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POSITIVE SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH UNBOUNDED GREEN'S KERNEL

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Sufficient conditions are obtained for the existence/nonexistence of at least two positive periodic solutions of a class of first order differential equations having an unbounded Green's function. An application to an ecological model with strong Allee effects is also given.

1. INTRODUCTION

This paper is concerned with the existence, nonexistence, and multiplicity of positive T-periodic solutions of a class of differential equations of the form

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(h(t))), \tag{1}$$

where λ is a positive parameter, a, b, and $h \in C(R, [0, \infty))$ are T-periodic functions, $\int_0^T a(t)\,\mathrm{d}t > 0$, $\int_0^T b(t)\mathrm{d}t > 0$, $f : [0, \infty) \to [0, \infty)$ is continuous with $f(x) > 0$ for $x > 0$, and $g : [0, \infty) \to [0, \infty)$ is a continuous function. Equations of the form (1) and

$$x'(t) = a(t, x(t))x(t) - \lambda b(t)f(t, x(h(t))) \tag{2}$$

have been considered by many authors for the purpose of proving the existence of one or more positive periodic solutions. See, for example, [**3, 5, 6, 7, 12, 18, 21, 22, 26, 27, 29**] and the references cited therein. In the above cited references it is often required that the functions $g(x)$ and $a(t, x)$ satisfy the following conditions:

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(H_1) $g : [0, \infty) \to [0, \infty)$ is continuous and there are constants $0 < l < L$ such that $0 < l \leq g(x) < L < \infty$ for $u \geq 0$;

(H_2) there exist continuous T-periodic functions c and d with $\int\limits_0^T c(t)\mathrm{d}t > 0$ such that $0 \leq c(t) \leq |a(t, x)| \leq d(t)$ for all T-periodic functions x.

Jin and Wang [**11**] studied the existence of a positive periodic solution of (1) in the case where $g(x)$ is unbounded. We note that the method used in [**11**] can be applied to (2) when $a(x, t)$ is unbounded.

There is a rich literature in the last two decades on the use of fixed point theorems (Krasnosel'skii's theorem, Leggett-Williams multiple fixed point theorem, upper-lower solution method, iterative techniques, etc.) to prove the existence of positive solutions to boundary value problems. The proofs of the existence of periodic solutions to the types of equations considered here is closely related to proving the existence of positive solutions to general boundary value problems. In most cases, once the equation is transformed into an equivalent integral equation via a Green's kernel, the arguments used are often quite similar. The assumptions on $g(x)$ and $a(t, x)$ in the papers referenced above usually leads to a Green's kernel in the equivalent integral equation that is bounded.

Our primary concern in this paper is the study of the positive periodic solutions of equations of the form (1) and in some cases allow for an unbounded Green's kernel. Clearly, (1) is equivalent to

$$x(t) = \lambda \int_t^{t+T} G_x(t, s) b(s) f(x(h(s)))\, \mathrm{d}s, \tag{3}$$

where

$$G_x(t, s) = \frac{e^{-\int_t^s a(\theta) g(x(\theta))\, \mathrm{d}\theta}}{1 - e^{-\int_0^T a(\theta) g(x(\theta))\, \mathrm{d}\theta}}. \tag{4}$$

If (H_1) holds, then $G_x(t, s)$ is bounded by

$$\frac{\sigma^L}{1 - \sigma^L} \leq G_x(t, s) \leq \frac{1}{1 - \sigma^L}, \quad t \leq s \leq t + T,$$

where

$$\sigma = e^{-\int_0^T a(s)\, \mathrm{d}s} < 1. \tag{5}$$

In [**11**], Jin and Wang used the following condition on $g(x)$

$$g(x) = \begin{cases} e^x, & 0 \leq x \leq L, \\ e^L, & x \geq L, \end{cases} \tag{6}$$

that increases with increasing $L > 0$. In this case, the Green's kernel $G_x(t,s)$ is bounded by

$$\frac{\sigma^{e^L}}{1-\sigma^{e^L}} \le G_x(t,s) \le \frac{1}{1-\sigma}, \quad t \le s \le t+T, \tag{7}$$

with σ given in (5).

The equation

$$x'(t) = a(t)x(t) - \lambda b(t) f(t, x(h(t))), \tag{8}$$

which is a particular case of (1) or (2), has been studied by many authors, for example, see [**1, 2, 9, 10, 14, 15, 16, 17, 19, 22, 23, 24, 25, 28, 30, 31**] and the references therein. The results obtained in some of these papers have been applied to mathematical models arising in biological and ecological systems, for example, the Lasota-Wazewska model, a Hematopoiesis model, Nicholson's Blowflies model, the Michaelis Menton model, age-structured population models, etc. The monograph by Kuang [**12**] gives a nice discussion of such models.

As pointed out above, a variety of authors have used fixed point methods to prove the existence of at least one positive periodic solution of equations of the forms (1), (2), and (8). For instance, see [**2, 3, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 24, 25, 26, 27, 28, 29, 30, 31**] and the references there in. The question of the existence of at least two or at least three positive periodic solutions of equations of the forms (1), (2), and (8) has also been studied. For example, the existence of at least three positive periodic solutions of (1), (2), and (8) is examined in [**1, 19, 20, 22**] where the function f is required to be unimodal, that is, f first increases and then decreases eventually to zero. The Leggett-Williams multiple fixed point theorem ([**13**, Theorem 3.3]) has been used to prove such results. The unimodal nature of the function f excludes many mathematical models in ecological and biological systems where the f is either nondecreasing or nonincreasing with respect to x for $x \ge 0$. Thus, an attempt was made in [**21**] to consider the case where f is not necessarily unimodal, and the results were applied to many mathematical models where f is nondecreasing. The Leggett-Williams multiple fixed point theorem ([**13**, Theorem 3.5]) was used there. In a recent work, Padhi et al. [**23**] used the Leggett-Williams multiple fixed point theorem to prove the existence of at least two positive periodic solutions of the equation

$$x'(t) = -A(t)x(t) + f(t, x(t)), \tag{9}$$

where $A \in C([0,\infty),[0,\infty))$ and $f \in C([0,\infty)\times[0,\infty),[0,\infty))$ are T-periodic functions (see Theorem 3 and Corollary 4 in [**23**]). Theorem 3 in [**23**] was applied to the study of existence of at least two positive periodic solutions of the equation representing the dynamics of a renewable resource x that is subjected to the Allee effect, namely,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x(t)(x(t)-b(t))(c(t)-x(t)) \tag{10}$$

where $0 < b(t) < c(t)$. It is assumed that $a(t)$, $b(t)$, and $c(t)$ are T-periodic positive functions and $\int\limits_0^T A(t)\,\mathrm{d}t > 0$. Equation (10) was transformed into the form of equation (9) with $A(t) = a(t)b(t)c(t) + \frac{c'(t)}{c(t)}$ and $f(t, x(t)) = a(t)c^2(t)((1 + k(t)) - x(t))x^2(t)$.

The motivation for the present work has come from the following observation. We can rewrite (10) in the form

$$x'(t) = B(t)x(t)x(t) - a(t)(x^2(t) + b(t)c(t))x(t) \tag{11}$$

where

$$B(t) = a(t)(b(t) + c(t)), \tag{12}$$

which is of the form of equation (1) with $g(x) = x$ being an unbounded function. Assuming x is periodic with period T, equation (11) can be written as the integral equation

$$x(t) = \int_t^{t+T} G_x(t, s)a(s)(x^2(s) + b(s)c(s))x(s)\,\mathrm{d}s \tag{13}$$

where

$$G_x(t, s) = \frac{e^{-\int_t^s B(\theta)x(\theta)\,\mathrm{d}\theta}}{1 - e^{-\int_0^T B(\theta)x(\theta)\,\mathrm{d}\theta}} \tag{14}$$

is the Green's kernel. Note that in this case we have

$$\frac{e^{-\int_0^T B(\theta)x(\theta)\,\mathrm{d}\theta}}{1 - e^{-\int_0^T B(\theta)x(\theta)\,\mathrm{d}\theta}} \le G_x(t, s) \le \frac{1}{1 - e^{-\int_0^T B(\theta)x(\theta)\,\mathrm{d}\theta}}. \tag{15}$$

The function $g(x)$ considered in (6) (see [**11**]) has the lower bound 1, whereas the lower bound of $g(x) = x$ considered in (11) is zero implying $G_x(t, s)$ in (15) is not necessarily bounded above. Thus, it would be interesting to make a further study of equations (10) or (11). Section 3 is devoted to this problem.

Jin and Wang [**11**] used fixed point index theory to prove the following theorem.

Theorem 1.1. *Suppose $g(x)$ satisfies condition* (6) *and let* $\lim\limits_{x\to 0+} \frac{f(x)}{x} = 0$. *For each $L > 0$, there exists* $\lambda_0 = \dfrac{L(1 - \sigma^{e^L})}{m(L)\sigma^{e^L} \int\limits_0^T b(s)\,ds} > 0$ *such that* (1) *has a positive T-periodic solution x with* $\sup\limits_{0\le t\le T} x(t) < L$ *for* $\lambda > \lambda_0$.

In this paper, we shall use the Leggett-Williams multiple fixed point theorem to show that (1) has at least two positive T-periodic solutions under conditions very similar to those used in Theorem 1.1. This result appears in Section 2.

The following concept is needed in order to apply the Leggett-Williams multiple fixed point theorem.

Let X be a Banach space and K be a cone in X. For $a > 0$, define $K_a = \{x \in K \ : \ \|x\| < a\}$. A mapping ψ is said to be a nonnegative continuous *concave* functional on K if $\psi : K \to [0, \infty)$ is continuous and

$$\psi(\mu x + (1-\mu)y) \geq \mu\psi(x) + (1-\mu)\psi(y) \quad \text{for} \quad x, y \in K \quad \text{and} \quad \mu \in [0,1].$$

Let b, $c > 0$ be constants with K and X as defined above. Define

$$K(\psi, b, c) = \{x \in K \ : \ \psi(x) \geq b \text{ and } \|x\| \leq c\}.$$

Theorem 1.2 (Leggett-Williams multiple fixed point Theorem [**13**, Theorem 3.5])**.** *Let $c_3 > 0$ be a constant. Assume that $A : \overline{K}_{c_3} \to K$ is completely continuous, there exists a concave nonnegative functional ψ with $\psi(x) \leq \|x\|$ for $x \in K$, and there are constants c_1 and c_2 with $0 < c_1 < c_2 < c_3$ such that the following conditions are satisfied*:

(i) $\{x \in K(\psi, c_2, c_3) \ : \ \psi(x) > c_2\} \neq \phi$ *and* $\psi(Ax) > c_2$ *if* $x \in K(\psi, c_2, c_3)$;

(ii) $\|Ax\| < c_1$ *if* $x \in \overline{K}_{c_1}$;

and

(iii) $\psi(Ax) > \dfrac{c_2}{c_3}\|Ax\|$ *for each* $x \in \overline{K}_{c_3}$ *with* $\|Ax\| > c_3$.

Then A has at least two fixed points x_1, x_2 in $\overline{K}_{c_3}$. Furthermore, $\|x_1\| \leq c_1 < \|x_2\| < c_3$.

2. MAIN RESULTS

In this section, we prove two theorems for the existence of at least two positive T-periodic solutions of (1) with different unbounded functions $g(x)$. In fact, as in [**11**], we consider the differential equation

$$x'(t) = a(t)g_L(x(t))x(t) - \lambda b(t)f(x(h(t))), \tag{16}$$

where a, b, λ, h and f are as defined earlier and $g_L(x)$ satisfies either (6) or

$$g_L(x) = \begin{cases} e^x - 1, & 0 \leq x \leq L, \\ e^L - 1, & x \geq L, \end{cases} \tag{17}$$

for each $L > 0$. Throughout the remainder of this paper, we assume that X is the Banach space of all continuous T-periodic functions endowed with the sup norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)|.$$

Furthermore, on a cone K in X, we define a concave functional ψ by

$$\psi(x) = \min_{t\in[0,T]} x(t).$$

Note that (16) is equivalent to

$$x(t) = \lambda \int_t^{t+T} G_L(t,s)b(s)f(x(h(s)))\,\mathrm{d}s,$$

where $G_L(t,s)$ is the Green's kernel defined by

$$G_L(t,s) = \frac{e^{-\int_t^s a(\theta)g_L(x(\theta))\,\mathrm{d}\theta}}{1 - e^{-\int_0^T a(\theta)g_L(x(\theta))\,\mathrm{d}\theta}} \tag{18}$$

First, suppose that $g_L(x)$ satisfies (6). In this case, we define

$$m(L) = \min\left\{ f(x) \;:\; \frac{\sigma^{e^L}(1-\sigma)}{1-\sigma^{e^L}} L \le x \le L \right\} > 0 \tag{19}$$

for every $L > 0$ where σ is given in (5).

Here is our first result.

Theorem 2.1. *Let $g_L(x)$ satisfy* (6). *If* $\lim_{x\to 0+} \frac{f(x)}{x} = 0$, *then for each $L > 0$, there exists a* $\lambda_1 = \dfrac{L(1-\sigma)}{m(L)\int_0^T b(s)\,ds} > 0$ *such that* (16) *has at least two positive T-periodic solutions for $\lambda > \lambda_1$.*

Proof. Define a cone K on X by

$$K = \left\{ x \in X : x(t) \ge \frac{\sigma^{e^L}(1-\sigma)}{1-\sigma^{e^L}} \|x\|,\ t \in [0,T] \right\},$$

and an operator E on X by

$$(Ex)(t) = \lambda \int_t^{t+T} G_L(t,s)b(s)f(x(h(s)))\,\mathrm{d}s, \tag{20}$$

where $G_L(t,s)$ is given in (18). Simple calculations show that $E(K) \subset K$ and $E : K \to K$ is completely continuous. Furthermore, the existence of a positive

T-periodic solution of (16) is equivalent to the existence of a fixed point of E on K. The Green's kernel satisfies the bounds

$$\frac{\sigma^{e^L}}{1-\sigma^{e^L}} \leq G_L(t,s) \leq \frac{1}{1-\sigma}, \quad t \leq s \leq t+T,$$

and hence

$$\frac{\sigma^{e^L}(1-\sigma)}{1-\sigma^{e^L}} < 1.$$

Set $c_2 = \dfrac{\sigma^{e^L}(1-\sigma)}{1-\sigma^{e^L}}L$ and $c_3 = L$. Then $0 < c_2 < c_3$. Clearly, $c_0 = \dfrac{c_2+c_3}{2} \in \{x \in K(\psi, c_2, c_3) \ : \ \psi(x) > c_2\} \neq \emptyset$. By the choice of c_2, c_3, and (19), we have that

$$m(L) = \min\{f(x) \ : \ c_2 \leq x \leq c_3\}.$$

Hence, for $x \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned}\psi(Ex) &= \lambda \min_{0\leq t\leq T} \int_t^{t+T} G_L(t,s)b(s)f(x(h(s)))\,\mathrm{d}s \geq \lambda \frac{\sigma^{e^L}}{1-\sigma^{e^L}} m(L) \int_0^T b(s)\,\mathrm{d}s \\ &> \frac{L(1-\sigma)}{m(L)\int\limits_0^T b(s)\,ds} . \frac{\sigma^{e^L}}{1-\sigma^{e^L}} m(L) \int_0^T b(s)\,\mathrm{d}s = \frac{L(1-\sigma)\sigma^{e^L}}{1-\sigma^{e^L}} = c_2,\end{aligned}$$

so condition (i) of Theorem 1.2 holds.

Since $\lim\limits_{x\to 0+} \dfrac{f(x)}{x} = 0$, we can choose $0 < c_1 < c_2$ and $\epsilon > 0$ with

$$\frac{\epsilon\lambda}{1-\sigma} \int_0^T b(s)\,\mathrm{d}s < 1$$

such that $f(x) \leq \epsilon x$ for $0 < x < c_1$. Thus, for $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}\|Ex\| &\leq \lambda \frac{1}{1-\sigma} \int_0^T b(s)f(x(h(s)))\,\mathrm{d}s \\ &\leq \frac{\lambda}{1-\sigma}\epsilon \int_0^T b(s)\|x\|\,\mathrm{d}s \leq c_1 \frac{\epsilon\lambda}{1-\sigma} \int_0^T b(s)\,\mathrm{d}s < c_1,\end{aligned}$$

so condition (ii) of Theorem 1.2 holds.

To complete the proof of the theorem, it suffices to show that condition (iii) of Theorem 1.2 holds. Suppose that $x \in \overline{K}_{c_3}$ with $\|Ex\| > c_3$. Then,

$$c_3 < \|Ex\| \leq \lambda \frac{1}{1-\sigma} \int_0^T b(s)f(x(h(s)))\,\mathrm{d}s$$

implies that

$$\psi(Ex) \geq \frac{\sigma^{e^L}}{1-\sigma^{e^L}} \lambda \int_0^T b(s)f(x(h(s)))\,\mathrm{d}s \geq \frac{(1-\sigma)\sigma^{e^L}}{1-\sigma^{e^L}} \|Ex\| = \frac{c_2}{c_3}\|Ex\|.$$

Hence, (16) has at least two positive T-periodic solutions completing the proof of the theorem. $\square$

Next, we give a nonexistence result for positive periodic solutions of (16) again with $g_L(x)$ satisfying (6).

Theorem 2.2. *Let $g_L(x)$ be defined as in (6) and assume that $\lim_{x\to 0+}\frac{f(x)}{x}=0$. Then, for each $L>0$, there exists $\lambda_1=\dfrac{L(1-\sigma)}{m(L)\int_0^T b(s)\,\mathrm{d}s}>0$ such that (16) has no positive T-periodic solutions for $\lambda\le\lambda_1$.*

Proof. Suppose to the contrary that $x(t)$ is a positive T-periodic solution of (16). Since $\lim_{x\to 0+}\frac{f(x)}{x}=0$, there exists $0<\epsilon<\frac{m(L)}{L}$ and $\mu_1>0$ such that $f(x)\le\epsilon x$ for $0<x<\mu_1$. Choose $\mu_2=\min\{\mu_1,\lambda_1\}$. Now x being a positive T-periodic solution of (16) implies that $x=Ex$. Hence, for $x\in\overline{K}_{\mu_2}$, we have, for $\lambda\le\lambda_1$,

$$
\begin{aligned}
\|x\|=\|Ex\|&=\lambda\sup_{0\le t\le T}\int_t^{t+T}G_L(t,s)b(s)f(x(h(s)))\,\mathrm{d}s\\
&\le\lambda\frac{1}{1-\sigma}\int_t^{t+T}b(s)f(x(h(s)))\,\mathrm{d}s\le\lambda\epsilon\|x\|\frac{1}{1-\sigma}\int_t^{t+T}b(s)\,\mathrm{d}s\\
&\le\lambda_1\epsilon\|x\|\frac{1}{1-\sigma}\int_t^{t+T}b(s)\,\mathrm{d}s\le\frac{L(1-\sigma)}{m(L)\int_0^T b(s)\,ds}\epsilon\|x\|\frac{1}{1-\sigma}\int_0^T b(s)\,\mathrm{d}s<\|x\|,
\end{aligned}
$$

which is a contradiction. Therefore, (16) has no positive T-periodic solutions for $\lambda\le\lambda_1$. $\square$

We now consider the case where $g_L(x)$ be defined as in (17). In this case, we observe that the Green's kernel given in (18) is not necessarily bounded above. Hence, we need a separate result for this situation. We set

$$
M(L)=\min\{f(x)\ :\ \sigma^{(e^L-1)}L\le x\le L\}>0 \tag{21}
$$

for every $L>0$, where σ is given in (5). We need to introduce the notation that for any continuous T-periodic function $p:[0,T]\to(-\infty,\infty)$, we set

$$
p_*=\min_{0\le t\le T}p(t)\quad\text{and}\quad p^*=\max_{0\le t\le T}p(t).
$$

Theorem 2.3. *Let $g_L(x)$ be given in (17). If $\lim_{x\to 0+}\frac{f(x)}{xg_L(x)}=0$, then for each $L>0$, there exists a $\lambda_2=\dfrac{La^*(e^L-1)\sigma^{(e^L-1)}}{b_*M(L)}>0$ such that (16) has at least two positive T-periodic solutions for $\lambda>\lambda_2$.*

Proof. We define a cone K on X by

$$K = \{x \in X \ : \ x(t) \geq 0, \ t \in [0,T]\}$$

and an operator E on X as given in (20). Simple calculations show that

$$\int_t^{t+T} G_L(t,s)a(s)g_L(x(s))\,\mathrm{d}s \equiv 1.$$

Choose $c_2 = L\sigma^{(e^L-1)}$ and $c_3 = L$; then $0 < c_2 < c_3$. It is clear that $\{x \in K(\psi, c_2, c_3) \ : \ \psi(x) > c_2\} \neq \emptyset$. Now, for $x \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned}
\psi(Ex) &= \lambda \min_{0\leq t\leq T} \int_t^{t+T} G_L(t,s)b(s)f(x(h(s)))\,\mathrm{d}s \\
&\geq \lambda M(L) \min_{0\leq t\leq T} \int_t^{t+T} G_L(t,s)b(s)\,\mathrm{d}s \geq \lambda \frac{b_*}{a^*} M(L) \min_{0\leq t\leq T} \int_t^{t+T} G_L(t,s)a(s)\,\mathrm{d}s.
\end{aligned}$$

Since $g_L(x)$ is an increasing function of x, for $c_2 \leq x \leq c_3$, we have $g_L(c_2) \leq g_L(x) \leq g_L(c_3) = e^{c_3} - 1 = e^L - 1$. Hence,

$$\begin{aligned}
\psi(Ex) &\geq \lambda M(L) \int_t^{t+T} G_L(t,s)b(s)\,\mathrm{d}s \\
&\geq \lambda \frac{b_*}{a^*} \frac{M(L)}{(e^L-1)} \min_{0\leq t\leq T} \int_t^{t+T} G_L(t,s)a(s)g_L(x(s))\,\mathrm{d}s \\
&= \lambda \frac{b_*}{a^*} \frac{M(L)}{(e^L-1)} > \frac{La^*(e^L-1)\sigma^{(e^L-1)}}{b_* M(L)} . \frac{b_*}{a^*} \frac{M(L)}{(e^L-1)} = L\sigma^{(e^L-1)} = c_2,
\end{aligned}$$

so condition (i) of Theorem 1.2 holds.

Next, we have that $\lim_{x\to 0+} \frac{f(x)}{xg_L(x)} = 0$ so there exists $\epsilon > 0$ with $\frac{\lambda b^* \epsilon}{a_*} < 1$ and $c_1 \in (0, c_2)$ such that $f(x) \leq \epsilon g_L(x)x$ for $0 < x < c_1$. For $x \in \overline{K}_{c_1}$, we have

$$\|Ex\| \leq \lambda \frac{b^*}{a_*} \epsilon \|x\| \sup_{0\leq t\leq T} \int_t^{t+T} G_L(t,s)a(s)g_L(x(s))\,\mathrm{d}s \leq \lambda \frac{b^*}{a_*} \epsilon c_1 < c_1,$$

i.e., condition (ii) of Theorem 1.2 holds.

To complete the proof of the theorem, it remains to show that condition (iii) in Theorem 1.2 holds. Since

$$c_3 < \|Ex\| < \lambda \int_0^T \frac{b(s)f(x(h(s)))}{1 - e^{-\int_0^T a(\theta)g_L(x(\theta))\,\mathrm{d}\theta}}\,\mathrm{d}s,$$

for $x \in \overline{K}_{c_3} = \overline{K}_L$, we have

$$\begin{aligned}\psi(Ex) &= \lambda \min_{0\le t\le T} \int_t^{t+T} \frac{e^{-\int_t^s a(\theta) g_L(x(\theta))\,\mathrm{d}\theta}}{1-e^{-\int_0^T a(\theta) g_L(x(\theta))\,\mathrm{d}\theta}} b(s) f(x(h(s)))\,\mathrm{d}s \\ &> \lambda \min_{0\le t\le T} \int_t^{t+T} \frac{e^{-(e^L-1)\int_0^T a(\theta)\,\mathrm{d}\theta}}{1-e^{-\int_0^T a(\theta) g_L(x(\theta))\,\mathrm{d}\theta}} b(s) f(x(h(s)))\,\mathrm{d}s \\ &> \lambda \sigma^{(e^L-1)} \int_0^T \frac{b(s) f(x(h(s)))}{1-e^{-\int_0^T a(\theta) g_L(x(\theta))\,\mathrm{d}\theta}}\,\mathrm{d}s > \sigma^{(e^L-1)}\|Ex\| = \frac{c_2}{c_3}\|Ex\|,\end{aligned}$$

completing the proof of the theorem.

3. APPLICATION TO A MODEL WITH ALLEE EFFECTS

In this section, we prove the existence of at least two positive T-periodic solutions to the model with Allee effects given in (10). Recall that (10) can be rewritten in the form (11) with $B(t)$ given in (12). We let X be a Banach space as in Section 2 above. Assuming that $x(t)$ is a T-periodic solution of (10), we can write (13) as the equivalent integral equation, where $G_x(t,s)$ is the Green's kernel given in (14). We note that the Green's kernel satisfies

$$\int_t^{t+T} G_x(t,s) B(s) x(s)\,\mathrm{d}s \equiv 1.$$

We define an operator N on X by

$$(Nx)(t) = \int_t^{t+T} G_x(t,s) a(s)(x^2(s) + b(s)c(s)) x(s)\,\mathrm{d}s$$

and a cone K on X by

$$K = \{x \in X \ : \ x(t) \ge 0, \ t \in [0,T]\}.$$

It is clear that $N(K) \subset K$ and $N : K \to K$ is completely continuous. Moreover, the existence of a positive T-periodic solution to (10) is equivalent to the existence of a fixed point of N in K. On the cone K, we define a concave continuous functional ψ by

$$\psi(x) = \min_{0\le t\le T} x(t).$$

In a recent work, Padhi et al. [**23**] obtained the following sufficient condition for the existence of at least two positive T-periodic solutions to (10). Here, we set

$$M_1 = \int_0^T a(t) c^2(t)\,\mathrm{d}t \quad \text{and} \quad N_1 = \int_0^T a(t) b(t) c(t)\,\mathrm{d}t.$$

Theorem 3.1. *Let* $\int\limits_0^T \Big(a(s)b(s)c(s) + \dfrac{c'(s)}{c(s)}\Big)\,\mathrm{d}s > 0$. *If*

$$\frac{(M_1+N_1)+\sqrt{(M_1+N_1)^2-4M_1\Big(\dfrac{e^{N_1}-1}{e^{N_1}}\Big)}}{2M_1} > \frac{e^{2N_1}-\dfrac{1}{e^{N_1}}}{M_1+N_1}, \tag{22}$$

then equation (10) *has at least two positive* T*-periodic solutions.*

We shall now apply Theorem 1.2 to equation (10) to obtain a new sufficient condition for the existence of at least two positive T-periodic solutions.

Theorem 3.2. *If* $(b_*+c_*)^2 > 4b^*c^*$, *then equation* (10) *has at least two positive* T*-periodic solutions.*

Proof. Set $c_1 = \dfrac{(b_*+c_*)+\sqrt{(b_*+c_*)^2-4b^*c^*}}{2}$, $c_2 = \dfrac{(b^*+c^*)+\sqrt{(b^*+c^*)^2-4b_*c_*}}{2}$, and $c_3 > 0$ such that

$$c_3 e^{-c_3 a^*(b^*+c^*)T} = \frac{(b^*+c^*)+\sqrt{(b^*+c^*)^2-4b_*c_*}}{2} = c_2.$$

Then $0 < c_1 < c_2 < c_3$. For $x \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned}\psi(Nx)(t) &= \min_{0\le t\le T}\int_t^{t+T} G_x(t,s)a(s)(x^2(s)+b(s)c(s))x(s)\,\mathrm{d}s\\ &\ge \frac{c_2^2+b_*c_*}{b^*+c^*}\min_{0\le t\le T}\int_t^{t+T} G_x(t,s)a(s)(b(s)+c(s))x(s)\,\mathrm{d}s = \frac{c_2^2+b_*c_*}{b^*+c^*} = c_2.\end{aligned}$$

Furthermore, for $x \in \overline{K}_{c_1}$,

$$\begin{aligned}\|Nx\| &= \sup_{0\le t\le T}\int_t^{t+T} G_x(t,s)a(s)(x^2(s)+b(s)c(s))x(s)\,\mathrm{d}s\\ &\le \frac{c_1^2+b^*c^*}{b_*+c_*}\sup_{0\le t\le T}\int_t^{t+T} G_x(t,s)a(s)(b(s)+c(s))x(s)\,\mathrm{d}s = \frac{c_1^2+b^*c^*}{b_*+c_*} = c_1.\end{aligned}$$

For these choices of c_2 and c_3, we shall show that the third condition in Theorem 1.2 is satisfied. For each $x \in \overline{K}_{c_3}$ with $\|Nx\| > c_3$, we have $0 < x \le c_3$,

$$c_3 < \|Nx\| < \int_0^T \frac{a(s)(x^2(s)+b(s)c(s))x(s)}{1-e^{-\int_0^T a(\theta)(b(\theta)+c(\theta))x(\theta)\,\mathrm{d}\theta}}\,\mathrm{d}s,$$

and

$$\begin{aligned}\psi(Nx) &= \min_{0\le t\le T}\int_t^{t+T} G_x(t,s)a(s)(x^2(s)+b(s)c(s))x(s)\,\mathrm{d}s\\ &> \min_{0\le t\le T}\int_t^{t+T}\frac{e^{-c_3\int_t^s a(\theta)(b(\theta)+c(\theta))\,\mathrm{d}\theta}}{1-e^{-\int_0^T a(\theta)(b(\theta)+c(\theta))x(\theta)\,\mathrm{d}\theta}}a(s)(x^2(s)+b(s)c(s))x(s)\,\mathrm{d}s\\ &> \min_{0\le t\le T}\int_t^{t+T}\frac{e^{-c_3\int_0^T a(\theta)(b(\theta)+c(\theta))\,\mathrm{d}\theta}}{1-e^{-\int_0^T a(\theta)(b(\theta)+c(\theta))x(\theta)\,\mathrm{d}\theta}}a(s)(x^2(s)+b(s)c(s))x(s)\,\mathrm{d}s\\ &> e^{-c_3a^*(b^*+c^*)T}\min_{0\le t\le T}\int_0^T\frac{a(s)(x^2(s)+b(s)c(s))x(s)}{1-e^{-\int_0^T a(\theta)(b(\theta)+c(\theta))x(\theta)\,\mathrm{d}\theta}}\,\mathrm{d}s\\ &> e^{-c_3a^*(b^*+c^*)T}\|Nx\| = \frac{c_2}{c_3}\|Nx\|.\end{aligned}$$

Hence, (10) has at least two positive T-periodic solutions, and this completes the proof of the theorem.

Remark 3.3. Let $a(t)\equiv a$, $b(t)\equiv b$, and $c(t)\equiv c$ be positive constants. Then (10) reduces to the constant coefficient equation

$$x'(t) = ax(t)(x(t)-b)(c-x(t)),\quad 0<b<c. \tag{23}$$

In this case $b^* = b_* = b$, $c^* = c_* = c$, and the condition $(b_*+c_*)^2 > 4b^*c^*$ becomes $(b+c)^2 > 4bc$, i.e., $(b-c)^2>0$, which is true for all $0<b<c$. Thus, we immediately have the following corollary.

Corollary 3.4. *Equation* (23) *has at least two positive T-periodic solutions.*

It is well known that (23) admits two positive solutions, namely $x(t)=b$ and $x(t)=c$ and one trivial solution as its equilibrium solution. This observation in a sense validates our Corollary 3.4. On the other hand, applying Theorem 3.1 to equation (23), we observe that (23) has at least two positive T-periodic solutions provided that

$$\frac{aT(b+c)+\sqrt{a^2T^2(b+c)^2-4aT\frac{(e^{abcT}-1)}{e^{abcT}}}}{2} > \frac{e^{2abcT}-e^{-abcT}}{b+c} \tag{24}$$

holds. In [**23**], the authors showed that the inequality

$$a^2T^2(b+c)^2 > 4aT\frac{(e^{abcT}-1)}{e^{abcT}}$$

holds for all a, b and $c>0$. However, it is possible to choose a, b and $c>0$ so that (24) fails. Thus, Corollary 3.4 gives a better result for the existence of at least two positive periodic solutions of (23) than a direct application of Theorem 3.1

The following example illustrates that our Theorem 3.2 above may apply when Theorem 3.1 fails to hold.

Example 3.5. Consider Eq. (10) with $a(t) = b(t) = \frac{1}{10}\Big(0.9999 + \frac{1}{10^4}\cos 8t\Big)$ and $c(t) = \frac{2}{10}\Big(0.9999 + \frac{1}{10^4}\sin 8t\Big)$. Here $b^* = 0.1, b_* = 0.09998, c^* = 0.2, c_* = 0.19996$ and $T = \frac{\pi}{4}$. Since

$$(b_* + c_*)^2 = 0.0899640036 > 0.08 = 4b^*c^*$$

holds, Theorem 3.2 can be applied to this example. On the other hand,

$$\begin{aligned} M_1 = \int_0^T a(t)c^2(t)\,\mathrm{d}t &= \frac{4}{1000}\int_0^{\frac{\pi}{4}} \Big(0.9999 + \frac{1}{10^4}\cos 8t\Big)\Big(0.9999 + \frac{1}{10^4}\sin 8t\Big)^2 \mathrm{d}t \\ &= 0.00314065 \end{aligned}$$

and

$$\begin{aligned} N_1 = \int_0^T a(t)b(t)c(t)\,\mathrm{d}t &= \frac{2}{1000}\int_0^{\frac{\pi}{4}} \Big(0.9999 + \frac{1}{10^4}\cos 8t\Big)^2\Big(0.9999 + \frac{1}{10^4}\sin 8t\Big) \mathrm{d}t \\ &= 0.001570325 \end{aligned}$$

implies that

$$\begin{aligned} \frac{(M_1 + N_1) + \sqrt{(M_1+N_1)^2 - 4M_1\Big(\frac{e^{N_1}-1}{e^{N_1}}\Big)}}{2M_1} &= 1.000783524 \\ &< 1.000786483 = \frac{e^{2N_1} - \frac{1}{e^{N_1}}}{M_1 + N_1}. \end{aligned}$$

Hence, Theorem 3.1 cannot be applied to this example.

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