Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **6** (2012), 287–303. doi:10.2298/AADM120619014K

FUSION FRAMES AND G-FRAMES IN TENSOR PRODUCT AND DIRECT SUM OF HILBERT SPACES

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In this paper we study fusion frames and g-frames for the tensor products and direct sums of Hilbert spaces. We show that the tensor product of a finite number of g-frames (resp. fusion frames, g-Riesz bases) is a g-frame (resp. fusion frame, g-Riesz basis) for the tensor product space and vice versa. Moreover we obtain some important results in tensor products and direct sums of g-frames, fusion frames, resolutions of the identity and duals.

1. INTRODUCTION

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [**10**] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [**8**]. Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory, sigma-delta quantization, signal and image processing and wireless communications.

Fusion frame is a generalization of frame which was introduced in [**5**] and investigated in [**2, 6, 21**]. Fusion frames have important applications e.g., in sensor networks and packet encoding.

Sun in [**23**] introduced g-frame as a generalization of frame. He showed that oblique frames, pseudo frames and fusion frames are special cases of g-frames.

Note that fusion frames and g-frames have been introduced in Hilbert C^*-modules and Banach spaces, see [**18, 19**].

2010 Mathematics Subject Classification. 41A58, 42C15.
Keywords and Phrases. Fusion frames, g-frames, tensor products, direct sums.

Let H be a Hilbert space and let $\mathcal{I}$ be a finite or countable index set. A family $\{f_i\}_{i\in\mathcal{I}} \subseteq H$ is a *frame* for H, if there exist $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{i\in\mathcal{I}} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. A family $\{f_i\}_{i\in\mathcal{I}} \subseteq H$ is *complete* if the span of $\{f_i\}_{i\in\mathcal{I}}$ is dense in H. We say that $\{f_i\}_{i\in\mathcal{I}}$ is a *Riesz basis* for H, if it is complete in H and there exist two constants $0 < A \leq B < \infty$, such that

$$A\sum_{i\in F} |c_i|^2 \leq \|\sum_{i\in F} c_i f_i\|^2 \leq B\sum_{i\in F} |c_i|^2,$$

for each sequence of scalars $\{c_i\}_{i\in F}$, where F is a finite subset of $\mathcal{I}$. For more study about frames see [**7**].

For each $i \in \mathcal{I}$, let H_i be a Hilbert space and $L(H, H_i)$ be the set of all bounded, linear operators from H to H_i. We call $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ a *g-frame* for H with respect to $\{H_i\}_{i\in\mathcal{I}}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2,$$

for each $f \in H$. In this case we say that Λ is an (A, B) g-frame. A and B are the lower and upper g-frame bounds, respectively. If $A = B$, then Λ is called an *A-tight* g-frame. We call Λ a *Parseval* g-frame if $A = B = 1$. If only the second inequality is required, we call it a *g-Bessel sequence*. If Λ is a g-Bessel sequence with upper bound B, then the *g-frame operator* S_Λ is defined by $S_\Lambda f = \sum_{i\in\mathcal{I}} \Lambda_i^*\Lambda_i f$. In this case S_Λ is bounded and $0 \leq S_\Lambda \leq B.Id_H$. If Λ is an (A, B) g-frame, then S_Λ is a bounded, positive and invertible operator such that $A.Id_H \leq S_\Lambda \leq B.Id_H$. Recall that if Λ is a g-Bessel sequence such that S_Λ is invertible, then Λ is a g-frame. In this case $\|S_\Lambda^{-1}\|^{-1}$ is a lower bound for Λ.

Let $\{H_i\}_{i\in\mathcal{I}}$ be a sequence of Hilbert spaces. Then by considering $K = \oplus_{i\in\mathcal{I}} H_i$, we can assume that each H_i is a closed subspace of K, therefore if $f_{i_1} \in H_{i_1}$ and $f_{i_2} \in H_{i_2}$, for $i_1, i_2 \in \mathcal{I}$, then $\langle f_{i_1}, f_{i_2}\rangle$ is well-defined.

We say that $\{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ is *g-complete* if $\{f : \Lambda_i f = 0, \forall i \in \mathcal{I}\} = \{0\}$, and we call it a *g-orthonormal basis* for H, if

$$\langle \Lambda_{i_1}^* f_{i_1}, \Lambda_{i_2}^* f_{i_2}\rangle = \delta_{i_1,i_2}\langle f_{i_1}, f_{i_2}\rangle, \quad i_1, i_2 \in \mathcal{I}, f_{i_1} \in H_{i_1}, f_{i_2} \in H_{i_2},$$

and

$$\sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 = \|f\|^2, \quad \forall f \in H.$$

$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ is a *g-Riesz basis* for H, if it is g-complete and there exist two constants $0 < A \leq B < \infty$, such that for each finite subset $F \subseteq \mathcal{I}$ and $f_i \in H_i, i \in F$,

$$A\sum_{i\in F} \|f_i\|^2 \leq \|\sum_{i\in F} \Lambda_i^* f_i\|^2 \leq B\sum_{i\in F} \|f_i\|^2.$$

In this case we say that Λ is an (A, B) g-Riesz basis.

Let $\{W_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of a Hilbert space H. Let $\{\omega_i\}_{i\in\mathcal{I}}$ be a family of weights, i.e., $\omega_i > 0$ for each $i \in \mathcal{I}$. Then $W = \{(W_i, \omega_i)\}_{i\in\mathcal{I}}$ is a *fusion frame*, if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathcal{I}} \omega_i^2 \|\pi_{W_i}(f)\|^2 \leq B\|f\|^2,$$

for each $f \in H$, where π_{W_i} is the orthogonal projection onto the subspace W_i. Hence $W = \{(W_i, \omega_i)\}_{i\in\mathcal{I}}$ is a fusion frame if and only if $\Lambda_W = \{\omega_i \pi_{W_i}\}_{i\in\mathcal{I}}$ is a g-frame for H and we say that W is an (A, B) *fusion frame* (resp. a *Bessel fusion sequence*, a *tight fusion frame*, a *Parseval fusion frame*) if Λ_W is an (A, B) g-frame (resp. a g-Bessel sequence, a tight g-frame, a Parseval g-frame). If W is a Bessel fusion sequence with upper bound B, then the *fusion frame operator* S_W is defined by $S_W(f) = S_{\Lambda_W}(f) = \sum_{i\in\mathcal{I}} {\omega_i}^2 \pi_{W_i}(f)$.

Tensor products of frames, fusion frames and g-frames have been studied by some authors recently, see [**15, 3, 17, 12**]. In this paper, by using operator theory methods, we present different proofs for the results obtained in the above papers and by these methods we get some important properties of the tensor products of fusion frames and g-frames.

Also direct sums of fusion frames and g-frames have been considered by some authors, see [**16, 21, 24, 1, 20**]. In this paper we get more useful information about them.

The content of the present note is as follows: In Section 2 we study tensor products of g-frames, fusion frames, g-orthonormal bases and g-Riesz bases and we obtain some relations between direct sums and tensor products of these concepts. Also we consider exact fusion frames and approximation method of the inverse frame operators.

In Section 3 we present some new examples of resolutions of the identity and atomic resolutions of the identity by using tensor products and direct sums of fusion frames and g-frames. We also consider tensor products and direct sums of atomic resolutions of the identity and their atomic resolution operators and we get some results in tensor products and direct sums of duals.

In this paper I, J and I_k, for each $1 \leq k \leq n$, are finite or countable index sets. H, H_j, H_k, H_{kj}, $H_{i(k)}$ and $H_{i(k)j}$ are separable Hilbert spaces for each $j \in J$, $k \in \{1, \ldots, n\}$ and $i(k) \in I_k$.

2. TENSOR PRODUCTS AND DIRECT SUMS OF FUSION FRAMES AND G-FRAMES

Recall that if H_k is a Hilbert space for each $1 \leq k \leq n$, then the (Hilbert) tensor product $\otimes_{k=1}^n H_k = H_1 \otimes \ldots \otimes H_n$ is a Hilbert space. The inner product for simple tensors is defined by $\langle \otimes_{k=1}^n f_k, \otimes_{k=1}^n g_k\rangle = \Pi_{k=1}^n \langle f_k, g_k\rangle$, where $f_k, g_k \in H_k$. If U_k is a bounded linear operator on H_k, then the tensor product $\otimes_{k=1}^n U_k$ is a bounded linear operator on $\otimes_{k=1}^n H_k$. Also $(\otimes_{k=1}^n U_k)^* = \otimes_{k=1}^n U_k^*$ and $\|\otimes_{k=1}^n U_k\| = \Pi_{k=1}^n \|U_k\|$. Note that if M_k is a closed subspace of H_k, for each $1 \le k \le n$, then it is easy to see that $\pi_{\otimes_{k=1}^n M_k} = \otimes_{k=1}^n \pi_{M_k}$.

Also recall that if $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, then $\mathfrak{A} \otimes \mathfrak{B}$ is a C^*-algebra with the spatial norm and for each $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, we have $\|a \otimes b\| = \|a\|\|b\|$. The multiplication and involution on simple tensors are defined by $(a\otimes b)(c\otimes d) = ac\otimes bd$ and $(a \otimes b)^* = a^* \otimes b^*$, respectively. As we know if $a, b \ge 0$, then $a \otimes b \ge 0$.

Tensor products have important applications, for example tensor products are useful in the approximation of multi-variate functions of combinations of univariate ones. For more results about tensor products see [**13, 14, 22**].

Note that if $\mathfrak{A}$ and $\mathfrak{A}'$ are unital C^*-algebras and $a \in \mathfrak{A}$, $a' \in \mathfrak{A}'$ with $0 \le a \le 1_{\mathfrak{A}}$, and $0 \le a' \le 1_{\mathfrak{A}'}$, then

$$0 \le a \otimes a' \le \|a \otimes a'\| 1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}'} = \|a\|\|a'\| 1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}'} \le 1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}'}.$$

Also note that if $L = \{\ell_1, \dots, \ell_p, \dots\}$ and $K = \{k_1, \dots, k_q, \dots\}$ are two index sets and $f_{\ell k} \in H$, for each $\ell \in L, k \in K$, then the series $\sum\limits_{(\ell,k)\in L\times K} f_{\ell k}$ is defined by $\lim\limits_{p,q} s(p,q)$, where $s(p,q) = \sum\limits_{r=1}^{p}\sum\limits_{t=1}^{q} f_{\ell_r k_t}$. If c_{lk} is a nonnegative number for each $\ell \in L$, $k \in K$, then we have

$$\sum_{(\ell,k)\in(L\times K)} c_{\ell k} = \sum_{\ell\in L}\sum_{k\in K} c_{\ell k} = \sum_{k\in K}\sum_{\ell\in L} c_{\ell k}.$$

In this paper $\Phi^{(k)} = \{\Lambda_{i(k)} \in L(H_k, H_{i(k)})\}_{i(k)\in I_k}$, $\mathcal{W}^{(k)} = \{(W_{i(k)}, \omega_{i(k)})\}_{i(k)\in I_k}$, where $W_{i(k)}$ is a closed subspace of H_k and we define $\otimes_{k=1}^n \Phi^{(k)}$, $\otimes_{k=1}^n \mathcal{W}^{(k)}$ by

$$\{\Lambda_{i(1)} \otimes \ldots \otimes \Lambda_{i(n)} \in L(\otimes_{k=1}^n H_k, H_{i(1)} \otimes \ldots \otimes H_{i(n)})\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)},$$

$$\{(W_{i(1)} \otimes \ldots \otimes W_{i(n)}, \omega_{i(1)} \ldots \omega_{i(n)})\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)},$$

respectively.

Now we consider tensor products of g-frames and fusion frames.

Theorem 2.1. (i) *$\Phi^{(k)}$ is a g-frame for each $1 \le k \le n$ if and only if $\otimes_{k=1}^n \Phi^{(k)}$ is a g-frame. In this case $S_{\otimes_{k=1}^n \Phi^{(k)}} = \otimes_{k=1}^n S_{\Phi^{(k)}}$. If A_k and B_k are lower and upper bounds of $\Phi^{(k)}$, respectively, then $\otimes_{k=1}^n \Phi^{(k)}$ is an $(\Pi_{k=1}^n A_k, \Pi_{k=1}^n B_k)$ g-frame.*

(ii) *$\mathcal{W}^{(k)}$ is a fusion frame for each $1 \le k \le n$ if and only if $\otimes_{k=1}^n \mathcal{W}^{(k)}$ is a fusion frame. In this case $S_{\otimes_{k=1}^n \mathcal{W}^{(k)}} = \otimes_{k=1}^n S_{\mathcal{W}^{(k)}}$. If A_k and B_k are lower and upper bounds of $\mathcal{W}^{(k)}$, respectively, then $\otimes_{k=1}^n \mathcal{W}^{(k)}$ is an $(\Pi_{k=1}^n A_k, \Pi_{k=1}^n B_k)$ fusion frame.*

Proof. (i) It is enough to prove the theorem for $n = 2$.

Let $\Phi^{(k)}$ be an (A_k, B_k) g-frame. Then $0 \leq \frac{1}{B_k} S_{\Phi^{(k)}} \leq Id_{H_k}$, and so we have $0 \leq S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}} \leq B_1 B_2 . Id_{(H_1 \otimes H_2)}$. Hence for each $z = \sum_{l=1}^{m} x_\ell \otimes y_\ell \in H_1 \otimes_{a\ell g} H_2$, we have

$$\begin{aligned} \sum_{(i(1),i(2)) \in (I_1 \times I_2)} \|(\Lambda_{i(1)} \otimes \Lambda_{i(2)})z\|^2 &= \sum_{i(1) \in I_1} \sum_{i(2) \in I_2} \|(\Lambda_{i(1)} \otimes \Lambda_{i(2)})z\|^2 \\ &= \left\langle \sum_{l=1}^{m} \sum_{i(1) \in I_1} \sum_{i(2) \in I_2} \Lambda^*_{i(1)} \Lambda_{i(1)} x_l \otimes \Lambda^*_{i(2)} \Lambda_{i(2)} y_l, z \right\rangle \\ &= \langle (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z, z \rangle \leq B_1 B_2 \|z\|^2. \end{aligned}$$

Now let $z \in H_1 \otimes H_2$, F_1 and F_2 be finite subsets of I_1 and I_2, respectively, and let $\{z_m\}_{m=1}^{\infty} \subseteq H_1 \otimes_{a\ell g} H_2$ such that $\lim_m z_m = z$. Then

$$\begin{aligned} \sum_{i(1) \in F_1} \sum_{i(2) \in F_2} \|(\Lambda_{i(1)} \otimes \Lambda_{i(2)})z\|^2 &= \lim_m \sum_{i(1) \in F_1} \sum_{i(2) \in F_2} \|(\Lambda_{i(1)} \otimes \Lambda_{i(2)})z_m\|^2 \\ &\leq \lim_m \langle (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z_m, z_m \rangle \leq B_1 B_2 \|z\|^2. \end{aligned}$$

Since F_1 and F_2 are arbitrary, then $\otimes_{k=1}^{2} \Phi^{(k)}$ is a g-Bessel sequence with upper bound $B_1 B_2$. For each $z \in H_1 \otimes_{alg} H_2$

$$\langle S_{\otimes_{k=1}^{2} \Phi^{(k)}} z, z \rangle = \sum_{i(1) \in I_1} \sum_{i(2) \in I_2} \|(\Lambda_{i(1)} \otimes \Lambda_{i(2)})z\|^2 = \langle (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z, z \rangle,$$

and since the operators are bounded, then $S_{\otimes_{k=1}^{2} \Phi^{(k)}} = \otimes_{k=1}^{2} S_{\Phi^{(k)}}$. Now since $S_{\Phi^{(1)}}$ and $S_{\Phi^{(2)}}$ are invertible, then $S_{\otimes_{k=1}^{2} \Phi^{(k)}}$ is invertible. Hence $\otimes_{k=1}^{2} \Phi^{(k)}$ is a g-frame with lower bound $\|S^{-1}_{\otimes_{k=1}^{2} \Phi^{(k)}}\|^{-1} = \|S^{-1}_{\Phi^{(1)}}\|^{-1} \|S^{-1}_{\Phi^{(2)}}\|^{-1}$, and since $A_1 \leq \|S^{-1}_{\Phi^{(1)}}\|^{-1}$ and $A_2 \leq \|S^{-1}_{\Phi^{(2)}}\|^{-1}$, then $A_1 A_2$ is a lower bound for $\otimes_{k=1}^{2} \Phi^{(k)}$.

Conversely let $\otimes_{k=1}^{2} \Phi^{(k)}$ be an (A, B) g-frame and let $x \in H_1$. Then for each $y \in H_2$, we have

$$\begin{aligned} A\|x\|^2 \|y\|^2 = A\|x \otimes y\|^2 &\leq \left(\sum_{i(1) \in I_1} \|\Lambda_{i(1)} x\|^2 \right) \left(\sum_{i(2) \in I_2} \|\Lambda_{i(2)} y\|^2 \right) \\ &= \sum_{i(1) \in I_1} \sum_{i(2) \in I_2} \|(\Lambda_{i(1)} \otimes \Lambda_{i(2)})(x \otimes y)\|^2 \leq B\|x \otimes y\|^2 = B\|x\|^2 \|y\|^2. \end{aligned}$$

Hence we can choose an element $y \in H_2$ such that $\|y\| = 1$ and $C = \sum_{i(2) \in I_2} \|\Lambda_{i(2)} y\|^2$ is a positive number. Thus $\Phi^{(1)}$ is an $\left(\frac{A}{C}, \frac{B}{C} \right)$ g-frame. Similarly $\Phi^{(2)}$ is a g-frame.

(ii) We can get the result by using the fact that $\Phi^{(k)} = \{\omega_{i(k)} \pi_{W_{i(k)}}\}_{i(k) \in I_k}$ is a g-frame for each $1 \leq k \leq n$ if and only if

$$\otimes_{k=1}^{n} \Phi^{(k)} = \{\omega_{i(1)} \ldots \omega_{i(n)} \pi_{(W_{i(1)} \otimes \ldots \otimes W_{i(n)})}\}_{(i(1), \ldots, i(n)) \in (I_1 \times \ldots \times I_n)}$$

is a g-frame. □

Now by using the above theorem we obtain the following result that will have useful consequences in the rest of this note.

Corollary 2.2. (i) *If* $\Phi^{(k)}$ *is a Parseval g-frame (resp. tight g-frame, g-Bessel sequence) for each* $1 \leq k \leq n$*, then* $\otimes_{k=1}^{n}\Phi^{(k)}$ *is a Parseval g-frame (resp. tight g-frame, g-Bessel sequence).*

(ii) *If* $\mathcal{W}^{(k)}$ *is a Parseval fusion frame (resp. tight fusion frame, Bessel fusion sequence) for each* $1 \leq k \leq n$*, then* $\otimes_{k=1}^{n}\mathcal{W}^{(k)}$ *is a Parseval fusion frame (resp. tight fusion frame, Bessel fusion sequence).*

Proof. (i) Since $\Phi^{(k)}$'s are Parseval, then $A_k = B_k = 1$, for each $1 \leq k \leq n$ in Theorem 2.1. Hence $\otimes_{k=1}^{n}\Phi^{(k)}$ is a Parseval g-frame. Also if $\Phi^{(k)}$ is an A_k-tight g-frame, for each $1 \leq k \leq n$, then $\otimes_{k=1}^{n}\Phi^{(k)}$ is $(\Pi_{k=1}^{n}A_k)$-tight. It is also obvious from the first part of the proof of Theorem 2.1 that if $\Phi^{(k)}$'s are g-Bessel sequences, then $\otimes_{k=1}^{n}\Phi^{(k)}$ is a g-Bessel sequence.

(ii) The result follows from part (i) as in Theorem 2.1. □

Note that [**17**, Theorem 3.7] is a special case of parts (ii) of Theorem 2.1 and Corollary 2.2.

Remark 2.3. If each $\Phi^{(k)}$ contains a nonzero operator and $\otimes_{k=1}^{n}\Phi^{(k)}$ is a g-Bessel sequence, then similar to the proof of Theorem 2.1, we can obtain that each $\Phi^{(k)}$ is a g-Bessel sequence. Now let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_m\}_{m=1}^{\infty}$ be sequences of positive numbers with $c = \sum_{n=1}^{\infty}\alpha_n^2 > 1$, $d = \sum_{m=1}^{\infty}\beta_m^2 < 1$ and $cd = 1$. Hence if $\Phi^{(1)} = \{\alpha_n \cdot Id_{H_1} : n \in \mathbb{N}\}$ and $\Phi^{(2)} = \{\beta_m \cdot Id_{H_2} : m \in \mathbb{N}\}$, then $\otimes_{k=1}^{2}\Phi^{(k)}$ is a Parseval g-frame, but $\Phi^{(1)}$ and $\Phi^{(2)}$ are not Parseval.

Proposition 2.4. *If* $\Phi^{(k)}$ *is a g-orthonormal basis for each* $1 \leq k \leq n$*, then* $\otimes_{k=1}^{n}\Phi^{(k)}$ *is a g-orthonormal basis for* $\otimes_{k=1}^{n}H_k$.

Proof. Let $n = 2$. By Corollary 2.2, $\otimes_{k=1}^{2}\Phi^{(k)}$ is a Parseval g-frame. Now let $(i(1), i(2)), (j(1), j(2)) \in I_1 \times I_2$, and let $z = \sum_{\ell=1}^{\infty} x_{1\ell} \otimes x_{2\ell} \in H_{i(1)} \otimes H_{i(2)}$, $w = \sum_{r=1}^{\infty} y_{1r} \otimes y_{2r} \in H_{j(1)} \otimes H_{j(2)}$. Then we have

$$\begin{aligned}
&\langle (\Lambda_{i(1)} \otimes \Lambda_{i(2)})^* z, (\Lambda_{j(1)} \otimes \Lambda_{j(2)})^* w \rangle \\
&= \sum_{\ell=1}^{\infty}\sum_{r=1}^{\infty} \langle \Lambda_{i(1)}^*(x_{1\ell}), \Lambda_{j(1)}^*(y_{1r})\rangle \langle \Lambda_{i(2)}^*(x_{2\ell}), \Lambda_{j(2)}^*(y_{2r})\rangle \\
&= \delta_{(i(1),i(2)),(j(1),j(2))}\langle z, w\rangle.
\end{aligned}$$

This means that $\otimes_{k=1}^{2}\Phi^{(k)}$ is a g-orthonormal basis for $\otimes_{k=1}^{2}H_k$.

Proposition 2.5. *$\Phi^{(k)}$ is a g-Riesz basis for each $1 \leq k \leq n$, if and only if $\otimes_{k=1}^{n}\Phi^{(k)}$ is a g-Riesz basis.*

Proof. Let $n = 2$ and $\Phi^{(k)}$ be a g-Riesz basis for each $k \in \{1, 2\}$. Hence by [**23**, Corollary 3.4], there is a g-orthonormal basis $\{Q_{i(k)}\}_{i(k)\in I_k}$, for H_k and an invertible operator U_k on H_k, such that $\Lambda_{i(k)} = Q_{i(k)}U_k$, for each $i(k) \in I_k$. Therefore we have $\Lambda_{i(1)} \otimes \Lambda_{i(2)} = (Q_{i(1)} \otimes Q_{i(2)})(U_1 \otimes U_2)$. It follows from Proposition 2.4 that $\{Q_{i(1)} \otimes Q_{i(2)}\}_{(i(1),i(2))\in I_1\times I_2}$ is a g-orthonormal basis for $\otimes_{k=1}^{2}H_k$, and it is clear that $U_1 \otimes U_2$ is an invertible operator on $\otimes_{k=1}^{2}H_k$. Now we can get the result by using Corollary 3.4 in [**23**].

Conversely suppose that $\otimes_{k=1}^{2}\Phi^{(k)}$ is an (A, B) g-Riesz basis and $f \in H_1$, such that $\Lambda_{i(1)}f = 0$, for each $i(1) \in I_1$. Let g be a nonzero element of H_2. Then we have $(\Lambda_{i(1)} \otimes \Lambda_{i(2)})(f \otimes g) = (\Lambda_{i(1)}f) \otimes (\Lambda_{i(2)}g) = 0$. Since $\otimes_{k=1}^{2}\Phi^{(k)}$ is g-complete, then $\|f\|\|g\| = \|f \otimes g\| = 0$. Hence $f = 0$, and this means that $\Phi^{(1)}$ is g-complete. Now let F_1 be a finite subset of I_1 and $g_{i(1)} \in H_{i(1)}$, for each $i(1) \in F_1$. Suppose that F_2 is a finite subset of I_2 and $g_{i(2)} \in H_{i(2)}$, for each $i(2) \in F_2$ such that $\sum_{i(2)\in F_2} \|g_{i(2)}\|^2 = 1$. Now we have

$$A\Big(\sum_{i(1)\in F_1} \|g_{i(1)}\|^2\Big)\Big(\sum_{i(2)\in F_2} \|g_{i(2)}\|^2\Big) = A \sum_{(i(1),i(2))\in F_1\times F_2} \|g_{i(1)} \otimes g_{i(2)}\|^2$$

$$\leq \Big\| \sum_{(i(1),i(2))\in F_1\times F_2} (\Lambda^*_{i(1)} \otimes \Lambda^*_{i(2)})(g_{i(1)} \otimes g_{i(2)})\Big\|^2$$

$$= \Big\|\sum_{i(1)\in F_1} \Lambda^*_{i(1)}g_{i(1)}\Big\|^2 \Big\|\sum_{i(2)\in F_2} \Lambda^*_{i(2)}g_{i(2)}\Big\|^2 \leq B\Big(\sum_{i(1)\in F_1} \|g_{i(1)}\|^2\Big)\Big(\sum_{i(2)\in F_2} \|g_{i(2)}\|^2\Big).$$

Meaning that $\Phi^{(1)}$ is an $\left(\frac{A}{C}, \frac{B}{C}\right)$ g-Riesz basis, where $C = \Big\|\sum_{i(2)\in F_2} \Lambda^*_{i(2)}g_{i(2)}\Big\|^2$. $\square$

Note that all of the results in tensor product of g-frames obtained in [**12**] are special cases of the above results. Also by using Theorem 2.1, Proposition 2.5 and [**23**, Examples 1.1 and 3.1], we can obtain the main results of [**3**] ([**3**, Theorem 4.1]):

Corollary 2.6. *Let $f^{(k)} = \{f_{i(k)}\}_{i(k)\in I_k}$ be a sequence in H_k. Then $f^{(k)}$ is a frame (resp. Riesz basis) for each $1 \leq k \leq n$ if and only if $\otimes_{k=1}^{n}f^{(k)}$ which is defined by $\{f_{i(1)} \otimes \ldots \otimes f_{i(n)}\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$ is a frame (resp. Riesz basis) for $\otimes_{k=1}^{n}H_k$.*

Let $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in \mathcal{I}\}$ be a g-Bessel sequence for H_j, $j \in J$, with upper bound B_j such that $B = \sup\{B_j : j \in J\} < \infty$. Then $\{\Phi_j\}_{j\in J}$ is called a *B-Bounded family of g-Bessel sequences* or shortly *B-BFGBS*.

Let $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in \mathcal{I}\}$ be an (A_j, B_j) g-frame (resp. g-Riesz basis) for H_j, $j \in J$, such that $A = \inf\{A_j : j \in J\} > 0$ and $B = \sup\{B_j : j \in$

$J\} < \infty$. Then we say that $\{\Phi_j\}_{j\in J}$ is an (A,B)-*bounded family of g-frames* (resp. *bounded family of g-Riesz bases*) or shortly (A,B)-*BFGF* (resp. *BFGRB*).

Note that a *B-bounded family of Bessel fusion sequences* or shortly *B-BFBFS* and an (A,B)-*bounded family of fusion frames* or shortly (A,B)-*BFFF* for a family of fusion frames can be defined by using the g-frames generated by the fusion frames. We denote a family of Parseval fusion frames by FPFF.
In the rest of this note $\Phi_j^{(k)} = \{\Lambda_{i(k)j} \in L(H_{kj}, H_{i(k)j})\}_{i(k)\in I_k}$, $\mathcal{W}_j = \{(W_{ij},\omega_i)\}_{i\in I}$, $\mathcal{W}_j^{(k)} = \{(W_{i(k)j},\omega_{i(k)})\}_{i(k)\in I_k}$, $\oplus_{j\in J}\mathcal{W}_j = \{(\oplus_{j\in J}W_{ij},\omega_i)\}_{i\in I}$, where W_{ij} and $W_{i(k)j}$ are closed subspaces of H_j and H_{kj}, respectively and

$$\oplus_{j\in J}\Phi_j^{(k)} = \{\oplus_{j\in J}\Lambda_{i(k)j} \in L(\oplus_{j\in J}H_{kj}, \oplus_{j\in J}H_{i(k)j})\}_{i(k)\in I_k}.$$

Note that if W_{ij}'s are closed subspaces of H_j, then it is clear that $\oplus_{j\in J}\pi_{W_{ij}} = \pi_{\oplus_{j\in J}W_{ij}}$, for each $i \in I$, now as a consequence of [**20**, Theorems 2.3 and 2.5] and the above results, we have the following. Part (iii) of the following corollary is also a generalization of [**16**, Theorem 2.3 and Corollary 2.2] to a countable number of fusion frames.

Corollary 2.7. (i) $\{\Phi_j^{(k)}\}_{j\in J}$ *is a BFGF* (*resp. BFGRB*), *for each* $1 \le k \le n$ *if and only if* $\otimes_{k=1}^n(\oplus_{j\in J}\Phi_j^{(k)})$ *is a g-frame* (*resp. g-Riesz basis*) *for* $\otimes_{k=1}^n(\oplus_{j\in J}H_{kj})$. *Also if* $\{\Phi_j^{(k)}\}_{j\in J}$ *is an* (A_k,B_k)*-BFGF, then* $\otimes_{k=1}^n(\oplus_{j\in J}\Phi_j^{(k)})$ *is an* $(\Pi_{k=1}^n A_k, \Pi_{k=1}^n B_k)$ *g-frame.*

(ii) *If* $\Phi_j^{(k)}$ *is a g-orthonormal basis for each* $j \in J$ *and* $1 \le k \le n$, *then* $\otimes_{k=1}^n(\oplus_{j\in J}\Phi_j^{(k)})$ *is a g-orthonormal basis for* $\otimes_{k=1}^n(\oplus_{j\in J}H_{kj})$.

(iii) $\{\mathcal{W}_j\}_{j\in J}$ *is a BFBFS* (*resp. BFFF, FPFF*) *if and only if* $\oplus_{j\in J}\mathcal{W}_j$ *is a Bessel fusion sequence* (*resp. fusion frame, Parseval fusion frame*) *for* $\oplus_{j\in J}H_j$. *In this case* $S_{\oplus_{j\in J}\mathcal{W}_j} = \oplus_{j\in J}S_{\mathcal{W}_j}$.

(iv) $\{\mathcal{W}_j^{(k)}\}_{j\in J}$ *is a BFFF, for each* $1 \le k \le n$, *if and only if* $\otimes_{k=1}^n(\oplus_{j\in J}\mathcal{W}_j^{(k)})$ *is a fusion frame for* $\otimes_{k=1}^n(\oplus_{j\in J}H_{kj})$.

Recall that a frame (resp. fusion frame) is *exact*, if it ceases to be a frame (resp. fusion frame) whenever any of its elements is removed. A frame is exact if and only if it is a Riesz basis (see [**4**, Proposition 4.3]). Hence by Corollary 2.6, the tensor product of a finite number of frames is exact if and only if each of the frames is exact. We show that the same result holds for fusion frames. Note that if $W = \{(W_i,\omega_i)\}_{i\in\mathcal{I}}$ is an exact fusion frame, then $W_i \neq (0)$, for each $i \in \mathcal{I}$ and if $\mathcal{J}$ is a proper subset of $\mathcal{I}$, then $\{(W_i,\omega_i)\}_{i\in\mathcal{J}}$ is not a fusion frame.

Proposition 2.8. *Let* $\mathcal{W}^{(k)}$ *be a fusion frame for each* $1 \le k \le n$. *Then* $\mathcal{W}^{(k)}$ *is exact for each* $k \in \{1,\dots,n\}$ *if and only if* $\otimes_{k=1}^n\mathcal{W}^{(k)}$ *is exact.*

Proof. Let $n = 2$ and $\mathcal{W}^{(1)}$, $\mathcal{W}^{(2)}$ be exact fusion frames. Suppose that $(i_1,i_2) \in I_1 \times I_2$ such that $\{(W_{i(1)} \otimes W_{i(2)}, \omega_{i(1)}\omega_{i(2)})\}_{(i(1),i(2))\in(I_1\times I_2-\{(i_1,i_2)\})}$ is a fusion frame with lower bound A. Since $\mathcal{W}^{(k)}$'s are exact, then by [**5**, Proposition 3.6], there exist two nonzero elements $f_1 \in H_1$ and $f_2 \in H_2$ which are orthogonal to $\overline{span}\{W_{i(1)}\}_{i(1)\in I_1-\{i_1\}}$ and $\overline{span}\{W_{i(2)}\}_{i(2)\in I_2-\{i_2\}}$, respectively. Therefore

$$A\|f_1\|^2\|f_2\|^2 \leq \Big(\sum_{i(1)\in I_1} \omega_{i(1)}^2\|\pi_{W_{i(1)}}f_1\|^2\Big)\Big(\sum_{i(2)\in I_2} \omega_{i(2)}^2\|\pi_{W_{i(2)}}f_2\|^2\Big) - \omega_{i_1}^2\omega_{i_2}^2\|\pi_{W_{i_1}}f_1\|^2\|\pi_{W_{i_2}}f_2\|^2 = 0,$$

which is a contradiction. Hence $\otimes_{k=1}^2\mathcal{W}^{(k)}$ is exact.

Conversely suppose that $\otimes_{k=1}^2\mathcal{W}^{(k)}$ is exact. If $\mathcal{W}^{(1)}$ is not exact, then there exists some $i_1 \in I_1$ such that $\mathcal{Z} = \{(W_{i(1)}, \omega_{i(1)})\}_{i(1)\in I_1-\{i_1\}}$ is a fusion frame. Hence by part (ii) of Theorem 2.1,

$$\mathcal{Z}\otimes\mathcal{W}^{(2)} = \{(W_{i(1)}\otimes W_{i(2)}, \omega_{i(1)}\omega_{i(2)})\}_{(i(1),i(2))\in(I_1-\{i_1\})\times I_2}$$

is a fusion frame which is a contradiction, since $\otimes_{k=1}^2\mathcal{W}^{(k)}$ is exact and $(I_1-\{i_1\})\times I_2$ is a proper subset of $I_1\times I_2$. □

Suppose that $W = \{(W_i, \omega_i)\}_{i\in\mathcal{I}}$ is a fusion frame for H such that W_i is a finite-dimensional subspace of H, for each $i \in \mathcal{I}$. Let $\{\mathcal{I}_m\}_{m=1}^\infty \subseteq \mathcal{I}$ such that $\mathcal{I}_m$ is finite for each $m \in \mathbb{N}$ and $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \ldots \subseteq \mathcal{I}_m \nearrow \mathcal{I}$.

For each $m \in \mathbb{N}$ we assume that $W_m = \{(W_i, \omega_i)\}_{i\in\mathcal{I}_m}$ is a fusion frame for $H_m = span\{W_i\}_{i\in\mathcal{I}_m}$ with operator $S_{W_m} : H_m \longrightarrow H_m$ which is defined by $S_{W_m}(f) = \sum_{i\in\mathcal{I}_m}\omega_i^2\pi_{W_i}(f)$. We say that the approximation method of S_W^{-1} works if

$$\lim_{m\to\infty} S_{W_m}^{-1}\pi_{H_m}(f) = S_W^{-1}(f),$$

for each $f \in H$. For more results see [**16**].

As we know the inverse of the frame operator plays an important role in frame theory mostly because of the reconstruction, but it is often difficult to find it. In this case if we can approximate this inverse, then we can approximately reconstruct the signals which is useful in applications. Here since H_m is finite-dimensional, then S_{W_m} can be inverted using linear algebra. Therefore if the approximation method of S_W^{-1} works, then $S_W^{-1}(f)$ can be approximated by the sequence $\{S_{W_m}^{-1}\pi_{H_m}(f)\}_{m=1}^\infty$, for each $f \in H$, and we have

$$\lim_{m\to\infty} S_W(S_{W_m}^{-1}\pi_{H_m}f) = S_WS_W^{-1}f = f = S_W^{-1}S_W(f) = \lim_{m\to\infty} S_{W_m}^{-1}\pi_{H_m}(S_Wf).$$

In the following proposition all of the subspaces in the fusion frames are finite-dimensional.

Proposition 2.9. *Suppose that $\mathcal{W}^{(k)}$'s are fusion frames. If the approximation method of $S_{\mathcal{W}^{(k)}}^{-1}$ works for each $k \in \{1,\ldots,n\}$, then the approximation method of $S_{\otimes_{k=1}^n\mathcal{W}^{(k)}}^{-1}$ works.*

Proof. Let $n = 2$ and let the approximation method of $S^{-1}_{\mathcal{W}^{(k)}}$ work for each $k \in \{1,2\}$. Then there exist $\{\Gamma^1_m\}_{m=1}^{\infty} \subseteq I_1$ and $\{\Gamma^2_m\}_{m=1}^{\infty} \subseteq I_2$ such that Γ^k_m is a finite set for each $m \in \mathbb{N}$ and $k \in \{1,2\}$,

$$\Gamma^1_1 \subseteq \Gamma^1_2 \subseteq \ldots \subseteq \Gamma^1_m \nearrow I_1, \quad \Gamma^2_1 \subseteq \Gamma^2_2 \subseteq \ldots \subseteq \Gamma^2_m \nearrow I_2,$$

and $W^k_m = \{(W_{i(k)}, \omega_{i(k)})\}_{i(k)\in\Gamma^k_m}$ is a fusion frame for $H^k_m = span\{W_{i(k)}\}_{i(k)\in\Gamma^k_m}$. Hence $(\Gamma^1_1 \times \Gamma^2_1) \subseteq (\Gamma^1_2 \times \Gamma^2_2) \subseteq \ldots \subseteq (\Gamma^1_m \times \Gamma^2_m) \nearrow (I_1 \times I_2)$, and it is easy to see that

$$H^1_m \otimes H^2_m = span\{W_{i(1)} \otimes W_{i(2)}\}_{(i(1),i(2))\in(\Gamma^1_m\times\Gamma^2_m)}.$$

Since W^1_m and W^2_m are fusion frames for H^1_m and H^2_m, respectively, then by part (ii) of Theorem 2.1, $W^1_m \otimes W^2_m = \{(W_{i(1)} \otimes W_{i(2)}, \omega_{i(1)}\omega_{i(2)})\}_{(i(1),i(2))\in(\Gamma^1_m\times\Gamma^2_m)}$ is a fusion frame for $H^1_m \otimes H^2_m$. Now by using part (iii) of [**16**, Theorem 3.2], we have

$$\begin{aligned}\sup_{m\in\mathbb{N}}\{\|S^{-1}_{(W^1_m\otimes W^2_m)}\pi_{(H^1_m\otimes H^2_m)}\|\} &= \sup_{m\in\mathbb{N}}\{\|(S^{-1}_{W^1_m}\pi_{H^1_m}) \otimes (S^{-1}_{W^2_m}\pi_{H^2_m})\|\}\\ &\le \sup_{m\in\mathbb{N}}\{\|S^{-1}_{W^1_m}\pi_{H^1_m}\|\} \sup_{m\in\mathbb{N}}\{\|S^{-1}_{W^2_m}\pi_{H^2_m}\|\} < \infty.\end{aligned}$$

Hence $\sup_{m\in\mathbb{N}}\{\|S^{-1}_{(W^1_m\otimes W^2_m)}\|\} = \sup_{m\in\mathbb{N}}\{\|S^{-1}_{(W^1_m\otimes W^2_m)}\pi_{(H^1_m\otimes H^2_m)}\|\} < \infty$, and the result follows from Theorem 3.2 in [**16**].

3. RESOLUTIONS OF THE IDENTITY AND DUALS

Resolution of the identity was defined in [**5**], and afterwards the first author and Asgari introduced atomic resolution of the identity in [**2**] for more applications in fusion frames:

Definition 3.1. *Let $\{\omega_i\}_{i\in\mathcal{I}}$ be a family of weights. A family of bounded operators $\{T_i\}_{i\in\mathcal{I}}$ on H is called an atomic (unconditional) resolution of the identity with respect to $\{\omega_i\}_{i\in\mathcal{I}}$ for H if there exist two positive numbers A and B such that for each $f \in H$,*

(i) $f = \sum_{i\in\mathcal{I}} T_i(f)$ (*and the series converges unconditionally*),

(ii) $A\|f\|^2 \le \sum_{i\in\mathcal{I}} \omega_i^2\|T_i(f)\|^2 \le B\|f\|^2$. *In this case we say that $\{(T_i,\omega_i)\}_{i\in\mathcal{I}}$ is an (A,B)* atomic (unconditional) resolution of the identity *or shortly an (A,B)* ARI(AURI). *If we only know that $\{T_i\}_{i\in\mathcal{I}}$ satisfies in (i), then $\{T_i\}_{i\in\mathcal{I}}$ is called a (unconditional) resolution of the identity.*

Let $\mathcal{T} = \{(T_i,\omega_i)\}_{i\in\mathcal{I}}$ be an (A,B) ARI for H. Then the *atomic resolution operator* $R_{\mathcal{T}} : H \longrightarrow H$ is defined by $R_{\mathcal{T}}(f) = \sum_{i\in\mathcal{I}} \omega_i^2 T_i^* T_i(f)$. By Theorem 3.4 in [**2**], we have $A.Id_H \le R_{\mathcal{T}} \le B.Id_H$.

If $\mathcal{T}_j = \{(T_{ij}, \omega_i)\}_{i\in I}$ is an (A_j, B_j) ARI (resp. AURI) for H_j such that $A = \inf\{A_j : j \in J\} > 0$ and $B = \sup\{B_j : j \in J\} < \infty$, then we call $\{\mathcal{T}_j\}_{j\in J}$ an (A, B)-*bounded family of atomic* (*unconditional*) *resolutions of the identity* or shortly an (A, B)-BFARI (resp. BFAURI).

Example 3.2. Let $\{\mathcal{W}_j\}_{j\in J}$ be a BFFF and $\mathcal{W}^{(k)}$ be a fusion frame, for each $1 \le k \le n$. Then by Theorem 2.1 and Corollary 2.7, $\otimes_{k=1}^n \mathcal{W}^{(k)}$ and $\oplus_{j\in J}\mathcal{W}_j$ are fusion frames for $\otimes_{k=1}^n H_k$ and $\oplus_{j\in J} H_j$, respectively. Therefore by [**2**, Proposition 3.7], $\{(T_i, \omega_i^{-1})\}_{i\in I}$ and $\{(T_i^*, \omega_i^{-1})\}_{i\in I}$ are AURI for $\oplus_{j\in J} H_j$, where

$$T_i = \omega_i^2 \pi_{\oplus_{j\in J} W_{ij}} S^{-1}_{\oplus_{j\in J}\mathcal{W}_j} = \oplus_{j\in J}(\omega_i^2 \pi_{W_{ij}} S^{-1}_{\mathcal{W}_j}).$$

Also

$$\{\big(U_{i(1),\dots,i(n)}, (\omega_{i(1)}\dots\omega_{i(n)})^{-1}\big)\}_{(i(1),\dots,i(n))\in(I_1\times\dots\times I_n)}$$

and

$$\{\big(U^*_{i(1),\dots,i(n)}, (\omega_{i(1)}\dots\omega_{i(n)})^{-1}\big)\}_{(i(1),\dots,i(n))\in(I_1\times\dots\times I_n)}$$

are AURI, for $\otimes_{k=1}^n H_k$, where

$$U_{i(1),\dots,i(n)} = (\omega_{i(1)}\dots\omega_{i(n)})^2 \pi_{(W_{i(1)}\otimes\dots\otimes W_{i(n)})} S^{-1}_{\otimes_{k=1}^n \mathcal{W}^{(k)}} = \otimes_{k=1}^n(\omega_{i(k)}^2 \pi_{W_{i(k)}} S^{-1}_{\mathcal{W}^{(k)}}).$$

Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ be an (A, B) g-frame. Then the *canonical dual* g-frame for Λ is defined by $\tilde{\Lambda} = \{\tilde{\Lambda_i} \in L(H, H_i) : i \in \mathcal{I}\}$, where $\tilde{\Lambda_i} = \Lambda_i S_\Lambda^{-1}$, which is an $\left(\dfrac{1}{B}, \dfrac{1}{A}\right)$ g-frame for H. If Λ is a g-Bessel sequence, then the g-Bessel sequence $\{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ is called a *g-dual* of Λ if $f = \sum_{i\in\mathcal{I}} \Gamma_i^* \Lambda_i f$, for each $f \in H$.

Example 3.3. Let $R_{i(k)}$'s be finite or countable index sets, $\Phi^{(k)}$ be a g-frame for H_k and $\{(W_{i(k)r(k)}, \omega_{i(k)r(k)})\}_{r(k)\in R_{i(k)}}$ be a Parseval fusion frame for $H_{i(k)}$. Then by Theorem 2.1, $\otimes_{k=1}^n \Phi^{(k)}$ is a g-frame for $\otimes_{k=1}^n H_k$. Also by Corollary 2.2,

$$\{(W_{i(1)r(1)} \otimes \dots \otimes W_{i(n)r(n)}, \omega_{i(1)r(1)}\dots\omega_{i(n)r(n)})\}_{(r(1),\dots,r(n))\in(R_{i(1)}\times\dots\times R_{i(n)})}$$

is a Parseval fusion frame for $\otimes_{k=1}^n H_{i(k)}$. Now define $T_{i(1)r(1),\dots,i(n)r(n)}$ by

$$(\omega_{i(1)r(1)}\dots\omega_{i(n)r(n)})^2 (\Lambda_{i(1)} \otimes \dots \otimes \Lambda_{i(n)})^* \pi_{(W_{i(1)r(1)}\otimes\dots\otimes W_{i(n)r(n)})} \Lambda_{i(1)} \widetilde{\otimes \dots \otimes} \Lambda_{i(n)},$$

where $\Lambda_{i(1)} \widetilde{\otimes \dots \otimes} \Lambda_{i(n)} = (\Lambda_{i(1)} \otimes \dots \otimes \Lambda_{i(n)}) S^{-1}_{\otimes_{k=1}^n \Phi^{(k)}}$. Hence by [**21**, Corollary 2.6],

$$\{T_{i(1)r(1),\dots,i(n)r(n)}\}_{(i(1),\dots,i(n))\in(I_1\times\dots\times I_n),(r(1),\dots,r(n))\in(R_{i(1)}\times\dots\times R_{i(n)})}$$

is a resolution of the identity for $\otimes_{k=1}^n H_k$.

Example 3.4. Let $\{e_{mj}\}_{m=1}^\infty$, $\{e_{m(k)}\}_{m(k)=1}^\infty$ be orthonormal bases for H_j, H_k and W_{mj}, $W_{m(k)}$ be the Hilbert spaces generated by e_{mj}, $e_{m(k)}$, respectively, for each $j \in J$, $1 \le k \le n$. If $\pi_{mj} = \pi_{W_{mj}}$ and $\pi_{m(k)} = \pi_{W_{m(k)}}$, then by using Example 3.2, we can see that $\{\big(\oplus_{j\in J}\pi_{mj}, 1\big)\}_{m=1}^\infty$ and $\{\big(\pi_{m(1)} \otimes \dots \otimes \pi_{m(n)}, 1\big)\}_{(m(1),\dots,m(n))\in(\mathbb{N}\times\dots\times\mathbb{N})}$ are AURI for $\oplus_{j\in J} H_j$ and $\otimes_{k=1}^n H_k$, respectively.

Note that in the above example each π_{mj} is a positive operator and if $\mathcal{T}_j = \{(\pi_{mj}, \omega_m)\}_{m=1}^{\infty}$, $\omega_m = 1$, for each $m \in \mathbb{N}$, then $\{\mathcal{T}_j\}_{j\in J}$ is a BFARI and we see that $\{(\oplus_{j\in J}\pi_{mj}, \omega_m)\}_{m=1}^{\infty}$ is an AURI for $\oplus_{j\in J}H_j$. The following proposition shows that this result holds for each BFARI with positive elements:

Proposition 3.5. *Let $T_{ij} : H_j \longrightarrow H_j$ be a positive operator, for each $i \in I$ and $j \in J$. If $\{\mathcal{T}_j = \{(T_{ij}, \omega_i)\}_{i\in I}\}_{j\in J}$ is an (A, B)-BFARI, then $\{(\oplus_{j\in J}T_{ij}, \omega_i)\}_{i\in I}$ is an (A,B) AURI for $\oplus_{j\in J}H_j$. Conversely if $\{(\oplus_{j\in J}T_{ij}, \omega_i)\}_{i\in I}$ is an AURI, then $\{\mathcal{T}_j\}_{j\in J}$ is a BFAURI.*

Proof. Note that $\sum_{i\in I} \|T_{ij}^{\frac{1}{2}} f_j\|^2 = \langle \sum_{i\in I} T_{ij} f_j, f_j\rangle = \|f_j\|^2$, for each $f_j \in H_j$.

Since $\{\mathcal{T}_j\}_{j\in J}$ is an (A,B)-BFARI, then $\sup\{\|T_{ij}\| : j \in J\} \leq \frac{\sqrt{B}}{\omega_i}$, for each $i \in I$, therefore $\oplus_{j\in J}T_{ij}$ is a bounded operator on $\oplus_{j\in J}H_j$. Let F be a finite subset of I. Then for each $f = \{f_j\}_{j\in J}, g = \{g_j\}_{j\in J} \in \oplus_{j\in J}H_j$, we have

$$\sum_{i\in F} |\langle(\oplus_{j\in J}T_{ij})f, g\rangle| = \sum_{i\in F} \left| \sum_{j\in J} \langle T_{ij}^{\frac{1}{2}} f_j, T_{ij}^{\frac{1}{2}} g_j\rangle \right| \leq \sum_{j\in J}\sum_{i\in F} \|T_{ij}^{\frac{1}{2}} f_j\|\|T_{ij}^{\frac{1}{2}} g_j\|$$
$$\leq \sum_{j\in J} \left(\Big(\sum_{i\in I} \|T_{ij}^{\frac{1}{2}} f_j\|^2\Big)^{\frac{1}{2}} \Big(\sum_{i\in I} \|T_{ij}^{\frac{1}{2}} g_j\|^2\Big)^{\frac{1}{2}} \right) = \sum_{j\in J} \|f_j\|\|g_j\| \leq \|f\|\|g\|.$$

So $\sum_{i\in I} (\oplus_{j\in J}T_{ij})f$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\oplus_{j\in J}H_j$ (see [**9**], page 44, Theorems 6 and 8). Also

$$\sup\left\{ \left| \Big\langle \sum_{i\in I}(\oplus_{j\in J}T_{ij})f, g \Big\rangle \right| : g \in \oplus_{j\in J}H_j, \|g\| = 1 \right\} \leq \|f\|,$$

thus the operator $\sum_{i\in I} (\oplus_{j\in J}T_{ij})$ which is defined on $\oplus_{j\in J}H_j$ by

$$\Big(\sum_{i\in I}(\oplus_{j\in J}T_{ij})\Big)(\{f_j\}_{j\in J}) = \sum_{i\in I}(\oplus_{j\in J}T_{ij})\{f_j\}_{j\in J},$$

is bounded. Now we have

$$\Big\langle \sum_{i\in I}(\oplus_{j\in J}T_{ij})f, f \Big\rangle = \sum_{i\in I}\sum_{j\in J}\langle T_{ij} f_j, f_j\rangle = \sum_{j\in J}\langle\sum_{i\in I} T_{ij} f_j, f_j\rangle$$
$$= \sum_{j\in J}\langle f_j, f_j\rangle = \langle \{f_j\}_{j\in J}, \{f_j\}_{j\in J}\rangle.$$

This means that $\sum_{i\in I} (\oplus_{j\in J}T_{ij})\{f_j\}_{j\in J}$ converges unconditionally to $\{f_j\}_{j\in J}$, for each $\{f_j\}_{j\in J} \in \oplus_{j\in J} H_j$. Also we have

$$\sum_{i\in I} \omega_i^2 \|(\oplus_{j\in J} T_{ij})(\{f_j\}_{j\in J})\|^2 = \sum_{i\in I}\sum_{j\in J} \omega_i^2 \|T_{ij} f_j\|^2 = \sum_{j\in J}\sum_{i\in I} \omega_i^2 \|T_{ij} f_j\|^2 \leq B \sum_{j\in J} \|f_j\|^2,$$

similarly $\sum_{i\in I} \omega_i^2 \|(\oplus_{j\in J} T_{ij})(\{f_j\}_{j\in J})\|^2 \geq A \sum_{j\in J} \|f_j\|^2$. So $\{(\oplus_{j\in J} T_{ij}, \omega_i)\}_{i\in I}$ is an (A,B) AURI. The converse is clear.

Proposition 3.6. *If $\mathcal{T}^{(k)} = \{(T_{i(k)}, \omega_{i(k)})\}_{i(k)\in I_k}$ is an AURI, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n \mathcal{T}^{(k)} = \{(T_{i(1)} \otimes \ldots \otimes T_{i(n)}, \omega_{i(1)} \ldots \omega_{i(n)})\}_{(i(1),\ldots i(n))\in(I_1\times\ldots\times I_n)}$ is an AURI, for $\otimes_{k=1}^n H_k$. In this case $R_{\otimes_{k=1}^n \mathcal{T}^{(k)}} = \otimes_{k=1}^n R_{\mathcal{T}^{(k)}}$.*

Proof. Let $n = 2$ and σ be a permutation of $I_1 \times I_2$. Then for each $z = \sum_{\ell=1}^{m} x_\ell \otimes y_\ell \in H_1 \otimes_{a\ell g} H_2$, we have

$$(1) \qquad \sum_{(i(1),i(2))\in\sigma} (T_{i(1)} \otimes T_{i(2)})z = \lim_{p,q} S(p,q,z) = \sum_{l=1}^{m} \left(\sum_{r=1}^{\infty} T_{\alpha_r} x_\ell\right) \otimes \left(\sum_{t=1}^{\infty} T_{\beta_t} y_\ell\right) = z,$$

where $S(p,q,z) = \sum_{r=1}^{p}\sum_{t=1}^{q} (T_{\alpha_r} \otimes T_{\beta_t})z$ and

$$\alpha = \{\alpha_1, \ldots, \alpha_p, \ldots\}, \beta = \{\beta_1, \ldots, \beta_q, \ldots\}$$

are permutations of I_1, I_2, respectively. Suppose that $S_{\alpha p} = \sum_{r=1}^{p} T_{\alpha_r}$ and $S_{\beta q} = \sum_{t=1}^{q} T_{\beta_t}$. Since $\mathcal{T}^{(k)}$'s are AURI, then $\lim_p S_{\alpha p} x = x$ and $\lim_q S_{\beta q} y = y$, for each $x \in H_1$ and $y \in H_2$. Hence by using the uniform boundedness principle, we can get that $K_\alpha = \sup_{p\in\mathbb{N}}\{\|S_{\alpha p}\|\} < \infty$ and $K_\beta = \sup_{q\in\mathbb{N}}\{\|S_{\beta q}\|\} < \infty$. Now for each $z \in H_1 \otimes H_2$, by choosing some element $z_0 \in H_1 \otimes_{alg} H_2$ close to z, and by using (1) and the inequality

$$\|(S_{\alpha p} \otimes S_{\beta q})z - z\| \leq K_\alpha K_\beta \|z - z_0\| + \|(S_{\alpha p} \otimes S_{\beta q})z_0 - z_0\| + \|z - z_0\|,$$

we obtain that $\lim_{p,q} (S_{\alpha p} \otimes S_{\beta q})z = z$, which is equivalent to the convergence of $\sum_{(i(1),i(2))\in\sigma} (T_{i(1)} \otimes T_{i(2)})z$ to z. This means that $\sum_{(i(1),i(2))\in(I_1\times I_2)} (T_{i(1)} \otimes T_{i(2)})z$ converges to z unconditionally. If B_k is an upper bound for $\mathcal{T}^{(k)}$, then similar to the proof of Theorem 2.1, we can obtain that $0 \leq R_{\mathcal{T}^{(1)}} \otimes R_{\mathcal{T}^{(2)}} \leq B_1 B_2 . Id_{(H_1\otimes H_2)}$ and for each $z \in H_1 \otimes H_2$,

$$\sum_{(i(1),i(2))\in(I_1\times I_2)} \omega_{i(1)}^2 \omega_{i(2)}^2 \|(T_{i(1)} \otimes T_{i(2)})z\|^2 = \langle (R_{\mathcal{T}^{(1)}} \otimes R_{\mathcal{T}^{(2)}})z, z\rangle \leq B_1 B_2 \|z\|^2.$$

Since $R_{\mathcal{T}^{(1)}}$ and $R_{\mathcal{T}^{(2)}}$ are positive and invertible, then $\otimes_{k=1}^{2} R_{\mathcal{T}^{(k)}}$ is also positive and invertible. Thus for each $z \in H_1 \otimes H_2$, we have

$$\left\|(\otimes_{k=1}^{2} R_{\mathcal{T}^{(k)}})^{-\frac{1}{2}}\right\|^{-2}\|z\|^2 \leq \left\|(R_{\mathcal{T}^{(1)}} \otimes R_{\mathcal{T}^{(2)}})^{\frac{1}{2}} z\right\|^2 = \sum_{i(1)\in I_1}\sum_{i(2)\in I_2} \omega_{i(1)}^2 \omega_{i(2)}^2 \|(T_{i(1)} \otimes T_{i(2)})z\|^2.$$

Therefore $\otimes_{k=1}^{2} \mathcal{T}^{(k)}$ is an AURI. From the above conclusions it is clear that

$$R_{\otimes_{k=1}^{2}\mathcal{T}^{(k)}} = \otimes_{k=1}^{2} R_{\mathcal{T}^{(k)}}.$$

Corollary 3.7. *Suppose that $T_{i(k)j} : H_{kj} \longrightarrow H_{kj}$ is a positive operator for each $j \in J$ and $1 \leq k \leq n$, such that $\{\mathcal{T}_j^{(k)} = \{(T_{i(k)j}, \omega_{i(k)})\}_{i(k)\in I_k}\}_{j\in J}$ is a BFARI, for each $k \in \{1, \ldots, n\}$. Then*

$$\left\{\left((\oplus_{j\in J} T_{i(1)j}) \otimes \ldots \otimes (\oplus_{j\in J} T_{i(n)j}), \omega_{i(1)} \ldots \omega_{i(n)}\right)\right\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$$

is an AURI for $\otimes_{k=1}^{n}(\oplus_{j\in J} H_{kj})$.

Let $V = \{(V_i, \upsilon_i)\}_{i\in\mathcal{I}}$ be a fusion frame and $W = \{(W_i, \omega_i)\}_{i\in\mathcal{I}}$ be a Bessel fusion sequence for H. If $f = \sum_{i\in\mathcal{I}} \upsilon_i \omega_i \pi_{W_i} S_V^{-1} \pi_{V_i} f$, for each $f \in H$, then W is called an *alternate dual* of V ([**11**, Definition 2.7]).

In the rest of this section $\mathcal{V}_j = \{(V_{ij}, \upsilon_i)\}_{i\in I}$, $\mathcal{V}^{(k)} = \{(V_{i(k)}, \upsilon_{i(k)})\}_{i(k)\in I_k}$, $\mathcal{V}_j^{(k)} = \{(V_{i(k)j}, \upsilon_{i(k)})\}_{i(k)\in I_k}$, $\Psi^{(k)} = \{\Gamma_{i(k)} \in L(H_k, H_{i(k)})\}_{i(k)\in I_k}$, $\Psi_j^{(k)} = \{\Gamma_{i(k)j} \in L(H_{kj}, H_{i(k)j})\}_{i(k)\in I_k}$, where V_{ij}, $V_{i(k)}$ and $V_{i(k)j}$ are closed subspaces of H_j, H_k and H_{kj}, respectively.

Corollary 3.8. (i) *Suppose that $\mathcal{W}^{(k)}$'s and $\mathcal{V}^{(k)}$'s are Bessel fusion sequences and fusion frames, respectively. If $\mathcal{W}^{(k)}$ is an alternate dual of $\mathcal{V}^{(k)}$, for each $k \in \{1, \ldots, n\}$, then $\otimes_{k=1}^{n} \mathcal{W}^{(k)}$ is an alternate dual of $\otimes_{k=1}^{n} \mathcal{V}^{(k)}$.*

(ii) *Suppose that $\Phi^{(k)}$'s and $\Psi^{(k)}$'s are g-Bessel sequences. If $\Phi^{(k)}$ is a g-dual of $\Psi^{(k)}$, for each $k \in \{1, \ldots, n\}$, then $\otimes_{k=1}^{n} \Phi^{(k)}$ is a g-dual of $\otimes_{k=1}^{n} \Psi^{(k)}$.*

(iii) *If $\Phi^{(k)}$'s are g-frames, then $\widetilde{\otimes_{k=1}^{n} \Phi^{(k)}} = \otimes_{k=1}^{n} \widetilde{\Phi^{(k)}}$ ($\sim$ is used for showing the canonical dual of a g-frame).*

Proof. (i) First note that by Theorem 2.1 and Corollary 2.2, we have that $\otimes_{k=1}^{n} \mathcal{V}^{(k)}$ and $\otimes_{k=1}^{n} \mathcal{W}^{(k)}$ are fusion frame and Bessel fusion sequence, respectively. Now if we define $T_{i(k)} = \upsilon_{i(k)} \omega_{i(k)} \pi_{W_{i(k)}} S_{\mathcal{V}^{(k)}}^{-1} \pi_{V_{i(k)}}$, then it can be obtained from [**5**, Lemma 3.9] that $\mathcal{T}^{(k)} = \{T_{i(k)}\}_{i(k)\in I_k}$ is an unconditional resolution of the identity for H_k. Now by using the first part of the proof of Proposition 3.6, $\otimes_{k=1}^{n} \mathcal{T}^{(k)}$ is an unconditional resolution of the identity for $\otimes_{k=1}^{n} H_k$, which is equivalent to say that $\otimes_{k=1}^{n} \mathcal{W}^{(k)}$ is an alternate dual of $\otimes_{k=1}^{n} \mathcal{V}^{(k)}$.

(ii) If we define $T_{i(k)} = \Gamma_{i(k)}^{*} \Lambda_{i(k)}$, then $\mathcal{T}^{(k)} = \{T_{i(k)}\}_{i(k)\in I_k}$ is an unconditional resolution of the identity for H_k, see [**21**, Definition 3.2]. Now similar to part (i), by using Proposition 3.6, we can get the result.

(iii) Since $S^{-1}_{\otimes_{k=1}^{n}\Phi^{(k)}} = \otimes_{k=1}^{n} S^{-1}_{\Phi^{(k)}}$, then for each $(i(1),\ldots,i(n)) \in I_1\times\ldots\times I_n$, we have

$$(\Lambda_{i(1)}\otimes\ldots\otimes\Lambda_{i(n)})S^{-1}_{\otimes_{k=1}^{n}\Phi^{(k)}} = (\Lambda_{i(1)}S^{-1}_{\Phi^{(1)}})\otimes\cdots\otimes(\Lambda_{i(n)}S^{-1}_{\Phi^{(n)}}).$$

This shows that $\widetilde{\otimes_{k=1}^{n}\Phi^{(k)}} = \otimes_{k=1}^{n}\widetilde{\Phi^{(k)}}$. □

Note that if $W = \{(W_i,\omega_i)\}_{i\in\mathcal{I}}$ is a Bessel fusion sequence with upper bound B and $V = \{(V_i,\upsilon_i)\}_{i\in\mathcal{I}}$ is a (C,D) fusion frame for H, then by [**5**, Lemma 3.9], for each $f\in H$, $\sum_{i\in\mathcal{I}}\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}f$ converges unconditionally and for each $f,g\in H$,

$$\begin{aligned}\left|\left\langle\sum_{i\in\mathcal{I}}\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}f,g\right\rangle\right| &\le \left(\sum_{i\in\mathcal{I}}\|S_V^{-1}\|^2\upsilon_i^2\|\pi_{V_i}f\|^2\right)^{\frac{1}{2}}\left(\sum_{i\in\mathcal{I}}\omega_i^2\|\pi_{W_i}g\|^2\right)^{\frac{1}{2}}\\ &\le \frac{\sqrt{BD}}{C}\|f\|\|g\|.\end{aligned}$$

Hence the operator $\sum_{i\in\mathcal{I}}\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}$ which is defined on H by

$$\left(\sum_{i\in\mathcal{I}}\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}\right)(f) = \sum_{i\in\mathcal{I}}\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}f,$$

is bounded.

Proposition 3.9. *Let $\{\mathcal{W}_j\}_{j\in J}$ be a BFBFS and $\{\mathcal{V}_j\}_{j\in J}$ be a BFFF. Then $\mathcal{W}_j$ is an alternate dual of $\mathcal{V}_j$, for each $j\in J$ if and only if $\oplus_{j\in J}\mathcal{W}_j$ is an alternate dual of $\oplus_{j\in J}\mathcal{V}_j$.*

Proof. By Corollary 2.7, $\oplus_{j\in J}\mathcal{V}_j$ and $\oplus_{j\in J}\mathcal{W}_j$ are fusion frame and Bessel fusion sequence, respectively. Let $\{f_j\}_{j\in J}, g = \{g_j\}_{j\in J} \in \oplus_{j\in J}H_j$ and let F be a finite subset of J. Put $f_F = \{\chi_F(j)f_j\}_{j\in J}$, then we have

$$\begin{aligned}\left\langle\sum_{i\in I}\upsilon_i\omega_i\pi_{\oplus_{j\in J}W_{ij}}S^{-1}_{\oplus_{j\in J}\mathcal{V}_j}\pi_{\oplus_{j\in J}V_{ij}}f_F,g\right\rangle &= \sum_{j\in F}\left\langle\sum_{i\in I}\upsilon_i\omega_i\pi_{W_{ij}}S^{-1}_{\mathcal{V}_j}\pi_{V_{ij}}f_j,g_j\right\rangle\\ &= \langle f_F,g\rangle.\end{aligned}$$

Since the operator $\left(\sum_{i\in I}\upsilon_i\omega_i\pi_{\oplus_{j\in J}W_{ij}}S^{-1}_{\oplus_{j\in J}\mathcal{V}_j}\pi_{\oplus_{j\in J}V_{ij}}\right)$ is bounded, then

$$\sum_{i\in I}\upsilon_i\omega_i\pi_{\oplus_{j\in J}W_{ij}}S^{-1}_{\oplus_{j\in J}\mathcal{V}_j}\pi_{\oplus_{j\in J}V_{ij}} = Id_{(\oplus_{j\in J}H_j)},$$

and the result follows. The converse is obvious. □

As a consequence of the above results and [**20**, Propositions 3.4 and 3.5], we have the following:

Corollary 3.10. (i) *Let* $\{\mathcal{W}_j^{(k)}\}_{j\in J}$ *and* $\{\mathcal{V}_j^{(k)}\}_{j\in J}$ *be BFBFS and BFFF, for each* $1 \le k \le n$*, respectively and let* $\mathcal{W}_j^{(k)}$ *be an alternate dual of* $\mathcal{V}_j^{(k)}$*, for each* $j \in J$ *and* $k \in \{1, \dots, n\}$*. Then* $\otimes_{k=1}^n(\oplus_{j\in J}\mathcal{W}_j^{(k)})$ *is an alternate dual of* $\otimes_{k=1}^n(\oplus_{j\in J}\mathcal{V}_j^{(k)})$.

(ii) *Let* $\{\Phi_j^{(k)}\}_{j\in J}$ *and* $\{\Psi_j^{(k)}\}_{j\in J}$ *be BFGBS, for each* $1 \le k \le n$ *and let* $\Phi_j^{(k)}$ *be a g-dual of* $\Psi_j^{(k)}$*, for each* $j \in J$ *and* $k \in \{1, \dots, n\}$*. Then* $\otimes_{k=1}^n(\oplus_{j\in J}\Phi_j^{(k)})$ *is a g-dual of* $\otimes_{k=1}^n(\oplus_{j\in J}\Psi_j^{(k)})$.

(iii) *Let* $\{\Phi_j^{(k)}\}_{j\in J}$ *be a BFGF, for each* $1 \le k \le n$*. Then* $\otimes_{k=1}^n(\oplus_{j\in J}\widetilde{\Phi_j^{(k)}})$ *is the canonical dual of* $\otimes_{k=1}^n(\oplus_{j\in J}\Phi_j^{(k)})$.

Acknowledgements. The authors would like to thank the referee for his/her careful reading of the paper and several valuable comments and suggestions which improved the manuscript.

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(Received November 26, 2011)
(Revised May 4, 2012)