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ON THE CAUCHY PROBLEM ASSOCIATED TO THE BRINKMAN FLOW IN $\mathbb{R}^n$

Michel Molina Del Sol, Eduardo Arbieto Alarcon,
Rafael José Iorio Junior

In this work we deal with the Cauchy problem associated to the Brinkman flow, which models fluid flow in certain types of porous media. We study local and global well-posedness in Sobolev spaces $H^s(\mathbb{R}^n), s > \dfrac{n}{2} + 1$, using Kato's theory for quasilinear equations and parabolic regularization.

1. INTRODUCTION

In this article we are interested in the properties of the real valued solutions to the Cauchy problem associated to the Brinkman flow ([**4**],[**28**]). Namely,

$$\begin{cases} \phi\,\partial_t \rho + \operatorname{div}(\rho\,\vec{\mathbf{v}}) = F(t,\rho),\ x \in \mathbb{R}^n,\ t \in (0, T_0] \\ \left(-\mu_{\text{eff}}\Delta + \dfrac{\mu}{k}\right)\vec{\mathbf{v}} = -\nabla P(\rho) \\ (\rho(0),\, \vec{\mathbf{v}}(0)) = (\rho_0,\, \vec{\mathbf{v}}_0). \end{cases} \tag{1}$$

This system models fluid flows in certain types of porous media. Here μ, k, and μ_{eff} denote the fluid viscosity, the porous media permeability and the pure fluid viscosity, respectively, while ρ is the fluid's density, $\vec{\mathbf{v}}$ its velocity, $P(\rho)$ is the pressure, F is an external mass flow rate, and ϕ is the porosity of the medium.

In what follows, to simplify the notation, we will choose all the coefficients in (1) to be equal to 1. At the moment we want to consider only the mathematical

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structure of the system. At a later stage, the constants should be put back in, and various limiting cases should be studied. Thus our problem becomes:

$$\left\{\begin{aligned} \partial_t\rho + \operatorname{div}(\rho\,\vec{\mathbf{v}}) &= F(t,\rho),\ x\in\mathbb{R}^n,\ t\in(0,T_0]\\ (-\Delta+1)\vec{\mathbf{v}} &= -\nabla P(\rho)\\ (\rho(0),\,\vec{\mathbf{v}}(0)) &= (\rho_0,\,\vec{\mathbf{v}}_0). \end{aligned}\right. \tag{2}$$

To handle (2), we compute $\vec{\mathbf{v}}(t,x)$ using the second equation, (usually referred to as Brinkman's condition) to get

$$\vec{\mathbf{v}} = -\left(1-\Delta\right)^{-1}\nabla P\left(\rho\right), \tag{3}$$

and substitute into the first one (which describes the variation of mass) to obtain the Cauchy problem for the Brinkman flow equation (BFE)

$$\left\{\begin{aligned} &\partial_t\rho = \operatorname{div}\left(\rho\left(1-\Delta\right)^{-1}\nabla P\left(\rho\right)\right) + F\left(t,\rho\right),\ t\in(0,T_0]\\ &\rho\left(0\right) = \rho_0. \end{aligned}\right. \tag{4}$$

Then we solve (4), and compute $\vec{\mathbf{v}}$ using (3). Of course, the following compatibility condition must be satisfied:

$$\vec{\mathbf{v}}_0 = -\left(1-\Delta\right)^{-1}\nabla P\left(\rho_0\right).$$

This work is organized as follows:

In Section 3, we analyze the local well-posedness of (4) using Kato's quasilinear theory ([**8**], [**9**], [**15**], [**17**]). We will prove that (4) is locally well-posed in the sense described in Section 2 if $s > \frac{n}{2}+1$. It should be noted that in [**1**] the authors proved that the problem is well-posed in the one dimensional case. As a immediate consequence of Kato's method they obtained continuous dependence of the solution with respect to the initial conditions.

Sections 4, 5 and 6, are dedicated to the study of the the same problem in the context of parabolic regularization in order to obtain global well-posedness. In [**1**], such global results for the Brinkman equation are obtained without using additional information on the equation, because for $n=1, (1-\Delta)^{-1}$ has a bounded kernel. In our case we need to obtain a Comparison Principle for the solutions (see Section 5) to obtain the global estimates in $H^s(\mathbb{R}^n), n>1$.

In Section 6, we will prove global estimates (in the cases of case $F(t,\rho) = 0, P(\rho)=\rho^{2k}, k=1,2,\ldots$).

2. SOME PRELIMINARIES

The initial value problem associated to the Brinkman flow equation (4), corresponds to general problems of the form:

$$\left\{\begin{aligned} &\partial_t u = G(t,u)\in X\\ &u(0)=u_0\in Y. \end{aligned}\right. \tag{5}$$

Here X and Y are Banach spaces and $G : (0, T_0] \times Y \to X$ is continuous with respect to the relevant topologies. In practice, one often takes X and Y to be Sobolev spaces of type L^2.

We will say that (5) is *locally well-posed* or, that the solutions of (5) define a *dynamical system*, if the following conditions are satisfied:

- (LWP-I) *Existence and Persistence*: There exists $T > 0$ and a function $u \in C([0,T], Y)$ satisfying the differential equation in (5), with the time derivative computed with respect to the norm of X and such that $u(0) = u_0$, i.e,

$$\lim_{h\to 0} \left\| \frac{u(t+h)-u(t)}{h} - G(t, u(t)) \right\|_X = 0.$$

- (LWP-II) *Uniqueness*: There is at most one solution to the problem at hand.
- (LWP-III) *Continuous dependence*: The map $u_0 \to u(t)$ is continuous with respect to the appropriate topologies. More precisely, if $(u_0)_n \to u_0$ in Y, then for any $T' \in [0,T)$, the solution corresponding to $(u_0)_n$, u_n, can be extended (if necessary) to $[0,T']$ for all n sufficiently large and

$$\lim_{n\to\infty} \sup_{[0,T']} \|u_n(t) - u(t)\|_Y = 0.$$

In the case that T can be taken arbitrarily large, we will say that (5) is *globally well-posed*. If any of those conditions is not satisfied, then (5) is *ill-posed*. It deserves remark that any of the above conditions, including persistence, mail fail.

Finally, we will introduce some notations and definitions that will be used throughout this work.

Let $s \in \mathbb{R}$, the Sobolev space type L^2, denoted by $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : (1+\xi^2)^{\frac{s}{2}} \hat{f}(\xi) \in L^2(\mathbb{R}^n) \right\},$$

where $S'(\mathbb{R}^n)$ represents the set of temperate distributions, while of $\hat{f}$ will denote the Fourier Transform of f, whatever context it occurs. In $L^1(\mathbb{R}^n)$ we write

$$\hat{f}(\xi) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi}\, \mathrm{d}x.$$

$H^s(\mathbb{R}^n)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_s = \int_{\mathbb{R}^n} (1+\xi^2)^s \hat{f}(\xi) \overline{\hat{g}(\xi)}\, \mathrm{d}\xi.$$

It is easy to see that if $s \geq r$ then $H^s(\mathbb{R}^n) \hookrightarrow H^r(\mathbb{R}^n)$ where the inclusion is continuous and dense. In particular, if $s \geq 0$ we are dealing with L^2 functions. As s increases things get better and better: if $k \geq 0$ is an integer, $f \in H^k$ if and only if $\partial^\alpha f \in L^2$ for all multi-indexes α such that $|\alpha| \leq k$.

According to the Sobolev's lemma ([**29**, Vol.II]), if $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$, then $f \in C_\infty(\mathbb{R}^n)$, the set of all continuous functions that tend to zero at infinity, and f satisfies

$$\|f\|_{L^\infty} \leq C_s(s,n)\|f\|_s.$$

In this case $H^s(\mathbb{R}^n)$ is a Banach algebra with respect to the usual multiplication of functions. In particular,

$$\|fg\|_s \leq C_s(s,n)\|f\|_s\|g\|_s,$$

where $C(s,n)$ is a constant depending only s and n.

Other notations that we will use are:

$\mathbb{R}$ - the real number.
$\| \bullet \|_s$ – the norm in a H^s space.
$\| \bullet \|$ – the norm in a L^2 space.
$\| \bullet \|_{L^\infty}$ – the norm in a L^∞ space.
$B(Y,X)$ – the space of all bounded linear operators from Y to X.
$\| \bullet \|_{B(Y,X)}$ – the operator norm in $B(Y,X)$.
$\partial_x = \dfrac{\partial}{\partial x}$, $\partial_t = \dfrac{\partial}{\partial t}$.
$D(A)$ – the domain of an operator A.
$R(A)$ – the range of an operator A.
$S(\mathbb{R}^n)$ – the Schwarz space of rapidly decreasing C^∞ functions.
$L^p = L^p(\mathbb{R}^n), 1 \leq p \leq \infty$.
$L^p_s = L^p_s(\mathbb{R}^n) = J^{-s}L^p(\mathbb{R}^n)$ with norm $\| \bullet \|_{L^p_s} = \| \bullet \|_{s,p}$.
$C(I,X)$ – the space of continuous functions on an interval I into a Banach space X. If I is compact, it is a Banach space. with a supremum norm.
$C_w(I,X)$ – the space of all weakly continuous functions from I to X.
$A \lesssim B$ – there exist a constant $c > 0$ such that $A \leq cB$.
$\to$ – strong convergence.
$\rightharpoonup$ – weak convergence.
$\hookrightarrow$ – inclusion continuous and dense.

3. LOCAL WELL-POSEDNESS

In this section we will make use of Kato's quasilinear theory in order to obtain local existence results to the problem in question.

We consider the Cauchy problem for the quasi-linear equation, that is:

$$\begin{cases} \partial_t u = G(t,u) = -A(u)u + F(t,u) \in X, \quad t \in (0, T_0] \\ u(0) = u_0 \in Y. \end{cases} \tag{6}$$

Here $A(u)$ is a linear operator depending of u, and u_0 is the initial value.

We will need the following assumptions:

($\boldsymbol{K}$1) X is a reflexive Banach space. There is another reflexive space $Y \hookrightarrow X$ and an isomorphism S from Y onto X such that $\|S\varphi\|_X = \|\varphi\|_Y, \forall\varphi \in Y$.
($\boldsymbol{K}$2) The linear operator $A(u) \in G(X, 1, \beta)$ for $u \in W \subset Y$, where W is an open ball in Y and β is the real number. In other words, for each $u \in W$, $-A(u)$ generates a C^0 semigroup such that

$$\|e^{-sA(u)}\|_{B(X)} \leq e^{\beta s}, s \in [0, \infty), u \in W.$$

($\boldsymbol{K}$3) For each $u \in W$ we have

$$SA(u)S^{-1} = A(u) + B(u),$$

where $B(u) = [S, A(u)]S^{-1} \in B(X)$ is uniformly bounded, that is, there is a constant μ_B such that

$$\|B(u)\|_{B(X)} \leq \mu_B.$$

($\boldsymbol{K}$4) $Y \subset D(A(u))$ (so that $A(u)|_Y \in B(Y, X)$ by the Closed Graph Theorem). The maps $u \in W \mapsto A(u)$ is Lipschitz continuous in the sense:

$$\|A(u) - A(v)\|_{B(Y,X)} \leq \mu_A\|u - v\|_X.$$

($\boldsymbol{K}$5) The function F satisfies:

(a) For each $u \in W$, the maps $t \in (0, T_0] \to F(t, u) \in X$ is continuous.

(b) For each $t \in (0, T_0]$, the maps $u \in W \to F(t, u) \in X$ is Lipschitz continuous in this topology, that is, there is a constant μ_F such that:

$$\|F(t, u) - F(t, v)\|_X \leq \mu_F\|u - v\|_X.$$

Remark about K2. In many cases $A(u)$ is defined for all $u \in Y$, so that, W may be chosen as an arbitrary ball centered in zero.

If $A(u) \in G(X, 1, 0)$, that is, if $(-A(u))$ generates a contraction semigroup, we say that $A(u)$ is *maximally accretive(or m-accretive)*. If $A(u) \in G(X, 1, \beta)$, $A(u)$ is said to be *quasi maximally accretive(or quasi m-accretive)*. If X is a Hilbert space, it can be shown that (see [**16**],[**27**] and [**29**, Vol.II]) $A(u) \in G(X, 1, \beta)$ if and only if

(a) $$\langle A(u)f, f\rangle \geq -\beta\|f\|^2, \forall f \in D(A(u)); u \in W \subset Y.$$

(b) $(A(u) + \lambda)$ is onto for some (and therefore all) $\lambda > \beta$.

Note that (a) means that $A(u)$ is accretive while (b) says that it is maximally so.

Theorem 3.1 (Abstract Local Theory for Quasilinear Equations)**.** *Assume that* $\boldsymbol{K}1$–$\boldsymbol{K}5$ *are satisfied. Then there exist* $T \in (0, T_0]$ *and a unique* $u \in C([0, T]; Y)$ *such that* (6) *is satisfied with the derivative taken with respect to the norm of* X.

This theory is studied in [**8**],[**9**],[**15**] and [**23**].

We are now in position to state the main result of this section.

Theorem 3.2 (Existence and Uniqueness)**.** *Let $\vec{\Theta}(\rho) = J^{-2}\nabla P(\rho)$, $J = (1-\Delta)^{\frac{1}{2}}$. Define*

$$A(\rho)f = -\operatorname{div}(f\, J^{-2}\nabla P(\rho)) = -\operatorname{div}(f\vec{\Theta}(\rho)),$$

so that the partial differential equation in (4) can be written as

$$\partial_t \rho + A(\rho)\rho = F(t,\rho).$$

Let $\rho_0 \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$ and assume that P and F satisfy the following assumptions:

(a) *P maps $H^s(\mathbb{R}^n)$ into itself, $P(0) = 0$ and is Lipschitz in the following senses:*

$$\|P(\rho) - P(\widetilde{\rho})\|_s \le L_s(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\|_s \tag{7}$$

$$\|P(\rho) - P(\widetilde{\rho})\| \le \widetilde{L_s}(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\|, \tag{8}$$

where $L_s, \widetilde{L_s} : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and monotone non-decreasing with respect to each of its arguments.

(b) *$F : [0,T_0] \times H^s(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n)$, $F(t,0) = 0$ and satisfies the following Lipschitz conditions:*

$$\|F(t,\rho) - F(t,\widetilde{\rho})\|_s \le M_s(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\|_s \tag{9}$$

$$\|F(t,\rho) - F(t,\widetilde{\rho})\| \le \widetilde{M_s}(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\|, \tag{10}$$

where $M_s, \widetilde{M_s} : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and monotone non-decreasing with respect to each of its arguments.

(c) *For each $\rho \in W$, the map $t \in (0,T_0] \to F(t,\rho)$ is continuous with respect to the topology of X.*

Then there exists $T \in (0,T_0]$ and unique $\rho \in C([0,T], H^s)$ such that (4) is satisfied with the derivative taken with respect to the norm of H^{s-1}.

Remark. The definition of the norm in H^s- space and the properties of the Fourier Transform prove that the operators $J^{-2} \in B(H^s, H^{s+2})$ and $\partial_{x_i} \in B(H^s, H^{s-1})$, $i = \overline{1,n}$. This facts are usually used in the proof of this theorem and in the rest of the work.

Proof. Let $S = (1-\Delta)^{\frac{s}{2}} = J^s$, $X = L^2(\mathbb{R}^n)$, $Y = H^s(\mathbb{R}^n)$, then we can verify the assumptions **(*K*1–*K*5)** as in [**1**].

(*K*1) We prove that S is an isomorphism from Y to X.
Let $f \in Y$, applying Parseval identity we have:

$$\|S(f)\| = \|J^s(f)\| = \left\|(1+\xi^2)^{\frac{s}{2}}\hat{f}(\xi)\right\| = \|f\|_s.$$

(*K*2) Since X is a Hilbert space, it is sufficient to prove that $A(\rho)$ is maximally accretive in X.

$$(a)\ \big\langle A(\rho)f, f\big\rangle \geq -\beta\|f\|^2, \forall f \in D(A(\rho)) = Y; \rho \in W \subset Y. \tag{11}$$

Integration by parts and Sobolev's lemma implies

$$\big\langle A(\rho)f, f\big\rangle = \big\langle -\operatorname{div}(f\,\vec{\Theta}(\rho)), f\big\rangle \geq -\underbrace{\frac{\|\operatorname{div}\vec{\Theta}(\rho)\|_{L^\infty}}{2}}_{\beta}\|f\|^2.$$

$$(b)\ Rg(A(\rho)+\lambda) = X = L^2(\mathbb{R}^n), \forall\lambda > \beta.$$

The fact that $A(\rho)$ is a closed operator combined with the inequality (11) shows that $(A(\rho)+\lambda)$ has closed range for all $\lambda > \beta$. Thus it enough to show that $(A(\rho)+\lambda)$ has dense range for $\lambda > \beta$. For this, it is sufficient to prove that $R(A(\rho)+\lambda)^\perp = \{0\}$, because $A(\rho)$ is a linear operator.

Let $g \in L^2(\mathbb{R}^n)$, we will prove that:

$$\big\langle (A(\rho)+\lambda)f, g\big\rangle = 0, \forall f \in D(A(\rho)) = H^s(\mathbb{R}^n).$$

Integrating by parts, we have

$$\begin{aligned}\big\langle (A(\rho)+\lambda)f, g\big\rangle = 0 &\Rightarrow \big\langle A(\rho)f, g\big\rangle + \big\langle \lambda f, g\big\rangle = 0 \Rightarrow \big\langle f, \nabla g\,\vec{\Theta}(\rho)\big\rangle + \big\langle \lambda f, g\big\rangle = 0\\ &\Rightarrow \big\langle f, \nabla g\,\vec{\Theta}(\rho) + \lambda g\big\rangle = 0,\ \forall f \in D(A(\rho)) = H^s(\mathbb{R}^n)\\ &\Rightarrow \nabla g\,\vec{\Theta}(\rho) + \lambda g = 0.\end{aligned}$$

Therefore, multiplying by g, integrating by parts, and using (11) we have:

$$\begin{aligned}g\nabla g\,\vec{\Theta}(\rho) + \lambda g^2 = 0 &\Rightarrow \frac{1}{2}\int \nabla(g^2)\,\vec{\Theta}(\rho)\,dx + \lambda\|g\|^2 = 0\\ &\Rightarrow \underbrace{-\frac{1}{2}\int g^2 \operatorname{div}\vec{\Theta}(\rho)\,dx}_{=\langle A(\rho)g,g\rangle} + \lambda\|g\|^2 = 0 \Rightarrow \big\langle A(\rho)g, g\big\rangle + \lambda\|g\|^2 = 0\\ &\Rightarrow 0 \geq -\beta\|g\|^2 + \lambda\|g\|^2 = (\lambda-\beta)\|g\|^2 \Rightarrow g = 0.\end{aligned}$$

(*K*3) At this point, we use the following Lemma:

Lemma 3.1. *Let* $J = (1-\Delta)^{\frac{1}{2}}, s > \frac{n}{2}+1$. *Then*

$$\big\|[J^s, M_f]g\big\| \leq c\|\nabla f\|_{s-1}\|g\|_{s-1}.$$

Proof. See [**19**, Appendix, pg. 122]. □

Let $W = \{\rho \in H^s(\mathbb{R}^n) : \|\rho\|_s \leq R\}$ and $B(\rho) = [S, A(\rho)]S^{-1}$, then:

$B(\rho) = [S, A(\rho)]S^{-1} \in B(L^2) \Leftrightarrow [S, A(\rho)] \in B(H^s, L^2)$, i.e, $\|[S, A(\rho)]\|_{B(H^s,L^2)} \leq \mu_B$.

Let $f \in H^s(\mathbb{R}^n)$ so that,

$$
\begin{aligned}
[S, A(\rho)]f = SA(\rho)f - A(\rho)Sf &= -J^s \mathrm{div}\,(f\vec{\Theta}(\rho)) + \mathrm{div}\,((J^s f)\vec{\Theta}(\rho)) \\
&= -J^s \Big[\sum_{i=1}^{n} \partial_{x_i}(f\Theta_i(\rho))\Big] + \sum_{i=1}^{n} \partial_{x_i}((J^s f)\Theta_i(\rho)) \\
&= \underbrace{-\sum_{i=1}^{n} \big[J^s, \partial_{x_i}\Theta_i(\rho)\big] f}_{A} \underbrace{- \sum_{i=1}^{n} \big[J^s, \Theta_i(\rho)\big] \partial_{x_i} f}_{B}\,.
\end{aligned}
$$

Using Lemma 3.1, $\|J^{-2}\partial_{x_i}\|_{B(H^s,H^{s+1})} \leq 1$ and (7), we obtain

$$
\|A\| \leq \sum_{i=1}^{n} \big\|[J^s, \partial_{x_i}\Theta_i(\rho)]f\big\| \leq c\, n\sqrt{n}\, L_s(\|\rho\|_s, 0)\|\rho\|_s\|f\|_s \leq \mu(R)\|f\|_s,
$$

$$
\|B\| \leq \sum_{i=1}^{n} \big\|[J^s, \Theta_i(\rho)]\partial_{x_i} f\big\| \leq c\, n\sqrt{n}\, L_s(\|\rho\|_s, 0)\|\rho\|_s\|f\|_s \leq \mu(R)\|f\|_s.
$$

Then

$$
\|[S, A(\rho)]f\| \leq \|A\| + \|B\| \leq 2\mu(R)\|f\| \Rightarrow \|[S, A(\rho)]\|_{B(H^s,L^2)} \leq 2\mu(R) = \mu_B(R).
$$

(K4) Let $D(A(\rho)) = H^s(\mathbb{R}^n)$, we must prove the following inequality

$$
\|A(\rho) - A(\widetilde{\rho})\|_{B(H^s,L^2)} \leqq \mu_A\|\rho - \widetilde{\rho}\|.
$$

Let $f \in H^s(\mathbb{R}^n)$,

$$
\begin{aligned}
\|(A(\rho) - A(\tilde{\rho}))f\| &= \|\mathrm{div}\,(f\vec{\Theta}(\tilde{\rho})) - \mathrm{div}\,(f\vec{\Theta}(\rho))\| \\
&\leq \sum_{i=1}^{n} \underbrace{\|(\partial_{x_i} f)(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\|}_{C} + \sum_{i=1}^{n} \underbrace{\|f\partial_{x_i}(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\|}_{D}\,.
\end{aligned}
$$

Sobolev's lemma, $\|J^{-2}\partial_{x_i}\|_{B(L^2,H^1)} \leq 1$ and (8), leads to

$$
C \lesssim \|f\|_s\, \widetilde{L_s}(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|\ ,\ D \lesssim \|f\|_s\, \widetilde{L_s}(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|.
$$

Then

$$
\begin{aligned}
\|(A(\rho) - A(\tilde{\rho}))f\| &\lesssim 2\|f\|_s\, \widetilde{L_s}(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\| \lesssim \mu_A(R)\|\rho - \tilde{\rho}\|\|f\|_s \\
&\Rightarrow \|A(\rho) - A(\widetilde{\rho})\|_{B(H^s,L^2)} \leq \mu_A(R)\|\rho - \widetilde{\rho}\|.
\end{aligned}
$$

(K5) This assumption is satisfied due to the conditions about F in Theorem 3.2 (b), (c). $\square$

Continuous dependence of the initial data are also obtained by Kato's theory (See [**26**, Section 2.3]).

4. PARABOLIC REGULARIZATION OF THE BFE

In this section we begin the analysis of the problem:

$$(12)\quad \begin{array}{l} \partial_t \rho_\mu = \mu\Delta\rho_\mu + \overbrace{\operatorname{div}\left[\rho_\mu J^{-2}\nabla P(\rho_\mu)\right] + F(t,\rho_\mu)}^{=\tilde{F}(t,\rho_\mu)} \in H^{s-2}(\mathbb{R}^n), t \in I = (0, T_0] \\ \rho_\mu(0) = \rho_0 \in H^s(\mathbb{R}^n) \end{array}$$

where $\mu > 0$ and the time derivative is computed in the norm of H^{s-2}.

The nonlinearity $\tilde{F}(t,\rho)$ has the following properties:

Lemma 4.1. *Let $s > \frac{n}{2} + 1$ be fixed and P, F satisfy* (7)–(10) *as in Theorem* 3.2. *Then $\tilde{F}(t,\rho)$ is a continuous map from $I \times H^s$ to H^{s-1} and satisfies the estimates*

$$(13)\qquad \|\tilde{F}(t,\rho) - \tilde{F}(t,\tilde{\rho})\|_{s-1} \leq \gamma(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s,$$

$$(14)\qquad \left\langle \rho - \tilde{\rho}, \tilde{F}(t,\rho) - \tilde{F}(t,\tilde{\rho}) \right\rangle \leq L_0(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|^2,$$

for all $\rho, \tilde{\rho} \in H^s$, where $\gamma, L_0 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous functions, monotone nondecreasing with respect to each of their arguments.

Proof. We have

$$\begin{aligned} \left\|\tilde{F}(t,\rho) - \tilde{F}(t,\tilde{\rho})\right\|_{s-1} \leq & \underbrace{\left\|\operatorname{div}\left[\rho(\vec{\Theta}(\rho) - \vec{\Theta}(\tilde{\rho}))\right]\right\|_{s-1}}_{=F1} + \underbrace{\left\|\operatorname{div}\left[(\rho - \tilde{\rho})\vec{\Theta}(\tilde{\rho})\right]\right\|_{s-1}}_{=F2} \\ & + \left\|F(t,\rho) - F(t,\tilde{\rho})\right\|_{s-1}. \end{aligned}$$

Applying the fact that $H^s(\mathbb{R}^n), s > \frac{n}{2}$ is a Banach algebra, $\|J^{-2}\partial_{x_i}\|_{B(H^s,H^{s+1})} \leq 1\,\forall i$ and (7) we get:

$$F1 = \Big\|\sum_{i=1}^{n} \partial_{x_i}\left[\rho(\Theta_i(\rho) - \Theta_i(\tilde{\rho}))\right]\Big\|_{s-1} \lesssim n\|\rho\|_s L_s(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s,$$

$$F2 = \left\|\sum_{i=1}^{n} \partial_{x_i}\left[(\rho - \tilde{\rho})\Theta_i(\tilde{\rho})\right]\right\|_{s-1} \lesssim nL_s(\|\tilde{\rho}\|_s, 0)\|\rho - \tilde{\rho}\|_s\|\tilde{\rho}\|_s.$$

Finally, using (9), we obtain

$$(15)\qquad \left\|\tilde{F}(t,\rho) - \tilde{F}(t,\tilde{\rho})\right\|_{s-1} \leq \gamma(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho - \tilde{\rho}\|_s$$

with

$$\gamma(\|\rho\|_s, \|\tilde{\rho}\|_s) = n\Big[\|\rho\|_s L_s(\|\rho\|_s, \|\tilde{\rho}\|_s) + \|\tilde{\rho}\|_s L_s(\|\tilde{\rho}\|_s, 0)\Big] + M_s(\|\rho\|_s, \|\tilde{\rho}\|_s).$$

The continuity of $\tilde{F}$ is a consequence of (15).

In order to prove (14), we proceed as follows

$$\begin{aligned}(16)\qquad & \left\langle \rho-\tilde{\rho}, \tilde{F}(t,\rho)-\tilde{F}(t,\tilde{\rho})\right\rangle \\ &= \underbrace{\left\langle \rho-\tilde{\rho}, \operatorname{div}\left[\rho\,\vec{\Theta}(\rho)-\tilde{\rho}\,\vec{\Theta}(\tilde{\rho})\right]\right\rangle}_{=C1} + \underbrace{\left\langle \rho-\tilde{\rho}, F(t,\rho)-F(t,\tilde{\rho})\right\rangle}_{=C2}.\end{aligned}$$

Using integration by parts, the Cauchy-Schwarz inequality, Sobolev's lemma, (8), positivity of the pressure and the inequalities $\|J^{-2}\partial_{x_i}\|_{B(L^2,H^1)} \le 1$, $\|J^{-2}\partial_{x_i}^2\|_{B(L^2)} \le 1, \|J^{-2}\|_{B(H^s,H^{s+2})} \le 1$, we obtain

$$C1 \lesssim \widetilde{C}(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho-\tilde{\rho}\|^2, \tag{17}$$

where

$$\widetilde{C}(\|\rho\|_s, \|\tilde{\rho}\|_s) = 2\,n\|\rho\|_s\widetilde{L_s}(\|\rho\|_s, \|\tilde{\rho}\|_s) + \frac{1}{2}\|\tilde{\rho}\|_s L_s(\|\tilde{\rho}\|_s, 0).$$

Considering (10) in the last term in (16) we have:

$$\begin{aligned}(18)\qquad C2 &\le \left|\left\langle \rho-\tilde{\rho}, F(t,\rho)-F(t,\tilde{\rho})\right\rangle\right| \le \|\rho-\tilde{\rho}\|\|F(t,\rho)-F(t,\tilde{\rho})\| \\ &\le \widetilde{M}_s(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho-\tilde{\rho}\|^2.\end{aligned}$$

Finally, substituting (17) and (18) in (16) we have

$$\left\langle \rho-\tilde{\rho}, \tilde{F}(t,\rho)-\tilde{F}(t,\tilde{\rho})\right\rangle \le L_0(\|\rho\|_s, \|\tilde{\rho}\|_s)\|\rho-\tilde{\rho}\|^2,$$

with

$$L_0(\|\rho\|_s, \|\tilde{\rho}\|_s) = \widetilde{C}(\|\rho\|_s, \|\tilde{\rho}\|_s) + \widetilde{M}_s(\|\rho\|_s, \|\tilde{\rho}\|_s).$$

This finishes the proof. (See details in [**26**, Lemma 3.1.1]). □

Applying the Fourier Transform to the linear part of (12), we have:

$$\widehat{\rho_\mu}(\xi) = e^{-\mu t\xi^2}\widehat{\rho}_0(\xi) \Rightarrow \rho_\mu(t) = U_\mu(t)\rho_0 = e^{\mu t\Delta}\rho_0 := (e^{-\mu t\xi^2}\widehat{\rho}_0(\xi))^\vee$$

In the next lemma, we will show the smoothing properties of the semigroup $U_\mu(t) = e^{\mu t\Delta}$.

Lemma 4.2. *Let $\lambda \in [0,\infty), s \in \mathbb{R}$.*

a) *$U_\mu(t) \in B(H^s(\mathbb{R}^n), H^{s+\lambda}(\mathbb{R}^n)), \forall t > 0$ and satisfies:*

$$\|U_\mu(t)(\varphi)\|_{s+\lambda} \le K_\lambda\left[1+\left(\frac{1}{2\mu t}\right)^\lambda\right]^{\frac{1}{2}}\|\varphi\|_s,$$

where $g_\mu(t) = K_\lambda\Big[1+\Big(\frac{1}{2\mu t}\Big)^\lambda\Big]^{\frac{1}{2}} \in L^1_{\ell oc}([0,\infty))$ if $\lambda < 2$, K_λ is a constant depending only on λ.

b) *The maps* $t \in (0,\infty) \to U_\mu(t)\varphi$ *is continuous with respect to the topology of* $H^{s+\lambda}(\mathbb{R}^n)$.

Proof. See [**6**],[**7**] and [**8**]. □

In [**26**] the reader can find a rigorous proof, similar to that in Chapter IV of [**11**], of the fact that the problem (12) is equivalent to the integral equation

$$\rho_\mu(t) = e^{\mu t\Delta}\rho_0 + \int_0^t e^{\mu(t-t')\Delta}\tilde{F}(t',\rho_\mu(t'))\,\mathrm{d}t'. \tag{19}$$

We will prove that the above integral equation, has a unique solution in $C([0,T^\mu];H^s)$ for any $0 < T^\mu \le T_0$ and for all $\mu > 0$.

Theorem 4.1. *Let* $\mu > 0$ *be fixed and* $\rho_0 \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$. *Then there exists* $T^\mu = T(s,\|\rho_0\|_s,\mu)$ *and a unique function* $\rho_\mu \in C([0,T^\mu],H^s) \cap C((0,T^\mu];H^\infty)$ *satisfying the integral equation* (19).

Proof. We have:

$$\rho_\mu(t) = \overbrace{\underbrace{e^{\mu t\Delta}\rho_0}_{\in H^s(\mathbb{R}^n)} + \int_0^t \underbrace{e^{\mu(t-t')\Delta}\ \underbrace{\left[\operatorname{div}\left[\rho_\mu(1-\Delta)^{-1}\nabla P(\rho_\mu)\right] + F(t',\rho_\mu(t'))\right]}_{\in H^{s-1}(\mathbb{R}^n)}\,\mathrm{d}t'}_{\in V = H^{s-1+\lambda}(\mathbb{R}^n)}}^{\in Y = H^s(\mathbb{R}^n),\ (\lambda=1)}.$$

Consider the spaces $V = H^{s-1+\lambda}(\mathbb{R}^n), Y = H^s(\mathbb{R}^n)$. Thus, we have that $V \subseteq Y \subseteq X = H^{s-2}(\mathbb{R}^n)$ if $\lambda \ge 1$. In the rest of the proof of the theorem, we use $\lambda = 1$ for simplicity.

Consider the map

$$B(v(t)) = U_\mu(t)\rho_0 + \int_0^t U_\mu(t-t')\tilde{F}(t',v(t'))\,\mathrm{d}t', \tag{20}$$

defined in the complete metric space

$$X_s(T_0) = \left\{v \in C([0,T_0],H^s(\mathbb{R}^n)) : \left\|v(t) - U_\mu(t)\rho_0\right\| \le M,\ \forall t \in [0,T_0]\right\},$$

when the topology in the space $X_s(T_0)$ is defined by the sup-norm, that is, $d(v,w) = \sup_{t\in[0,T_0]} \|v(t) - w(t)\|_s$, with $v, w \in X_s(T_0)$.

In the proof, we will show that by taking T^μ sufficiently small, the map (20) is a contraction in $X_s(T_0)$. Once this is established, we will show that this is in fact the only possible solution in $C([0,T^\mu],H^s(\mathbb{R}^n))$.

Let $v(t) \in X_s(T_0)$, it is easy to see that $\|v(t)\|_s \le M + \|\rho_0\|_s$. The continuity of the semigroup $U_\mu(t)$ and $\tilde{F}(t,\rho)$ implies that $B(v(t)) \in C([0,T_0],H^s(\mathbb{R}^n))$.

On the other hand, using the properties of the semigroup $U_\mu(t)$ in Lemma 3.1.2 and $\tilde{F}(t, v(t))$, we obtain

$$\begin{aligned}\|B(v(t)) - U_\mu(t)\rho_0\|_s &\le \int_0^t \|U_\mu(t-t')\tilde{F}(t', v(t'))\|_s \,\mathrm{d}t' \\ &\le (M + \|\rho_0\|_s)\gamma(M + \|\rho_0\|_s, 0)\int_0^{T_0} g_\mu(r)\,\mathrm{d}r.\end{aligned}$$

As $g_\mu(r) \in L^1_{loc}([0,\infty))$,

$$\gamma(M + \|\rho_0\|_s, 0)\int_0^{T_0} g_\mu(r)\,\mathrm{d}r \longrightarrow 0, \text{ as } T_0 \to 0.$$

Then

$$\exists\, \tau \in (0, T_0] : \gamma(M + \|\rho_0\|_s, 0)\int_0^{\tau} g_\mu(r)\,\mathrm{d}r \le \frac{M}{M + \|\rho_0\|_s} \le 1.$$

Therefore, we have

$$\exists\, \tau \in (0, T_0] : \|B(v(t)) - U_\mu(t)\rho_0\|_s \le M \Rightarrow B(v(t)) \in X(\tau).$$

Next, we will prove that this map is a contraction: Let $v(t), w(t) \in X(\tau)$

$$\begin{aligned}\|B(v(t)) - B(w(t))\|_s &\le \int_0^t \left\|U_\mu(t-t')[\tilde{F}(t', v(t')) - \tilde{F}(t', w(t'))]\right\|_s \mathrm{d}t' \\ &\le \left[\gamma(M + \|\rho_0\|_s, M + \|\rho_0\|_s)\int_0^t g_\mu(r)\,\mathrm{d}r\right] d(v, w).\end{aligned}$$

Then

$$d(B(v), B(w)) \le \left[\gamma(M + \|\rho_0\|_s, M + \|\rho_0\|_s)\int_0^\tau g_\mu(r)\,\mathrm{d}r\right] d(v, w).$$

Similarly

$$\gamma(M + \|\rho_0\|_s, M + \|\rho_0\|_s)\int_0^\tau g_\mu(r)\,\mathrm{d}r \longrightarrow 0, \text{ as } \tau \to 0.$$

Then

$$\exists\, T^\mu \in (0, \tau] : \gamma(M + \|\rho_0\|_s, M + \|\rho_0\|_s)\int_0^{T^\mu} g_\mu(r)\,\mathrm{d}r = \delta < 1.$$

Therefore, we have

$$\exists\, T^\mu \in (0, \tau] : d(B(v), B(w)) \le \delta\, d(v, w).$$

Existence and uniqueness in $X_s(T_0)$ is a usual application of Banach's Fixed Point theorem. This gives us T^μ and $\rho_\mu \in C([0, T^\mu], H^s(\mathbb{R}^n))$. The fact that $\rho_\mu \in C((0, T^\mu], H^\infty(\mathbb{R}^n))$ now follows from the integral equation using a simple bootstrapping argument with $\lambda \in (1, 2)$.

Next, we deal with uniqueness in $C([0, T^\mu], H^s(\mathbb{R}^n))$. This is an immediate consequence of the following weak continuous dependence result (weak in the sense that we consider the same intervals of existence of solutions).

Lemma 4.3. *Let $\mu > 0$ and $\rho_\mu, \tilde{\rho}_\mu$ be solutions of* (12) *in* $C([0,T^\mu], H^s(\mathbb{R}^n))$ *with initial condition data* $\rho_0, \tilde{\rho}_0$ *respectively. Then*

$$\|\rho_\mu(t) - \tilde{\rho}_\mu(t)\|_s \le e^{\gamma(\widehat{M},\widehat{M})}\|\rho_0 - \tilde{\rho}_0\|_s,$$

where $\widehat{M} = \max\left[\sup_{t\in[0,T^\mu]}\|\rho_\mu(t)\|_s, \sup_{t\in[0,T^\mu]}\|\tilde{\rho}_\mu(t)\|_s\right]$.

Proof. Let $\rho_\mu(t), \tilde{\rho}_\mu(t) \in C((0,T^\mu], H^s(\mathbb{R}^n))$, with initial conditions $\rho_0, \tilde{\rho}_0$ respectively. Then

$$\begin{aligned}
(21) \quad & \|\rho_\mu(t) - \tilde{\rho}_\mu(t)\|_s \\
& \le \|U_\mu(t)(\rho_0 - \tilde{\rho}_0)\|_s + \left\| \int_0^t U_\mu(t-t')\left[\tilde{F}(t', \rho_\mu(t')) - \tilde{F}(t', \tilde{\rho}_\mu(t'))\right] \mathrm{d}t' \right\|_s \\
& \le \|\rho_0 - \tilde{\rho}_0\|_s + \gamma(\widehat{M},\widehat{M}) \int_0^t g_\mu(t-t')\|\rho_\mu(t') - \tilde{\rho}_\mu(t')\|_s \,\mathrm{d}t'.
\end{aligned}$$

Applying Gronwall's inequality in (21)

$$\|\rho_\mu(t) - \tilde{\rho}_\mu(t)\|_s \le e^{\gamma(\widehat{M},\widehat{M})}\|\rho_0 - \tilde{\rho}_0\|_s. \qquad \square$$

Finally, uniqueness of solution in $C([0,T^\mu], H^s(\mathbb{R}^n))$ follows of the above inequality taking the same initial conditions for the solution, i.e,

$$\|\rho_\mu(t) - \tilde{\rho}_\mu(t)\|_s \le 0 \Rightarrow \rho_\mu(t) = \tilde{\rho}_\mu(t), \ \forall t \in [0, T^\mu].$$

This finishes the proof. (See details in [**26**, Theorem 3.1.2]). $\square$

In order to take the limit as μ tends to 0, one must show that it is possible to choose intervals of existence independent of μ. We have:

Lemma 4.4. *Assume that $\mu > 0$ and that P, F satisfy (7),(8) and (9),(10) respectively for some fixed $s > \frac{n}{2}$. Then there exists $\tilde{T}_s = \tilde{T}(s, \|\rho_0\|_s)$ independent of* $\mu > 0$, *such that all solutions $\rho_\mu(t)$ can be extended, if necessary, to $(0, \tilde{T}_s]$ satisfying* $\left\|\rho_\mu(t)\right\|_s^2 \le h(t);\ t \in [0, \tilde{T}_s]$.

Proof. We will show the crucial estimate for the proof (For details see [**26**, Lemma 3.1.4]).

$$\begin{aligned}
\partial_t\|\rho_\mu(t)\|_s^2 &\lesssim M_s\big(\|\rho_\mu(t)\|_s, 0\big)\|\rho_\mu(t)\|_s^2 + \|\rho_\mu(t)\|_s^3 L_s\big(\|\rho_\mu(t)\|_s, 0\big) \\
&= M_s\Big((\|\rho_\mu(t)\|_s^2)^{\frac{1}{2}}, 0\Big)\|\rho_\mu(t)\|_s^2 + (\|\rho_\mu(t)\|_s^2)^{\frac{3}{2}} L_s\Big((\|\rho_\mu(t)\|_s^2)^{\frac{1}{2}}, 0\Big) \\
&= G\Big(\|\rho_\mu(t)\|_s^2\Big).
\end{aligned}$$

Let $h(t)$ be the maximal solution ([**3**]) of initial value problem for ordinary differential equation:

$$\begin{cases} \partial_t h(t) = G(h(t)) \\ h(0) = \|\rho_0\|_s^2 \end{cases}$$

Then $\left\|\rho_\mu(t)\right\|_s^2 \leq h(t);\ t \in [0,\tilde{T}_s];\ \tilde{T}_s \in [0,T_0)$, whenever both sides are defined. This finishes the proof since $h(t)$ not depends of μ and we can extend $\rho_\mu(t)$ to interval $[0,\tilde{T}_s]$. $\square$

We are now in position to state and prove the main results of this Section. For this, we consider the class

$$\Omega(\tilde{T}_s) = C([0,\tilde{T}_s], L^2(\mathbb{R}^n)) \cap C_w([0,\tilde{T}_s], H^s(\mathbb{R}^n)) \cap AC([0,\tilde{T}_s], H^{s-1}(\mathbb{R}^n)).$$

Theorem 4.2. *Let $\rho_0 \in H^s(\mathbb{R}^n),\ s > \dfrac{n}{2}+1$. Then there exists $\tilde{T}_s = \tilde{T}(s,\|\rho_0\|_s) > 0$ and unique $\rho \in \Omega(\tilde{T}_s)$. Moreover $\rho(t)$ satisfies $\partial_t\rho \in C_w([0,\tilde{T}_s], H^{s-1}(\mathbb{R}^n))$, $\|\rho(t)\|_s^2 \leq h(t)$, and the initial value problem (4).*

Proof. We choose any such interval as in the preceding lemma, and write $\rho = \rho_\mu(t),\ \tilde{\rho} = \rho_\nu(t);\ \mu,\nu > 0;\ \rho_\mu(0) = \rho_\nu(0) = \rho_0$. Let $M^2 = \sup_{t\in[0,\tilde{T}_s]} h(t)$, and note that ρ and $\tilde{\rho}$ belong to $H^\infty(\mathbb{R}^n)$ in view of Theorem 4.1.

$$\begin{aligned} \partial_t\|\rho-\tilde{\rho}\|^2 &= 2\big\langle \rho-\tilde{\rho}, \partial_t(\rho-\tilde{\rho})\big\rangle \\ &= 2\Big[\big\langle \rho-\tilde{\rho}, \tilde{F}(t,\rho)-\tilde{F}(t,\tilde{\rho})\big\rangle + \underbrace{\big\langle \rho-\tilde{\rho}, \mu\Delta\rho-\nu\Delta\tilde{\rho}\big\rangle}_{=A}\Big]. \end{aligned} \tag{22}$$

Integration by parts and Cauchy-Schwarz inequality imply that

$$\begin{aligned} A &= \big\langle \rho-\tilde{\rho}, \mu\Delta\rho-\nu\Delta\tilde{\rho}\big\rangle = \big\langle \rho-\tilde{\rho}, (\mu-\nu)\Delta\rho\big\rangle - \overbrace{\nu\big\langle \rho-\tilde{\rho}, H_0(\rho-\tilde{\rho})\big\rangle}^{\geq 0} \\ &\leq |\mu-\nu|\Big|\sum_{i=1}^n \Big\langle \partial_{x_i}\rho, \partial_{x_i}(\rho-\tilde{\rho}\Big\rangle\Big| \\ &\leq |\mu-\nu|\sum_{i=1}^n \overbrace{\|\partial_{x_i}\rho\|}^{\leq\|\rho\|_1\leq\|\rho\|_s\leq M}\Big(\overbrace{\|\partial_{x_i}\rho\|}^{\leq\|\rho\|_1\leq\|\rho\|_s\leq M} + \overbrace{\|\partial_{x_i}\tilde{\rho}\|}^{\leq\|\tilde{\rho}\|_1\leq\|\tilde{\rho}\|_s\leq M}\Big) \\ &\leq 2nM^2|\mu-\nu|. \end{aligned} \tag{23}$$

Finally, substituting (14), (23) in (22); we have:

$$\begin{aligned} \partial_t\|\rho-\tilde{\rho}\|^2 &\leq 4nM^2|\mu-\nu| + 2\,L_0(\|\rho\|_s,\|\tilde{\rho}\|_s)\|\rho-\tilde{\rho}\|^2 \\ &\leq 4nM^2|\mu-\nu| + 2\,L_0(M,M)\|\rho-\tilde{\rho}\|^2. \end{aligned} \tag{24}$$

Integrating the last estimate from 0 to t:

$$\|\rho_\mu(t)-\rho_\nu(t)\|^2 \leq 4\,n\,M^2\tilde{T}_s|\mu-\nu| + \int_0^t 2\,L_0(M,M)\|\rho_\mu(\tau)-\rho_\nu(\tau)\|^2 \mathrm{d}\tau.$$

Gronwall's inequality then shows that

$$\|\rho_\mu(t)-\rho_\nu(t)\|^2 \leq 4\,n\,M^2\tilde{T}_s|\mu-\nu|\,e^{2\,\tilde{T}_s L_0(M,M)},$$

then

$$\lim_{\mu\to 0,\,\nu\to 0} \|\rho_\mu(t)-\rho_\nu(t)\|^2 = 0 \Rightarrow \rho_\mu(t) \longrightarrow \rho_\nu(t) \text{ in } L^2,\ t\in[0,\tilde{T}_s].$$

Now, $\rho_\mu(t)$ is a Cauchy net in the space $L^2(\mathbb{R}^n)$, which is complete. Therefore, there exists $\rho(t)\in C([0,\tilde{T}_s],L^2(\mathbb{R}^n))$ that satisfies

$$\lim_{\mu\to 0}\sup_{[0,\tilde{T}_s]}\|\rho_\mu(t)-\rho(t)\| = 0.$$

Thus $t\in[0,\tilde{T}_s]\to\rho_\mu(t)$ is continuous and uniformly bounded in $L^2(\mathbb{R}^n)$.

We claim that $\{\rho_\mu(t)\}_{\mu>0}$ is a weak Cauchy net in $H^s(\mathbb{R}^n)$ uniformly with respect to $t\in[0,\tilde{T}_s]$. Indeed, given $\varphi\in H^s(\mathbb{R}^n)$ and $\epsilon>0$, choose $\varphi_\epsilon\in S(\mathbb{R}^n)$ such that $\|\varphi-\varphi_\epsilon\|_s<\epsilon$.

$$\begin{aligned}\langle\rho_\mu(t)-\rho_\nu(t),\varphi\rangle_s &= \langle\rho_\mu(t)-\rho_\nu(t),\varphi-\varphi_\epsilon\rangle_s + \langle\rho_\mu(t)-\rho_\nu(t),\varphi_\epsilon\rangle_s\\ &\le |\langle\rho_\mu(t)-\rho_\nu(t),\varphi-\varphi_\epsilon\rangle_s| + |\langle J^s(\rho_\mu(t)-\rho_\nu(t)),J^s\varphi_\epsilon\rangle|\\ &\le \|\rho_\mu(t)-\rho_\nu(t)\|_s\|\varphi-\varphi_\epsilon\|_s + \|\rho_\mu(t)-\rho_\nu(t)\|\|\varphi_\epsilon\|_{2s}\\ &\le 2\,M\,\epsilon + \|\rho_\mu(t)-\rho_\nu(t)\|\|\varphi_\epsilon\|_{2s}.\end{aligned}$$

So that $\lim\limits_{\mu\to 0,\nu\to 0}\sup_{[0,\tilde{T}_s]}\langle\rho_\mu(t)-\rho_\nu(t),\varphi\rangle_s = 0$ uniformly.

Since $H^s(\mathbb{R}^n)$ is reflexive, it is weakly complete ([**5**],[**31**]), and there exists $v(t)\in C_w([0,\tilde{T}_s],H^s(\mathbb{R}^n))$ satisfying

$$\lim_{\mu\to 0}\langle\rho_\mu(t),\varphi\rangle_s = \langle v(t),\varphi\rangle_s\ \forall\varphi\in S(\mathbb{R}^n).$$

It is easy to see that $v(t)=\rho(t)\,\forall t\in[0,\tilde{T}_s]$, as a consequence of uniqueness of weakly limit. In particular, $\rho(t)$ is weakly continuous and uniformly bounded by the function $\sqrt{h(t)}$. Indeed,

$$\begin{aligned}\|\rho(t)\|_s &= \sup_{\|\psi\|_s=1}|\langle\rho(t),\psi\rangle_s| = \sup_{\|\psi\|_s=1}\lim_{\mu\to 0}|\langle\rho_\mu(t),\psi\rangle_s|\\ &\le \sup_{\|\psi\|_s=1}\lim_{\mu\to 0}\|\rho_\mu(t)\|_s\|\psi\|_s \le \sqrt{h(t)}.\end{aligned}$$

It remains to prove that $\partial_t\rho\in C_w([0,\tilde{T}_s],H^{s-1}(\mathbb{R}^n))$. Let $\psi\in H^{s-1}(\mathbb{R}^n)$,

$$\langle\rho_\mu(t),\psi\rangle_{s-1} = \langle U_\mu(t)\rho_0,\psi\rangle_{s-1} + \int_0^t \langle\tilde{F}(t',\rho_\mu(t')),\psi\rangle_{s-1}\,\mathrm{d}t',\forall t\in[0,\tilde{T}_s]. \tag{25}$$

Since $\rho_\mu(t)\rightharpoonup\rho(t)$ in $H^s(\mathbb{R}^n)$, it follows that, $\tilde{F}(t,\rho_\mu(t))\rightharpoonup\tilde{F}(t,\rho(t))$ uniformly in $H^{s-1}(\mathbb{R}^n)$, therefore, taking the limit as $\mu\to 0$ in (25) we obtain

$$\langle\rho(t),\psi\rangle_{s-1} = \langle\rho_0,\psi\rangle_{s-1} + \int_0^t \langle\tilde{F}(t',\rho(t')),\psi\rangle_{s-1}\,\mathrm{d}t',\forall t\in[0,\tilde{T}_s]. \tag{26}$$

As the integrand on the right-hand side of (26) is a continuous function, from the Fundamental Theorem of Calculus, follows that:

$$\langle \partial_t \rho(t), \psi\rangle_{s-1} = \langle \tilde{F}(t,\rho(t)), \psi\rangle_{s-1}, \forall t \in [0,\tilde{T}_s].$$

Since the map $t \in [0,\tilde{T}_s] \longrightarrow \tilde{F}(t,\rho(t))$ is weakly continuous and uniformly bounded, Petti's Theorem ([**31**, Chap.V]) implies that it is strongly measurable. Thus we may define a Bochner integral

$$\int_0^t \tilde{F}(t',\rho(t'))\,\mathrm{d}t'.$$

Combining this remark with (26) we conclude

$$\rho(t) = \rho_0 + \int_0^t \tilde{F}(t',\rho(t'))\,\mathrm{d}t'.$$

Thus $\rho(t) \in AC([0,\tilde{T}_s], H^{s-1}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n)$. Therefore $\partial_t\rho(t)$ exists almost everywhere in $[0,\tilde{T}_s]$ and is given by

$$\partial_t\rho(t) = \tilde{F}(t,\rho(t)) = \operatorname{div}\left[\rho(t)J^{-2}\nabla P(\rho(t))\right] + F(t,\rho(t)), a.e., \; t\in[0,\tilde{T}_s].$$

Next we claim that there is only such function in the class

$$\Omega(\tilde{T}_s) = C([0,\tilde{T}_s], L^2(\mathbb{R}^n)) \cap C_w([0,\tilde{T}_s], H^s(\mathbb{R}^n)) \cap AC([0,\tilde{T}_s], H^{s-1}(\mathbb{R}^n)).$$

Let $\rho(t), \eta(t) \in \Omega(\tilde{T}_s)$ with $\rho(0) = \eta(0) = \rho_0$, a calculation similar to that leading to (24) implies

$$\partial_t\|\rho(t)-\eta(t)\|^2 \le 2\,L_0(M,M)\|\rho(t)-\eta(t)\|^2.$$

Integrating from 0 to t

$$\|\rho(t)-\eta(t)\|^2 \le \underbrace{\|\rho(0)-\eta(0)\|^2}_{=0} + \int_0^t 2\,L_0(M,M)\|\rho(t')-\eta(t')\|^2\mathrm{d}t'$$

Applying Gronwall's lemma in the last estimate, we have:

$$\|\rho(t)-\eta(t)\|^2 \le 0 \Rightarrow \rho(t) = \eta(t) \in \Omega(\tilde{T}_s).$$

Corollary 4.1. *Let $\rho_0 \in H^s(\mathbb{R}^n)$, $s > \dfrac{n}{2}+1$. Then there exists $0 < T'(s,\|\rho_0\|_s) < \tilde{T}_s = \tilde{T}(s,\|\rho_0\|_s)$ such that the initial value problem* (4) *is locally well posed in the space $C([0,T'], H^s(\mathbb{R}^n))$ and the solution satisfies $\|\rho(t)\|_s^2 \le h(t)$.*

Proof. Let $T' \le \min\{T,\tilde{T}_s\}$ where T is the existence time obtained by Kato's theory in Theorem 3.1. Due to the uniqueness established in this Theorem, it follows that the solution obtained by parabolic regularization method, coincides with the one obtained through Kato's theory. □

5. COMPARISON PRINCIPLE FOR THE BFE

Consider the initial value problem (BFE) with $F(t,\rho) = 0$, $P(\rho) = \rho^{2k}, k = 1, 2, 3, \dots$

$$
\left\{\begin{array}{l} \partial_t \rho + \operatorname{div}(\rho\, \vec{\mathbf{v}}) = 0,\ x \in \mathbb{R}^n,\ t \in (0, T_0] \\ \vec{\mathbf{v}} = -\nabla(-\Delta + 1)^{-1}\rho^{2k} = -\vec{\Theta}(\rho) \\ (\rho(0),\ \vec{\mathbf{v}}(0)) = (\rho_0,\ \vec{\mathbf{v}}_0) \end{array}\right. \tag{27}
$$

Theorem 5.1 (Comparison Principle)**.** *Let* $(\rho, \vec{\mathbf{v}})$ *and* $(\eta, \vec{\mathbf{w}})$ *be solutions of* (*BFE*) *with* $P(\rho) = \rho^{2k}, P(\eta) = \eta^{2k}, k = 1, 2, 3. \dots;$ *and initial values* $(\rho_0, \vec{\mathbf{v}}_0)$ *and* $(\eta_0, \vec{\mathbf{w}}_0)$ *respectively. Then*

$$
0 \le \eta_0(x) \le \rho_0(x) \text{ in } \mathbb{R}^n \Rightarrow 0 \le \eta(x,t) \le \rho(x,t) \text{ in } \mathbb{R}^n \times [0, T_0]
$$

Proof. In this proof, we consider the same idea, that was employed by Blanco in the study of Camassa Holm equation ([**2**]).

Let $R(t,y) = \rho(\vec{\phi}(t,y), t); S(t,y) = \eta(\vec{\psi}(t,y), t)$, and $Q(t,y) = R(t,y) - S(t,y)$ where $\vec{\phi}(t,y)$ and $\vec{\psi}(t,y)$ satisfy the following equations respectively,

$$
\left\{\begin{array}{l} \dfrac{\partial \vec{\phi}}{\partial t}(t,y) = \vec{\mathbf{v}}(\vec{\phi}(t,y), t) \\ \vec{\phi}(0,y) = y \end{array}\right. \qquad \begin{array}{c} \vec{\phi}(t,y) = (\phi_1(t,y), \phi_2(t,y), \dots, \phi_n(t,y)) \\ v_i = -\partial_{x_i}(1-\Delta)^{-1}\rho^{2k} \end{array} \tag{28}
$$

and

$$
\left\{\begin{array}{l} \dfrac{\partial \vec{\psi}}{\partial t}(t,y) = \vec{\mathbf{w}}(\vec{\psi}(t,y), t) \\ \vec{\psi}(0,y) = y \end{array}\right. \qquad \begin{array}{c} \vec{\psi}(t,y) = (\psi_1(t,y), \psi_2(t,y), \dots, \psi_n(t,y)) \\ w_i = -\partial_{x_i}(1-\Delta)^{-1}\eta^{2k}. \end{array} \tag{29}
$$

Combining (27) with (28) and (27) with (29), we have that $R(t)$ and $S(t)$ satisfy the following differential equations,

$$
\left\{\begin{array}{ll} \dfrac{\mathrm{d}R}{\mathrm{d}t} = -R \operatorname{div} \vec{\mathbf{v}} & \dfrac{\mathrm{d}S}{\mathrm{d}t} = -S \operatorname{div} \vec{\mathbf{w}} \\ R(0,y) = \rho_0(y) & S(0,y) = \eta_0(y). \end{array}\right. \tag{30}
$$

Solving (30), we obtain:

$$
R(t) = R(0) \exp\left[-\int_0^t \operatorname{div} \vec{\mathbf{v}}(\vec{\phi}(s,y), s)\, \mathrm{d}s \right] \overset{\rho_0(y) \ge 0}{\Longrightarrow} R(t) \ge 0.
$$

Analogously we have that:

$$
S(t) = S(0) \exp\left[-\int_0^t \operatorname{div} \vec{\mathbf{w}}(\vec{\psi}(s,y), s) \mathrm{d}s \right] \overset{\eta_0(y) \ge 0}{\Longrightarrow} S(t) \ge 0.
$$

On the other hand, differentiating $Q(t)$:

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\mathrm{d}R}{\mathrm{d}t} - \frac{\mathrm{d}S}{\mathrm{d}t} = (-\mathrm{div}\,\vec{\mathbf{v}})R(t) + (\mathrm{div}\,\vec{\mathbf{w}})S(t) \\ &= -\rho\,\mathrm{div}\,\vec{\mathbf{v}} + \eta\,\mathrm{div}\,\vec{\mathbf{w}} = -(\rho-\eta)\mathrm{div}\,\vec{\mathbf{v}} + \eta(\mathrm{div}\,\vec{\mathbf{w}} - \mathrm{div}\,\vec{\mathbf{v}}) \\ &= -Q(t)(\mathrm{div}\,\vec{\mathbf{v}}) + S(t)(\mathrm{div}\,\vec{\mathbf{w}} - \mathrm{div}\,\vec{\mathbf{v}}),\end{aligned} \tag{31}$$

where

$$\mathrm{div}\,\vec{\mathbf{v}} = \rho^{2k} - (1-\Delta)^{-1}\rho^{2k}, \quad \mathrm{div}\,\vec{\mathbf{w}} = \eta^{2k} - (1-\Delta)^{-1}\eta^{2k}. \tag{32}$$

Substituting (32) in (31), we obtain a new ordinary differential equation for $Q(t)$, i.e:

$$\begin{cases} \dfrac{\mathrm{d}Q}{\mathrm{d}t} = -\big[\mathrm{div}\,\vec{\mathbf{v}} + S(t)P(R(t),S(t))\big]Q(t) + B(t,Q(t)) \\ Q(0) = \rho_0(y) - \eta_0(y), \end{cases} \tag{33}$$

with

$$P(R(t),S(t)) = P(\rho,\eta) = \sum_{i=0}^{2k-1} \rho^{2k-1-i}\eta^i$$

and

$$B(t,Q(t)) = S(t)(1-\Delta)^{-1}\Big[Q(t)P(R(t),S(t))\Big].$$

Applying the method of variation of parameters in (33), the integral solution is:

$$Q(t) = U(t,0)Q(0) + \int_0^t U(t,s)B(s,Q(s))\,\mathrm{d}s,$$

where

$$U(t,s) = \exp\left[-\int_s^t [\mathrm{div}\,(\vec{\mathbf{v}}(\vec{\phi}(\tau,y),\tau)) + S(\tau)P(R(\tau),S(\tau))]\mathrm{d}\tau\right].$$

Consider the sequence

$$\begin{aligned} Q_{n+1}(t) &= U(t,0)Q(0) + \int_0^t U(t,s)B(s,Q_n(s))\mathrm{d}s \qquad n = 0,1,2,\ldots. \\ Q_0(t) &= Q(0) = \rho_0(y) - \eta_0(y). \end{aligned}$$

If $Q(0) \geq 0,$ then $Q_n(t) \geq 0,$ for all n. Thus

$$Q(t) = \rho\big(\vec{\phi}(t,y),t\big) - \eta\big(\vec{\psi}(t,y),t\big) = \lim_{n\to\infty} Q_n(t) \geq 0.$$

To complete the proof we need to show that the functions $y \in \mathbb{R}^n \to \vec{\phi}(t,y) \in \mathbb{R}^n$ and $y \in \mathbb{R}^n \to \vec{\psi}(t,y) \in \mathbb{R}^n$ are onto. To do this, we analyze in detail the map $y \in \mathbb{R}^n \to \vec{\phi}(t,y) \in \mathbb{R}^n$.

Integrating (28) from 0 to t, we obtain:

$$\phi_i(t) - y_i = \int_0^t v_i(\vec{\phi}(s,y),s)\,\mathrm{d}s\,;\; i = 1,2,\ldots.$$

Then

$$|\phi_i(t) - y_i| \leq \int_0^t |v_i(\vec{\phi}(s,y),s)|\,\mathrm{d}s \leq a_i(\|\rho_0\|_s,t)t\;, i = 1,2,\ldots\;;\; s > \frac{n}{2}$$

$$y_i - a_i(\|\rho_0\|_s,t) \leq \phi_i(t,y) \leq y_i + a_i(\|\rho_0\|_s,t),\ \forall y_i \in \mathbb{R}.$$

Taking $z_i \in \mathbb{R}$; $y_i^{(1)} << 0, y_i^{(2)} >> 0$ such that $z_i \in (y_i^{(1)}, y_i^{(2)})$ we have:

$$y_i^{(1)} + a_i(\|\rho_0\|_s,t) < z_i < y_i^{(2)} - a_i(\|\rho_0\|_s,t)$$

Therefore

$$\phi_i(t,y_i^{(1)}) < z_i < \phi_i(t,y_i^{(2)}).$$

The Mean Value theorem for continuous functions, applied to ϕ_i, implies that exists $y_i \in (y_i^{(1)}, y_i^{(2)})$ and satisfies $\phi_i(t,y_i) = z_i$.

A similar argument, proves that the map $y \in \mathbb{R}^n \longrightarrow \vec{\psi}(t,y)$ is onto.

6. GLOBAL ESTIMATES IN $H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$

In this section we obtain the global H^s-estimate for the solution of the Brinkman flow equation. This will be a consequence of global-well posedness of the regularized problem.

First, we will introduce the following estimates.

Lemma 6.1. *If $s > 0$ and $1 < p < \infty$, then*

$$\Big\| \sum_{k=1}^n \big[\partial_{x_k} J^s(g\partial_{x_k} f) - \partial_{x_k} f(\partial_{x_k} J^s g)\big] \Big\|_{L^p} \leq c\Big(\|J^2 f\|_{L^\infty}\|J^s g\|_{L^p} + \|J^{s+2} f\|_{L^p}\|g\|_{L^\infty}\Big).$$

Proof. The proof of this lemma is similar to that of Lemma X1 in [**22**], and is based on the following result due to R. R.Coifman and Y. Meyer (Lemma A.1.2) (See [**26**, Lemma A.1.3]). □

Lemma 6.2. *If $s > 0$ and $1 < p < \infty$, then $L_s^p \cap L^\infty$ is a Banach Algebra. Moreover*

$$\|fg\|_{s,p} \leq c(\|f\|_{L^\infty}\|g\|_{L^p} + \|f\|_{L^p}\|g\|_{L^\infty}).$$

Proof. See [**22**] □

Lemma 6.3. *Let $f \in H^s = L_s^2, s > \frac{n}{2}, k = 1,2,\ldots.$ Then*

$$\|f^{2k}\|_s \lesssim \|f\|_{L^\infty}^{2k-1}\|f\|_s.$$

Proof. Since $f \in H^s, s > \dfrac{n}{2}$, Sobolev's lemma implies that $f \in L^\infty$. Then $f \in L^2_s \cap L^\infty, s > \dfrac{n}{2} > 0$. Combining induction principle, and Lemma 6.2, we have the desired estimate. For details, see [**26**, Corollary A.1.1]) □

Now, we are ready to establish the following result.

Theorem 6.1 (Global Solution)**.** *Let* $s > \dfrac{n}{2} + 1,\ P(\rho) = \rho^{2k},\ F \equiv 0$ *and* $\rho_0 \in H^s(\mathbb{R}^n)$ *with* $0 \leq \rho_0(x) \leq 1$ *in* $\mathbb{R}^n$*. Then* (27) *is globally well-posed in the sense described in Section* 2 *and satisfies* $0 \leq \rho(x,t) \leq 1,\ \forall t \geq 0$.

Proof. From the Comparison principle follows that $0 \leq \rho(x,t) \leq 1$. Using the regularized initial value problem, with the simplified notations $\rho_\mu(t) \equiv \tilde{\rho}$; $\vec{\mathbf{v}}_\mu(t) \equiv \vec{\mathbf{v}}$.

$$
\begin{cases}
\partial_t \tilde{\rho} - \mu \Delta \tilde{\rho} + \operatorname{div}[\tilde{\rho}\,\vec{\mathbf{v}}] = 0 \\
\vec{\mathbf{v}} - \Delta \vec{\mathbf{v}} = -\nabla \tilde{\rho}^{2k} \\
(\tilde{\rho}(0), \vec{\mathbf{v}}(0)) = (\tilde{\rho}_0, \vec{\mathbf{v}}_0)
\end{cases} \tag{34}
$$

We have that $\vec{\mathbf{v}} = -(1-\Delta)^{-1}\nabla\tilde{\rho}^{2k} = -J^{-2}\nabla\tilde{\rho}^{2k} = -\vec{\Theta}(\tilde{\rho})$.

Applying J^s to the regularized equation:

$$
\frac{\mathrm{d}}{\mathrm{d}t}(J^s\tilde{\rho}) - \mu(J^s\Delta\tilde{\rho}) + J^s \operatorname{div}(\tilde{\rho}\,\vec{\mathbf{v}}) = 0. \tag{35}
$$

Multiplying (35) by $J^s\tilde{\rho}$ and integrate over $\mathbb{R}^n$

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (J^s\tilde{\rho})^2\,\mathrm{d}x = \mu\int (J^s\tilde{\rho})J^s(\Delta\tilde{\rho})\,\mathrm{d}x - \int (J^s\tilde{\rho})(J^s\operatorname{div}(\tilde{\rho}\,\vec{\mathbf{v}}))\,\mathrm{d}x,
$$

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (J^s\tilde{\rho})^2\,\mathrm{d}x = \underbrace{\mu\int (J^s\tilde{\rho})\Delta(J^s\tilde{\rho})\,\mathrm{d}x}_{\leq 0} - \sum_{i=1}^{n}\int (J^s\tilde{\rho})\partial_{x_i}J^s(\tilde{\rho}\,v_i)\,\mathrm{d}x.
$$

Using the commutator $[\partial_{x_i}J^s, v_i]\tilde{\rho} = \partial_{x_i}J^s(\tilde{\rho}\,v_i) - v_i\partial_{x_i}J^s\tilde{\rho}$, we obtain:

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (J^s\tilde{\rho})^2\,\mathrm{d}x \leq -\sum_{i=1}^{n}\int (J^s\tilde{\rho})[\partial_{x_i}J^s, v_i]\tilde{\rho}\,\mathrm{d}x - \sum_{i=1}^{n}\int (J^s\tilde{\rho})v_i\partial_{x_i}J^s\tilde{\rho}\,\mathrm{d}x. \tag{36}
$$

Integrating (36) by parts,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (J^s\tilde{\rho})^2\,\mathrm{d}x \leq -\sum_{i=1}^{n}\int (J^s\tilde{\rho})[\partial_{x_i}J^s, v_i]\tilde{\rho}\,\mathrm{d}x + \frac{1}{2}\int (J^s\tilde{\rho})^2\operatorname{div}\vec{\mathbf{v}}\,\mathrm{d}x. \tag{37}
$$

Using (32) in (37)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (J^s\tilde{\rho})^2\,\mathrm{d}x \le -\sum_{i=1}^{n}\int (J^s\tilde{\rho})[\partial_{x_i}J^s, v_i]\tilde{\rho}\,\mathrm{d}x + \frac{1}{2}\int (J^s\tilde{\rho})^2\tilde{\rho}^{2k}\,\mathrm{d}x - \frac{1}{2}\int (J^s\tilde{\rho})^2(1-\Delta)^{-1}\tilde{\rho}^{2k}\,\mathrm{d}x. \tag{38}$$

From the second equation in (34) we have $v_i = -\partial_{x_i}(1-\Delta)^{-1}\tilde{\rho}^{2k}$. Substituting in (38)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (J^s\tilde{\rho})^2\,\mathrm{d}x \le \int (J^s\tilde{\rho})^2\tilde{\rho}^{2k}\,\mathrm{d}x - \overbrace{\int (J^s\tilde{\rho})^2(1-\Delta)^{-1}\tilde{\rho}^{2k}\,\mathrm{d}x}^{\ge 0} + 2\int (J^s\tilde{\rho})\Big(\sum_{i=1}^{n}[\partial_{x_i}J^s, \partial_{x_i}(1-\Delta)^{-1}\tilde{\rho}^{2k}]\tilde{\rho}\Big)\,\mathrm{d}x$$

Observing that the third term it is non negative, and applying the Cauchy-Schwarz inequality in the fourth term; we observe:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (J^s\tilde{\rho})^2\,\mathrm{d}x \le \|\tilde{\rho}^{2k}\|_{L^\infty}\int (J^s\tilde{\rho})^2\,\mathrm{d}x + \ 2\ \|J^s\tilde{\rho}\|\Big\|\sum_{i=1}^{n}\big[\partial_{x_i}J^s, \partial_{x_i}(1-\Delta)^{-1}\tilde{\rho}^{2k}\big]\tilde{\rho}\Big\|. \tag{39}$$

Using Lemma 6.1 in (39), with $f = (1-\Delta)^{-1}\tilde{\rho}^{2k}$ and $g = \tilde{\rho}$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\rho}\|_s^2 \le \|\tilde{\rho}^{2k}\|_{L^\infty}\|\tilde{\rho}\|_s^2 + 2c\|\tilde{\rho}\|_s\Big[\|\tilde{\rho}^{2k}\|_{L^\infty}\|\tilde{\rho}\|_s + \|\tilde{\rho}^{2k}\|_s\|\tilde{\rho}\|_{L^\infty}\Big]. \tag{40}$$

Applying Lemma 6.3 in (40):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\rho}\|_s^2 \lesssim \|\tilde{\rho}\|_{L^\infty}^{2k}\|\tilde{\rho}\|_s^2.$$

In the following we need to estimate $\|\tilde{\rho}\|_{L^\infty}$. Applying the Comparison principle for ρ and Sobolev's lemma we have

$$\|\tilde{\rho}\|_{L^\infty} \le \|\tilde{\rho}-\rho\|_{L^\infty} + \|\rho\|_{L^\infty} \lesssim 1 + \|\tilde{\rho}-\rho\|_s.$$

Next, we compute $\|\tilde{\rho}-\rho\|_s$ using $\|\tilde{\rho}-\rho\|_s = \sup_{\|\varphi\|_s=1}|\langle\tilde{\rho}-\rho,\varphi\rangle_s|$. To do this, recall that in the analysis of weak convergence of sequence ρ_μ (in the proof of Theorem 4.2) that we have obtained the following estimate

$$\begin{aligned} |\langle\rho_\mu(t)-\rho_\nu(t),\varphi\rangle_s| &\le \|\rho_\mu(t)-\rho_\nu(t)\|_s\|\varphi-\varphi_\epsilon\|_s + \|\rho_\mu(t)-\rho_\nu(t)\|\|\varphi_\epsilon\|_{2s} \\ &\le 2M\epsilon + \|\rho_\mu(t)-\rho_\nu(t)\|\|\varphi_\epsilon\|_{2s}. \end{aligned} \tag{41}$$

Applying limit as $\nu \to 0$ in (41)

$$|\langle \rho_\mu(t) - \rho(t), \varphi\rangle_s| \leq 2M\epsilon + \|\rho_\mu(t) - \rho(t)\| \|\varphi_\epsilon\|_{2s}. \tag{42}$$

Considering that $\|\rho_\mu(t) - \rho_\nu(t)\| \leq 2M\sqrt{n\tilde{T}_s|\mu-\nu|}e^{\tilde{T}_s L_0(M,M)}$ (see Theorem 4.2) and taking the limit as $\nu \to 0$

$$\|\rho_\mu(t) - \rho(t)\| \leq 2M\sqrt{n\,\tilde{T}_s\,\mu}\,e^{\tilde{T}_s L_0(M,M)} = \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}. \tag{43}$$

Substituting (43) in (42) and considering that $\|\varphi_\epsilon\|_{2s} \leq \epsilon^{-s}\|\varphi\|_s$ with φ_ϵ constructed as in [**10**, Lemma 2.6, pg 900], yields

$$|\langle \rho_\mu(t) - \rho(t), \varphi\rangle_s| \leq 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\|\varphi\|_s.$$

Then

$$\|\tilde{\rho} - \rho\|_s = \sup_{\|\varphi\|_s=1} |\langle \tilde{\rho} - \rho, \varphi\rangle_s| \leq 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}$$

and

$$\|\tilde{\rho}\|_{L^\infty} \lesssim 1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}, \forall \epsilon > 0.$$

Let $r(\tau) = \tau^{2k}$ a non-decreasing function, it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\rho}\|_s^2 \lesssim r(1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s})\|\tilde{\rho}\|_s^2. \tag{44}$$

Integrating (44) from 0 to t, we get

$$\|\tilde{\rho}\|_s^2 \lesssim \|\rho_0\|_s^2 + r(1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}) \int_0^t \|\tilde{\rho}(\tau)\|_s^2 \,\mathrm{d}\tau. \tag{45}$$

From Gronwall's inequality in (45), follows a priori-estimate in $H^s(\mathbb{R}^n)$; $s > \dfrac{n}{2} + 1$,

$$\|\tilde{\rho}\|_s^2 \lesssim \|\rho_0\|_s^2 \; e^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s}, \; \forall \tilde{T}_s > 0, \forall \epsilon > 0. \tag{46}$$

Finally, applying [**31**, Theo. 1, pg 120] in (46) we obtain the final estimate

$$\begin{aligned}
\|\rho(t)\|_s^2 &\leq \liminf_{\mu\to 0} \|\rho_\mu(t)\|_s^2 \\
&\leq \liminf_{\mu\to 0} \|\rho_0\|_s^2 \; e^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s} \\
&= \lim_{\mu\to 0} \|\rho_0\|_s^2 \; e^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s} \\
&= \|\rho_0\|_s^2 \; e^{r(1+2M\epsilon)\tilde{T}_s} \; \forall \epsilon > 0.
\end{aligned}$$

Therefore, applying limit as ϵ tends to zero, the final estimate follows, that is,

$$\|\rho(t)\|_s^2 \leq \|\rho_0\|_s^2 \; e^{\tilde{T}_s}, \; \forall t \in [0, \tilde{T}_s].$$

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Facultad de Ciencias,
Departamento de Matemática,
Universidad Católica del Norte (UCN),
Avenida Angamos 0610, Antofagasta
Chile
E-mail: mmolina01@ucn.cl

Universidade Federal de Goiás,
IME, GO
Brasil
E-mail: alarcon@mat.ufg.br

Instituto Nacional de Matemática Pura e Aplicada (IMPA)
RJ, Brazil
E-mail: rafael@impa.br