

MINIMAL DOUBLY RESOLVING SETS AND THE STRONG METRIC DIMENSION OF HAMMING GRAPHS

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We consider the problem of determining the cardinality $\psi(H_{2,k})$ of minimal doubly resolving sets of Hamming graphs $H_{2,k}$. We prove that for $k \geq 6$ every minimal resolving set of $H_{2,k}$ is also a doubly resolving set, and, consequently, $\psi(H_{2,k})$ is equal to the metric dimension of $H_{2,k}$, which is known from the literature. Moreover, we find an explicit expression for the strong metric dimension of all Hamming graphs $H_{n,k}$.

1. INTRODUCTION

The metric dimension problem, introduced independently by SLATER [22] and HARARY [8], has been widely investigated [1, 3, 5–7, 9–13]. It arises in many diverse areas including network discovery and verification [2], geographical routing protocols [18], the robot navigation, connected joints in graphs, chemistry, etc.

Given a simple connected undirected graph $G = (V_G, E_G)$, where $V_G = \{1, 2, \dots, n\}$, $|E_G| = m$, $d(u, v)$ denotes the distance between vertices u and v , i.e. the length of a shortest $u - v$ path. A vertex x of graph G is said to resolve two

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vertices u and v of G if $d(u, x) \neq d(v, x)$. A subset $B = \{x_1, x_2, \dots, x_k\}$ of V_G is a *resolving set* of G if every two distinct vertices of G are resolved by some vertex of B . Given a vertex t , the k -tuple $r(t, B) = (d(t, x_1), d(t, x_2), \dots, d(t, x_k))$ is called the *vector of metric coordinates* of t with respect to B . A *metric basis* of G is a resolving set of the minimum cardinality. The *metric dimension* of G , denoted by $\beta(G)$, is the cardinality of its metric basis.

CACERES et al. [3] define the notion of a doubly resolving set as follows. Vertices x, y of graph G ($n \geq 2$) are said to doubly resolve vertices u, v of G if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A subset $D = \{x_1, x_2, \dots, x_\ell\}$ of V_G is a *doubly resolving set* of G if every two distinct vertices of G are doubly resolved by some two vertices of D . The minimal doubly resolving set problem consists of finding a doubly resolving set of G with the minimum cardinality, denoted by $\psi(G)$. Note that if x, y doubly resolve u, v then $d(u, x) - d(v, x) \neq 0$ or $d(u, y) - d(v, y) \neq 0$, and hence x or y resolves u, v . Therefore, a doubly resolving set is also a resolving set and $\beta(G) \leq \psi(G)$.

The strong metric dimension problem was introduced by A. SEBO and E. TANNIER [21] and further investigated by O. R. OELLERMANN and J. PETERS-FRANSEN [20]. Recently, the strong metric dimension of distance hereditary graphs has been studied by T. MAY and O. R. OELLERMANN [19]. A vertex w strongly resolves two vertices u and v if u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path. A subset S of V_G is a *strong resolving set* of G if every two distinct vertices of G are strongly resolved by some vertex of S . A *strong metric basis* of G is a strong resolving set of the minimum cardinality. Now, the *strong metric dimension* of G , denoted by $\text{sdim}(G)$ is defined as the cardinality of its strong metric basis. It is easy to see that if a vertex w strongly resolves vertices u and v then w also resolves these vertices. Hence every strong resolving set is a resolving set and $\beta(G) \leq \text{sdim}(G)$.

All three previously defined problems are NP-hard in general case. The proofs of NP-hardness are given for the metric dimension problem in [13], for the minimal doubly resolving set problem in [16] and for the strong metric dimension problem in [20]. The first metaheuristic approaches to all three problems have been proposed in [14–16].

If S is a strong resolving set of G , then set $\{r(v, S) | v \in V\}$ uniquely determines graph G in the following sense. If G' is a graph with $V_{G'} = V_G$ such that S strongly resolves G' , and if for all vertices v we have $r_G(v, S) = r_{G'}(v, S)$, then $G = G'$. If S is a resolving set or a doubly resolving set, then set S need not uniquely determine G .

The next example illustrates previously defined invariants $\beta(G)$, $\psi(G)$ and $\text{sdim}(G)$.

EXAMPLE 1. Consider graph G_1 of Figure 1. It is easy to check that $\beta(G_1)$, $\psi(G_1)$ and $\text{sdim}(G_1)$ are all different. Namely, $\{A, B\}$ is a minimal resolving set, $\{A, B, E, F\}$ is a minimal doubly resolving set, while $\{A, B, E\}$ is a minimal strong resolving set of G_1 . Therefore, $\beta(G_1) = 2$, $\psi(G_1) = 4$ and $\text{sdim}(G_1) = 3$.

For some simple classes of graphs it is possible to determine $\beta(G)$, $\psi(G)$ and $\text{sdim}(G)$ explicitly: the complete graph K_n with n vertices has $\beta(K_n) = \psi(K_n) = \text{sdim}(K_n) = n - 1$, the cycle C_n with n vertices has $\beta(C_n) = 2, \psi(C_n) = 3$, while $\text{sdim}(C_n) = \lceil n/2 \rceil$.

The Cartesian product $G \square H$ of graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, is the graph with the vertex set $V_G \times V_H = \{(a, v) \mid a \in V_G, v \in V_H\}$, where (a, v) is adjacent to (b, w) whenever $a = b$ and $\{v, w\} \in E_H$, or $v = w$ and $\{a, b\} \in E_G$.

We consider here a special class of Cartesian products of graphs, the so called Hamming graphs. The Hamming graph $H_{n,k}$ is defined as:

$$(1) \quad H_{n,k} = \underbrace{K_k \square K_k \square \dots \square K_k}_n$$

where K_k denotes the complete graph with k vertices. The vertices of Hamming graphs can be considered also as n -dimensional vectors, with coordinates which take values from set $\{0, 1, \dots, k - 1\}$. Two vertices are adjacent if they differ in exactly one coordinate. Specially, for $k = 2$, graph $H_{n,2}$ can be interpreted as the hypercube Q_n of order n , whose vertices are n -dimensional binary vectors which are adjacent if they have only one different coordinate.

It is obvious that $H_{n,k}$ has k^n vertices and every vertex has the n -dimensional neighborhood with $k - 1$ neighbors with respect to each coordinate, so the overall number of edges is $k^n \cdot n \cdot (k - 1)/2$.

EXAMPLE 2. Hamming graph $H_{4,3}$ has $3^4 = 81$ vertices, which are all ternary vectors of length 4. For example, vertex $(0,1,1,0)$ has adjacent vertices $(0,1,1,1)$, $(0,1,1,2)$, $(0,1,0,0)$, $(0,1,2,0)$, $(0,0,1,0)$, $(0,2,1,0)$, $(1,1,1,0)$, $(2,1,1,0)$.

CHVÁTAL in [4] has obtained an upper bound on the number of guesses for the game *Mastermind*, which also gives an upper bound for metric dimension of Hamming graphs. This is an asymptotic result which can be formulated as follows:

For each $\epsilon > 0$, there exists an integer $n(\epsilon)$ such that if $n > n(\epsilon)$ and $k < n^{1-\epsilon}$, then

$$\beta(H_{n,k}) \leq (2 + \epsilon) \cdot n \cdot \frac{1 + 2 \cdot \log_2 k}{\log_2 n - \log_2 k}$$

It has been shown in [3] that the metric dimension of Hamming graphs $H_{2,k}$ for all $k \geq 2$, is equal to

$$(2) \quad \beta(H_{2,k}) = \left\lfloor \frac{4k - 2}{3} \right\rfloor$$

In this paper we study the minimal cardinality $\psi(H_{2,k})$ of doubly resolving sets of Hamming graphs $H_{2,k}$ and prove that for $k \geq 5$ it is equal to $\beta(H_{2,k})$.

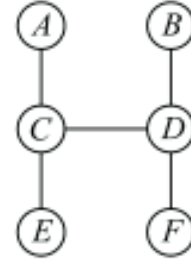


Figure 1. Graph G_1

Moreover, we show that the strong metric dimension for all Hamming graphs $H_{n,k}$ is $\text{sdim}(H_{n,k}) = (k-1) \cdot k^{n-1}$.

The paper is organized as follows. Section 2 is devoted to minimal doubly resolving sets of Hamming graphs $H_{2,k}$, while in Section 3 we find the exact value of $\text{sdim}(H_{n,k})$. Finally, Section 4 summarizes conclusions of the paper.

2. MINIMAL DOUBLY RESOLVING SETS OF $H_{2,k}$

In order to find the cardinality of minimal doubly resolving sets in Hamming graphs $H_{2,k}$ we will use here the following definitions and results from CACERES et al. [3], which are related to the Cartesian product $G \square H$ of arbitrary graphs G and H .

Let S be a subset of $V_{G \square H}$. The *projection of S onto G* is the set of all vertices $u \in V_G$ such that there exists a vertex $(u, v) \in S$. Similarly, *the projection of S onto H* contains all vertices $v \in V_H$ for which there exists a vertex $(u, v) \in S$. A *column* of $G \square H$ is the set of vertices $\{(u, v) | v \in V_H\}$ for a fixed vertex $u \in V_G$ and a *row* of $G \square H$ is the set $\{(u, v) | u \in V_G\}$ for a fixed vertex $v \in V_H$.

Lemma 1. ([3]) *For arbitrary graphs G and H and for every resolving set S of $G \square H$, the projection of S onto G is a resolving set in G and the projection of S onto H is a resolving set in H .*

Let us consider the Hamming graph $H_{2,k} = K_k \square K_k$. Its vertices can be represented as all two-dimensional vectors with coordinates from set $\{0, 1, \dots, k-1\}$, where two vertices are adjacent if they differ in exactly one coordinate. For a subset S of $V_{H_{2,k}}$, the projection of S onto the first K_k is the set of the first coordinates of all vertices from S , while the projection of S onto the second K_k is the set of the second coordinates of these vertices. A row of $H_{2,k}$ contains k vertices with the same second coordinate, and a column of $H_{2,k}$ consists of k vertices with the same first coordinate. Therefore, each row or column of $H_{2,k}$ represents a clique. Moreover, two vertices of $H_{2,k}$ are adjacent if and only if they belong to the same row or the same column. Otherwise, they are at distance two. Vertices of $H_{2,k}$ can be represented by the points of the $k \times k$ grid, where rows and columns of the grid correspond to rows and columns of $H_{2,k}$.

Now we will prove the following lemma.

Lemma 2. *For any Hamming graph $H_{2,k}$, where $k \geq 6$, every minimal resolving set is also a doubly resolving set, and consequently $\psi(H_{2,k}) \leq \beta(H_{2,k})$.*

Proof. Let B be a minimal resolving set of $H_{2,k}$, and let us suppose that it is not a doubly resolving set. Then, there exist two distinct vertices $u, v \in V_{H_{2,k}}$, $u \neq v$ and some integer $q \geq 0$ such that

$$(3) \quad r(u, B) - r(v, B) = (q, q, \dots, q)$$

Since $\text{Diam}(H_{2,k}) = 2$, the vectors $r(u, B)$ and $r(v, B)$ can contain only values 0, 1 or 2, and therefore, $q \in \{0, 1, 2\}$.

Value q cannot be equal to 0, because in that case there would not exist a vertex from B resolving u and v , which is in contradiction with the fact that B is a resolving set.

For $q = 2$ vectors $r(u, B)$ and $r(v, B)$ could take only values $r(u, B) = (2, 2, \dots, 2)$ and $r(v, B) = (0, 0, \dots, 0)$. But, the vector of metric coordinates of a vertex can contain at most one value 0, which appears only when this vertex belongs to B . As $r(v, B)$ contains more than one 0, the case $q = 2$ is also impossible.

Let us suppose that $q = 1$. Without loss of generality, we can assume that $r(v, B)$ has coordinates in non-decreasing order. Taking into account that $r(v, B)$ has at most one zero coordinate and cannot have any coordinate greater than 1, there exist only two possibilities:

Case 1: $r(u, B) = (2, 2, 2, \dots, 2)$ and $r(v, B) = (1, 1, 1, \dots, 1)$. As all coordinates of $r(v, B)$ are equal to 1, then $v \notin B$ and each vertex from B belongs to the same row $r(v)$ or the same column $c(v)$ as vertex v . Let us consider projection $P_1(B)$ onto the first set K_k from the Cartesian product $H_{2,k} = K_k \square K_k$, which contains the first coordinates of all vertices from B . According to Lemma 1, $P_1(B)$ is a resolving set of K_k and, therefore, $|P_1(B)| \geq \beta(K_k) = k - 1$. As the first coordinates of vertices from $B \cap r(v)$ are all different, while the first coordinates of vertices from $B \cap c(v)$ are all equal to the first coordinate of vertex v , which does not belong to B , then $|P_1(B)| = |B \cap r(v)| + 1$. Therefore, there should be at least $k - 2$ vertices from B in row $r(v)$. Considering projection $P_2(B)$ onto the second set K_k from $H_{2,k} = K_k \square K_k$, we can conclude in a similar way that $|P_2(B)| = |B \cap c(v)| + 1 \geq k - 1$, and there should be at least $k - 2$ vertices from B in column $c(v)$. As $v \notin B$, it follows that $|B| \geq 2k - 4$. Since B is a minimal resolving set, then, according to (2), $|B| = \left\lfloor \frac{4k-2}{3} \right\rfloor$.

But, for $k \geq 6$ it holds that $\left\lfloor \frac{4k-2}{3} \right\rfloor < 2k - 4$, which is a contradiction.

Case 2: $r(u, B) = (1, 2, 2, \dots, 2)$ and $r(v, B) = (0, 1, 1, \dots, 1)$. As $r(v, B)$ contains value 0 and all other its coordinates are equal to 1, then $v \in B$ and each other vertex from B belongs to row $r(v)$ or column $c(v)$. Using similar arguments as in Case 1, we can conclude that $|B| \geq 2n - 3$. But, for $k \geq 4$ it holds that $\left\lfloor \frac{4k-2}{3} \right\rfloor < 2k - 3$, which is a contradiction. Furthermore, it can be shown that in this case B cannot be even a resolving set. Actually, from the additional fact that $r(u, B) = (1, 2, 2, \dots, 2)$ it follows that $u \in r(v)$ or $u \in c(v)$, and vertex u does not belong to the row and the column of any vertex from $B \setminus \{v\}$. Let us suppose, without loss of generality, that $u \in r(v)$. Then, consequently, all vertices of B belong to $c(v)$. But, for such a B and $k \geq 3$, each two vertices from $V_{H_{2,k}} \setminus c(v)$ which are at the same row, are not resolved by any vertex from B . Therefore, B is not a resolving set.

Since for all possible values of q we have obtained contradictions, there are no two vertices $u, v \in V_{H_{2,k}}$ which satisfy condition (3), and, consequently, B is a doubly resolving set. \square

Now the minimal cardinality $\psi(H_{2,k})$ of doubly resolving sets in $H_{2,k}$ can be obtained by the following theorem.

$$\textbf{Theorem 3. } \psi(H_{2,k}) = \begin{cases} 3, & k = 2, 3 \\ 5, & k = 4 \\ \lfloor \frac{4k-2}{3} \rfloor, & k \geq 5. \end{cases}$$

Proof. As $\psi(H_{2,k}) \geq \beta(H_{2,k})$ and $\beta(H_{2,k}) = \lfloor \frac{4k-2}{3} \rfloor$, then it follows directly from Lemma 2 that $\psi(H_{2,k}) = \lfloor \frac{4k-2}{3} \rfloor$ for $k \geq 6$. Values $\psi(H_{2,2}) = 3$, $\psi(H_{2,3}) = 3$, $\psi(H_{2,4}) = 5$ and $\psi(H_{2,5}) = 6$ are obtained by total enumeration. \square

Let us mention that condition $k \geq 6$ in the statement of Lemma 2 cannot be omitted. Namely, as $\beta(H_{2,2}) < \psi(H_{2,2})$ and $\beta(H_{2,4}) < \psi(H_{2,4})$, then, obviously, this statement does not hold for $k = 2$ and $k = 4$. It also does not hold for $k = 5$, which is shown in the following example.

EXAMPLE 3. Let us consider graph $H_{2,5}$ and subset B of its vertices, indicated by bolded points in the grid representation of $H_{2,5}$ on Figure 2. It is easy to see that B is a resolving set. As $\lfloor \frac{4k-2}{3} \rfloor = 6$ for $k = 5$, B represents a metric basis of $H_{2,5}$. But, for vertices u and v , indicated in Figure 2, $r(u, B) = (2, 2, 2, 2, 2, 2)$ and $r(v, B) = (1, 1, 1, 1, 1, 1)$ and, consequently, B is not a doubly resolving set.

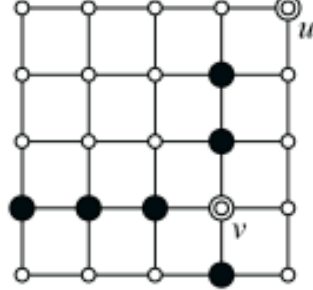


Figure 2. Graph $H_{2,5}$

3. STRONG METRIC DIMENSION OF $H_{n,k}$

In this section we obtain the exact value of the strong metric dimension for any Hamming graph $H_{n,k}$.

Theorem 4. For any Hamming graph $H_{n,k}$, $\text{sdim}(H_{n,k}) = (k-1) \cdot k^{n-1}$.

In order to prove Theorem 4 we use the following lemma from [17], which is related to a strong resolving set of an arbitrary graph G .

Lemma 5. ([17]) If S is a strong resolving set of graph G , then, for every two vertices $u, v \in V_G$ such that $d(u, v) = \text{Diam}(G)$, $u \in S$ or $v \in S$.

Next, we state and prove several lemmas which support the proof of Theorem 4 as follows.

Lemma 6. For any set $S \subseteq V_{H_{n,k}}$, where $|S| > k^{n-1}$, there exist two vertices u and v from S such that $d(u, v) = \text{Diam}(H_{n,k})$.

Proof. Let us define

$$S(x_2, x_3, \dots, x_n) = \{(i, (i + x_2) \bmod k, \dots, (i + x_n) \bmod k) \mid i = 0, \dots, k - 1\}.$$

Since each two distinct vertices $u, v \in S(x_2, x_3, \dots, x_n)$ have translated coordinates, different at all positions, then $d(u, v) = n = \text{Diam}(G)$.

Let us consider the family $F = \{S(x_2, x_3, \dots, x_n) \mid 0 \leq x_i < k, 1 \leq i \leq n\}$. It can be proved that F is a partition of set $V_{H_{n,k}}$. Each pair of sets in F are disjoint. Indeed, if $S(x_2, x_3, \dots, x_n) \cap S(y_2, y_3, \dots, y_n) \neq \emptyset$ for some distinct (x_2, x_3, \dots, x_n) and (y_2, y_3, \dots, y_n) , then there exist integers a and b from $\{0, 1, \dots, k - 1\}$ such that

$$(a, (a + x_2) \bmod k, \dots, (a + x_n) \bmod k) = (b, (b + y_2) \bmod k, \dots, (b + y_n) \bmod k).$$

It follows that $a = b$. Moreover, from $(a + x_i) \bmod k = (b + y_i) \bmod k$, $a = b$ and $x_i, y_i \in \{0, 1, \dots, k - 1\}$, it follows that $x_i = y_i$ for all $i = 2, \dots, n$. Therefore, $S(x_2, x_3, \dots, x_n) \cap S(y_2, y_3, \dots, y_n) = \emptyset$. Also, if $u = (a, a_2, \dots, a_n)$ is a vertex from $V_{H_{n,k}}$, then there is a set from family F which contains u . Indeed, for $i = 2, \dots, n$ the equality $(a + x) \bmod k = a_i$ with unknown x has the unique solution x_i^* in the set $\{0, 1, \dots, k - 1\}$. Namely, $x_i^* = \begin{cases} a_i - a, & a_i \geq a \\ k + a_i - a, & a_i < a. \end{cases}$

Therefore, $u \in S(x_2^*, x_3^*, \dots, x_n^*)$.

Let S be an arbitrary subset of $V_{H_{n,k}}$, where $|S| > k^{n-1}$. As F is a partition of $V_{H_{n,k}}$ and $|F| = k^{n-1}$, then there exists a set $S(x_2, x_3, \dots, x_n)$ from F which has at least two common members with S . If $u, v \in S \cap S(x_2, x_3, \dots, x_n)$, then, from $u, v \in S(x_2, x_3, \dots, x_n)$, it follows that $d(u, v) = \text{Diam}(H_{n,k})$. Since $u, v \in S$, the proof is completed.

Lemma 7. *If S is a strong resolving set of $H_{n,k}$ then $|S| \geq (k - 1) \cdot k^{n-1}$.*

Proof. Let us assume that $|S| < (k - 1) \cdot k^{n-1}$. Then, for set $\bar{S} = V_{H_{n,k}} \setminus S$, it follows that $|\bar{S}| > k^{n-1}$, and due to Lemma 6, there exist $u, v \in \bar{S}$ such that $d(u, v) = \text{Diam}(H_{n,k})$. But, according to Lemma 5, $u \in S$ or $v \in S$, which is a contradiction.

Lemma 8. *The set $S = \{(x_1, x_2, \dots, x_n) \mid 1 \leq x_1 < k, 0 \leq x_i < k \text{ for } 2 \leq i \leq n\}$ is a strong resolving set of $H_{n,k}$.*

Proof. According to the definition of S , set $V_{H_{n,k}} \setminus S$ contains all vertices of $H_{n,k}$ with the first coordinate equal to 0. Let us consider an arbitrary pair of vertices $u, v \in V_{H_{n,k}} \setminus S$. If $v = (0, x_2, \dots, x_n)$, then there exists a vertex $w \in S$ such that $w = (1, x_2, \dots, x_n)$. As $d(v, w) = 1$ and $d(u, w) = d(u, v) + 1$, then $d(u, w) = d(u, v) + d(v, w)$, i. e. vertex v belongs to a shortest $u - w$ path, and, therefore, w strongly resolves vertices u and v . Also, for $u \in S$ or $v \in S$, vertex u or vertex v obviously strongly resolves pair u, v . Therefore, S is a strong resolving set.

Proof of Theorem 4. As the set S from Lemma 8 has cardinality equal to $(k-1) \cdot k^{n-1}$, then it follows directly from Lemma 7 and Lemma 8 that S is a strong metric basis of $H_{n,k}$, and consequently $\text{sdim}(H_{n,k}) = (k-1) \cdot k^{n-1}$.

Corollary 9. *For all n , the strong metric dimension of the hypercube Q_n is equal to 2^{n-1} .*

Proof. Since $Q_n = H_{n,2}$, the statement follows directly from Theorem 4.

4. CONCLUSIONS

In this paper we have studied minimal doubly resolving sets and the strong metric dimension of Hamming graphs. We have proved that for $k \geq 5$ the cardinality of minimal doubly resolving sets of $H_{2,k}$ is equal to the metric dimension of $H_{2,k}$. In addition, the explicit expression for the strong metric dimension of $H_{n,k}$ is conjectured and proved.

Interesting open problems are to find the metric dimension and/or the cardinality of minimal doubly resolving sets (or a tight upper bound) for some other classes of Hamming graphs $H_{n,k}$, $n \geq 3$. Future work can also be directed towards obtaining the cardinality of minimal doubly resolving sets and/or the strong metric dimension of some other challenging classes of graphs.

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