cases} 1 & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ e^{n(1+2^n)} & \text{if } m = 2q+1 \text{ and } n = 2p \\ e^{-m(1+2^m)} & \text{if } m = 2q \text{ and } n = 2p+1 \\ e^{n(1+2^n)}e^{-m(1+2^m)} & \text{if } m = 2q \text{ and } n = 2p \end{cases}$$

Suppose that the system $(\mathfrak{A})$ is P-e.d. There exist some constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$e^{\alpha(m-n)}\left(c^{m-n}a_{mn}|x_1| + |x_2|\right) \leq N\left(e^{\beta n}|x_1| + \frac{e^{\beta m}}{c^{m-n}}|x_2|\right)$$

for all $(m,n,x_1,x_2) \in \Delta \times \mathbb{R}^2$. Further, if we consider $x_1 \neq 0$ and $x_2 = 0$ it follows that

$$(e^{\alpha}c)^{m-n}\, a_{mn} \leq Ne^{\beta n}. \tag{5.4}$$

There are three different cases at this point.

Case 1. If $e^{\alpha}c = 1$ then $a_{mn} \leq Ne^{\beta n}$ which for $n = 2p$ and $m = n+1$ leads to

$$1 + 2^n \leq \frac{\ln N}{n} + \beta$$

which is false.

Case 2. If $e^{\alpha}c > 1$ then for $m = 2q+1$ and $n = 1$ relation (5.4) became

$$(e^{\alpha c})^{2q} \leq Ne^{\beta}$$

which is false.

Case 3. If $0 < e^{\alpha}c < 1$ then for $n = 2p$ and $m = n+1$ relation (5.4) became

$$1 + 2^n \leq \frac{\ln(N/e^{\alpha}c)}{n} + \beta$$

which is also false. Hence, system $(\mathfrak{A})$ is not P-e.d.

Finally, we remark that for $c = \dfrac{1}{e}$, $\alpha = 1$ and $N(n) = e^{n(1+2^n)}$ the inequality (4.2) is satisfied and thus by Remark 4.1 the system $(\mathfrak{A})$ is P-n.e.d. □

6. STRONG EXPONENTIAL DICHOTOMY

A particular concept of P-exponential dichotomy is introduced by

Definition 6.1. *The linear discrete-time system* $(\mathfrak{A})$ *is said to be P-strongly exponentially dichotomic* (*and denote P-s.e.d.*) *if there exist the constants* $N \geq 1$, $\alpha > 0$ *and* $\beta \in [0,\alpha)$ *such that*

$$e^{\alpha(m-n)}\left(\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|\right) \leq N\left(e^{\beta n}\|P(n)x\| + e^{\beta m}\|\mathcal{A}_Q(m,n)x\|\right) \tag{6.1}$$

for all $(m,n,x) \in \Delta \times X$.

Remark 6.1. The linear discrete-time system $(\mathfrak{A})$ is P-strongly exponentially dichotomic if and only if there exist the constants $N \geq 1$, $\alpha > 0$ and $\beta \in [0, \alpha)$ such that

$$e^{\alpha(m-n)} \left(\|\mathcal{A}_P(m,p)x\| + \|\mathcal{A}_Q(n,p)x\| \right) \leq \ \leq N \left(e^{\beta n} \|\mathcal{A}_P(n,p)x\| + e^{\beta m} \|\mathcal{A}_Q(m,p)x\| \right)$$

for all $(m, n, p, x) \in T \times X$.

A characterization of the strong exponential dichotomy is given by

Proposition 6.1. *The linear discrete-time system* $(\mathfrak{A})$ *is P-strongly exponentially dichotomic if and only if there exist the constants* $D \geq 1$, $d > 0$ *and* $0 \leq c < d$ *such that relation* (5.3) *it is true for all* $(m, n, p, x) \in T \times X$.

Proof. It results from Definition 6.1 and the proof of Theorem 5.2.

Remark 6.2. A P-uniformly exponentially dichotomic system $(\mathfrak{A})$ is P-strongly exponentially dichotomic. The reciprocal statements are not true, as shown in what follows.

Example 6.1. Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$A(n)(x_1, x_2) = (c_1 a_n x_1, c_2 a_n x_2)$$

for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$, where c_1 and c_2 are two positive constants and

$$a_n = \begin{cases} e^{-n} & \text{if } n = 2k \\ e^{n+1} & \text{if } n = 2k+1. \end{cases}$$

Then for $P : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$P(n)(x_1, x_2) = (x_1, 0)$$

for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$, we have that

$$\mathcal{A}_P(m,n)(x_1,x_2) = \begin{cases} \left(c_1^{m-n} a_{mn} x_1, 0\right) & m > n \\ (x_1, 0) & m = n \end{cases},$$

$$\mathcal{A}_Q(m,n)(x_1,x_2) = \begin{cases} \left(0, c_2^{m-n} a_{mn} x_2\right) & m > n \\ (0, x_2) & m = n \end{cases},$$

where

$$a_{mn} = \begin{cases} 1 & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ e^{-n} & \text{if } m = 2q+1 \text{ and } n = 2p \\ e^{m} & \text{if } m = 2q \text{ and } n = 2p+1 \\ e^{m-n} & \text{if } m = 2q \text{ and } n = 2p \end{cases}.$$

If we suppose that system $(\mathfrak{A})$ is P-u.e.d. then there exist two constants $\alpha > 0$ and $N \geq 1$ such that

$$e^{\alpha(m-n)} \left(c_1^{m-n} a_{mn} |x_1| + |x_2| \right) \leq N \left(|x_1| + c_2^{m-n} a_{mn} |x_2| \right)$$

for all $(m, n, x_1, x_2) \in \Delta \times \mathbb{R}^2$. But for $m = 2k+2$, $n = 2k+1$, $x_1 \neq 0$ and $x_2 = 0$ we have that

$$e^{\alpha} c_1 e^{2k+2} \leq N$$

which is a contradiction.

Further, for $c_1 = \frac{1}{e^4}$, $c_2 = e^2$, $\alpha = 2$, $\beta = 1$ and $N = e$ we have that

$$e^{\alpha(m-n)}\left(\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|\right) = e^{\alpha(m-n)}\left(c_1^{m-n}a_{mn}|x_1| + |x_2|\right)$$

$$= \begin{cases} e^{\alpha(m-n)}\left(c_1^{m-n}|x_1| + |x_2|\right) & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ e^{\alpha(m-n)}\left(c_1^{m-n}e^{-n}|x_1| + |x_2|\right) & \text{if } m = 2q+1 \text{ and } n = 2p \\ e^{\alpha(m-n)}\left(c_1^{m-n}e^{m}|x_1| + |x_2|\right) & \text{if } m = 2q \text{ and } n = 2p+1 \\ e^{\alpha(m-n)}\left(c_1^{m-n}e^{m-n}|x_1| + |x_2|\right) & \text{if } m = 2q \text{ and } n = 2p \end{cases}$$

$$\leq \begin{cases} e^{-(m-n)}|x_1| + c_2^{m-n}|x_2| & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ e^{-(m-n)}|x_1| + c_2^{m-n}|x_2| & \text{if } m = 2q+1 \text{ and } n = 2p \\ e^{n}|x_1| + c_2^{m-n}|x_2| & \text{if } m = 2q \text{ and } n = 2p+1 \\ |x_1| + c_2^{m-n}|x_2| & \text{if } m = 2q \text{ and } n = 2p \end{cases}$$

$$\leq \begin{cases} N\left(e^{\beta n}|x_1| + e^{\beta m}c_2^{m-n}|x_2|\right) & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ N\left(e^{\beta n}|x_1| + e^{\beta m}c_2^{m-n}e^{-n}|x_2|\right) & \text{if } m = 2q+1 \text{ and } n = 2p \\ N\left(e^{\beta n}|x_1| + e^{\beta m}c_2^{m-n}e^{m}|x_2|\right) & \text{if } m = 2q \text{ and } n = 2p+1 \\ N\left(e^{\beta n}|x_1| + e^{\beta m}c_2^{m-n}e^{m-n}|x_2|\right) & \text{if } m = 2q \text{ and } n = 2p \end{cases}$$

$$= N(e^{\beta n}\|P(n)x\| + e^{\beta m}\|\mathcal{A}_Q(m,n)x\|), \tag{6.2}$$

for all $(m,n,x) \in \Delta \times X$.

Hence, the system $(\mathfrak{A})$ is P-s.e.d., which completes the proof □

Remark 6.3. It is obvious that a P-strongly exponentially dichotomic system $(\mathfrak{A})$ is P-exponentially dichotomic. The following example emphasizes the difference between these concepts and presents that inverse implication does not hold.

Example 6.2. Let $(\mathfrak{A})$ be the linear discrete-time system and $P : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ the projections family considered in Example 6.1 with $c_1 = e^{-\frac{3}{2}}$ and $c_2 = e^{\frac{1}{2}}$.

If we suppose that system $(\mathfrak{A})$ is P-s.e.d. then there exist the constants $N \geq 1$, $\alpha > 0$, $\beta \geq 0$ with $0 \leq \beta < \alpha$ such that

$$e^{\alpha(m-n)}\left(\|\mathcal{A}_P(m,n)x\| + \|Q(n)x\|\right) = e^{\alpha(m-n)}\left(e^{-\frac{3(m-n)}{2}}a_{mn}|x_1| + |x_2|\right)$$

$$= \begin{cases} e^{(\alpha-3/2)(m-n)}|x_1| + e^{\alpha(m-n)}|x_2| & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ e^{(\alpha-3/2)(m-n)}e^{-n}|x_1| + e^{\alpha(m-n)}|x_2| & \text{if } m = 2q+1 \text{ and } n = 2p \\ e^{(\alpha-3/2)(m-n)}e^{m}|x_1| + e^{\alpha(m-n)}|x_2| & \text{if } m = 2q \text{ and } n = 2p+1 \\ e^{(\alpha-3/2)(m-n)}e^{m-n}|x_1| + e^{\alpha(m-n)}|x_2| & \text{if } m = 2q \text{ and } n = 2p \end{cases}$$

$$\leq \begin{cases} N\left(e^{\beta n}|x_1| + e^{\beta m}e^{\frac{m-n}{2}}|x_2|\right) & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ N\left(e^{\beta n}|x_1| + e^{\beta m}e^{\frac{m-n}{2}}e^{-n}|x_2|\right) & \text{if } m = 2q+1 \text{ and } n = 2p \\ N\left(e^{\beta n}|x_1| + e^{\beta m}e^{\frac{m-n}{2}}e^{m}|x_2|\right) & \text{if } m = 2q \text{ and } n = 2p+1 \\ N\left(e^{\beta n}|x_1| + e^{\beta m}e^{\frac{m-n}{2}}e^{m-n}|x_2|\right) & \text{if } m = 2q \text{ and } n = 2p \end{cases}$$

$$= N(e^{\beta n}\|P(n)x\| + e^{\beta m}\|\mathcal{A}_Q(m,n)x\|), \tag{6.3}$$

for all $(m,n,x) \in \Delta \times X$.

In particular, for $x_1 \neq 0$, $x_2 = 0$, $m = 2q+2$ and $n = 2p+1$ it follows that

$$e^{(\alpha-1/2)(m-n)}e^n \leq Ne^{\beta n}$$

which is true for $\alpha - \dfrac{1}{2} \leq 0$, $N \geq e$ and $\beta \geq 1$. It results that $0 < \alpha \leq \frac{1}{2} < 1 \leq \beta$ which is a contradiction with the hypotheses that $0 \leq \beta < \alpha$. Hence, $(\mathfrak{A})$ is not P-s.e.d.

Finally, we observe that for $\alpha = \frac{1}{2}$, $\beta = 1$ and $N = e$ relation (6.3) it is verified and thus the system $(\mathfrak{A})$ is P-e.d. □

Acknowledgements. The authors would like to express their sincere thanks to the reviewer for the valuable suggestions and comments which have improved the quality of the paper.

This work is financially supported from the Exploratory Research Grant PN II ID 1080 No. 508/2009 of the Romanian Ministry of Education, Research and Innovation.

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(Received September 5, 2011)
(Revised March 1, 2012)

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