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COMPOSITIONS AS NON-ALTERNATING SEQUENCES OF PARTITIONS

Aubrey Blecher, Charlotte Brennan, Toufik Mansour

Compositions are conceptualized as non alternating sequences of blocks of non-decreasing and strictly decreasing partitions. We find the generating function $F(x, y, q)$ where x marks the size of the composition, y the number of parts and q the number of such partition blocks minus 1. We form these blocks starting on the left-hand-side of the composition and maximizing the size of each block. We also find the generating function for the total number of such blocks.

1. INTRODUCTION

The splitting of positive integers into the sum of "smaller" positive integers has been intensively studied in terms of partitions [**1, 2**] and compositions [**6, 7**]. Recently, some work has been done on the unification of these (usually) disjoint theories. See, for example, [**3, 4, 5, 6, 8**] where general compositions have been described as alternating sequences of increasing/decreasing partitions.

Possibly, a natural view of compositions is as a sequence of (non-alternating) partitions. The advantage of such a view is that it suggests a basis for unifying the study of different combinatorial objects (partitions and compositions). This is achieved by allowing the techniques usually reserved for one or other of these objects to cross the boundary and be used in the study of the other. As such, we hope it will be a useful addition to the armoury of other researchers.

In this paper, we develop the generating function expressing an arbitrary composition as a non-alternating sequence of non-decreasing or strictly decreasing partitions.

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A *composition* $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ of a positive integer n is an ordered collection of one or more positive integers whose sum is n. Each summand σ_i is called a part of the composition. A *partition* of a positive integer n is either a non-increasing or a non-decreasing composition of n. For example, for the partition of n with k parts, $n = a_1 + a_2 + \cdots + a_k$ either

$$a_1 \geq a_2 \geq \cdots \geq a_k \geq 1 \quad \text{or} \quad 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k.$$

Consider arbitrary compositions of n. We split them into blocks of weakly increasing (non-decreasing) or strictly decreasing partitions, beginning on the left and maximizing the size of each block. The blocks do not have to alternate, as was the case in [**5**] and [**6**]. The block division is constructed as follows. The first block is made up of maximal number of weakly increasing or strictly decreasing parts. The next part (if it exists) forms the beginning of the next block which again consists of the maximal number of weakly increasing or strictly decreasing parts. This process is iterated until we reach the last part of the composition.

For example, consider the composition of 29: 3 3 4 2 5 2 5 4 1. It can be split correctly as (3 3 4)(2 5)(2 5)(4 1) starting with a weakly increasing partition and incorrectly as (3)(3 4)(2 5)(2 5)(4 1) starting with a strictly decreasing block. The former is chosen as it has a larger initial block.

Our main result is given in Theorem 2 and can be formulated as follows. Define $F(x, y, q) := \sum_{\pi} x^n\, y^m\, q^r$, where the sum is over all non-empty compositions π, x marks the size of π, y the number of parts of π and q the number of partition blocks minus one in π. Then Theorem 2 expresses F in terms of q-series. To achieve this we define $P(t) = \prod_{j\geq 0}(1 - tx^j) = (t; x)_\infty$ and $H_{a,b}(t) = aP(t) + bP(-t)$, for any constants a, b. Then Theorem 2 gives

$$F(x, y, q) = \frac{A}{q'^3 H_{(1+q')^2,-(1-q')^2}(xyq') - qA},$$

where $q' = \sqrt{1 - 2q}$ and

$$A = 4(1-q)\big(1 + P(xyq')P(-xyq')\big) - \left(\frac{q'^2xy}{1-x} + 2 + 2q'^2\right) H_{1+q',1-q'}(xyq') - \frac{q'xy}{1-x} H_{1+q',-1+q'}(xyq').$$

We expand this generating function as the following series up to the term x^8.

$$\begin{aligned}F(x, y, q) &= xy + x^2y(1 + y) + x^3y(1 + y)^2 + x^4y(1 + 3y + y^2(1 + 2q) + y^3)\\ &+ x^5y(1 + 4y + 2y^2(1 + 2q) + y^3(1 + 3q) + y^4)\\ &+ x^6y(1 + 5y + 2y^2(2 + 3q) + 2y^3(1 + 4q) + y^4(1 + 4q) + y^5)\\ &+ x^7y(1 + 6y + 5y^2(1 + 2q) + y^3(3 + 17q) + y^4(2 + 10q + 3q^2)\end{aligned}$$

$$+ y^5(1+5q) + y^6) + x^8 y \left(1 + 7y + 7y^2(1+2q) + 5y^3(1+6q) + y^4(3+22q)\right.$$
$$\left. + 10y^4q^2 + y^5(2+13q+6q^2) + y^6(1+6q) + y^7\right) + \cdots .$$

We illustrate the splitting of the compositions of $n = 1$ to 5 into blocks of weakly increasing and strictly decreasing partitions in the table below:

n	Compositions of n split into blocks	Term in series
1	(1)	xy
2	(2)	x^2y
	(11)	x^2y^2
3	(3)	x^3y
	(12),(21)	$2x^3y^2$
	(111)	x^3y^3
4	(4)	x^4y
	(22), (13), (31)	$3x^4y^2$
	(112)	x^4y^3
	(12)(1),(21)(1)	$2x^4y^3q$
	(1111)	x^4y^4
5	(5)	x^5y
	(14), (41), (23), (32)	$4x^5y^2$
	(122), (113)	$2x^5y^3$
	(22)(1), (21)(2), (13)(1), (31)(1)	$4x^5y^3q$
	(1112)	x^5y^4
	(112)(1), (12)(11), (21)(11)	$3x^5y^4q$
	(11111)	x^5y^5

Table 1. Blocks in the compositions of n for $n = 1$ to 4

If we are not interested in the number of parts in these compositions, we let $y = 1$ and obtain the following coefficients of x^iq^j in $F(x, 1, q)$ for $n = 0$ to 13 and $q = 0$ to 4:

$q\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1	2	4	6	9	14	19	27	37	51	67	91	118
1	0	0	0	0	2	7	18	42	85	162	288	499	827	1342
2	0	0	0	0	0	0	0	3	16	57	169	428	991	2126
3	0	0	0	0	0	0	0	0	0	0	4	30	139	505
4	0	0	0	0	0	0	0	0	0	0	0	0	0	5

Table 2. The coefficients of x^iq^j in $F(x, 1, q)$

After this, in Corollary 1, we deduce the generating function for the total number of blocks in an arbitrary composition of n with m parts.

2. GENERATING FUNCTIONS

In order to find an explicit formula for the generating function $F(x,y,q)$ we need the following notation. Let $F_a := F_a(x,y,q) = \sum_{\pi} x^n\, y^m\, q^r$ be the generating function for all compositions π of n with exactly m parts and exactly $r+1$ partition blocks, where the first part is a. Hence, considering all possible starting value for a, we have

$$F(x,y,q) = \sum_{a\geq 1} F_a \tag{1}$$

which excludes the empty composition. We extend this notation to $F_{a_1a_2\cdots a_s}(x,y,q)$ as the generating function for the compositions starting with $a_1a_2\cdots a_s$. Let $I_a(x,y,q)$ be the generating function for compositions π starting with the letter a and a weak increase. Similarly, let $D_a(x,y,q)$ be the generating function for compositions π starting with the letter a and a strict decrease. In both generating functions x marks the size of π, y the number of parts of π and q the number of partition blocks minus one in π.

At first, we define

$$\begin{aligned} F(t) = F(t;x,y,q) &= \sum_{a\geq 1} F_a(x,y,q)t^a,\\ I(t) = I(t;x,y,q) &= \sum_{a\geq 1} I_a(x,y,q)t^a,\\ D(t) = D(t;x,y,q) &= \sum_{a\geq 1} D_a(x,y,q)t^a. \end{aligned}$$

In this section, we find the generating function F_a for all compositions beginning with a. These consist of the one letter word or the words starting with an increase or a decrease. Thus

$$F_a(x,y,q) = x^a y + I_a(x,y,q) + D_a(x,y,q),$$

which leads to

$$F(t) = \frac{xyt}{1-xt} + I(t) + D(t). \tag{2}$$

First, we find $I_a = I_a(x,y,q)$. We consider the two letter words aj separately and split the larger word into two cases as follows:

$$\begin{aligned} I_a &= \sum_{j\geq a} F_{aj} = \sum_{j\geq a} x^{a+j}y^2 + \sum_{j\geq a}\sum_{i\geq j} F_{aji} + \sum_{j\geq a}\sum_{i<j} F_{aji}\\ &= \frac{x^{2a}y^2}{1-x} + x^a y \sum_{j\geq a}\sum_{i\geq j} F_{ji} + qy^2 \sum_{j\geq a} x^{a+j} \sum_{i<j} F_i \end{aligned}$$

$$= \frac{x^{2a}y^2}{1-x} + x^a y \sum_{j \geq a} I_j + qy^2 \sum_{j \geq a} x^{a+j} \sum_{i<j} F_i$$
$$= \frac{x^{2a}y^2}{1-x} + x^a y \sum_{j \geq a} I_j + qy^2 \left(x^{2a} \sum_{i<a} F_i + x^{2a+1} \sum_{i<a+1} F_i + x^{2a+2} \sum_{i<a+2} F_i + \cdots \right).$$

Thus

$$\begin{aligned} I(t) &= \frac{x^2y^2t}{(1-x)(1-x^2t)} + \sum_{a \geq 1} \left(t^a x^a y \sum_{j \geq a} I_j \right) \\ &+ qy^2 \sum_{a \geq 1} t^a \left(x^{2a} \sum_{i<a} F_i + x^{2a+1} \sum_{i<a+1} F_i + x^{2a+2} \sum_{i<a+2} F_i + \cdots \right) \\ &= \frac{x^2y^2t}{(1-x)(1-x^2t)} + xyt \sum_{j \geq 1} \frac{1-x^jt^j}{1-xt} I_j \\ &+ qy^2 \left(x^2t \frac{1-xt}{1-xt} \sum_{i<1} F_i + x^3t \frac{1-x^2t^2}{1-xt} \sum_{i<2} F_i + x^4t \frac{1-x^3t^3}{1-xt} \sum_{i<3} F_i + \cdots \right) \\ &= \frac{x^2y^2t}{(1-x)(1-x^2t)} + xyt \sum_{j \geq 1} \frac{1-x^jt^j}{1-xt} I_j \\ &+ \frac{qx^2y^2t}{1-xt} \left(\frac{1}{1-x} \sum_{j \geq 1} F_j x^j - \frac{xt}{1-x^2t} \sum_{j \geq 1} F_j (x^2t)^j \right), \end{aligned}$$

which implies

$$\begin{aligned} I(t) &= \frac{x^2y^2t}{(1-x)(1-x^2t)} + \frac{xyt}{1-xt}(I(1) - I(xt)) \\ &+ \frac{qx^2y^2t}{(1-xt)(1-x)} F(x) - \frac{qx^3y^2t^2}{(1-xt)(1-x^2t)} F(x^2t). \end{aligned} \tag{3}$$

Similarly for $D_a := D_a(x,y,q)$ we have

$$\begin{aligned} D_a &= \sum_{j<a} F_{aj} = \sum_{j<a} x^{a+j} y^2 + \sum_{j<a} \sum_{i \geq j} F_{aji} + \sum_{j<a} \sum_{i<j} F_{aji} \\ &= \frac{(x^{a+1} - x^{2a})y^2}{1-x} + q \sum_{j<a} x^{a+j} y^2 \sum_{i \geq j} F_i + x^a y \sum_{j<a} D_j, \end{aligned}$$

which leads to

$$\begin{aligned} D(t) &= \frac{x^3y^2t^2}{(1-xt)(1-x^2t)} + \frac{xyt}{1-xt} \sum_{j \geq 1} D_j x^j t^j + \frac{qx^3y^2t^2}{1-xt} \sum_{j \geq 1} \frac{1-(x^2t)^j}{1-x^2t} F_j \\ &= \frac{x^3y^2t^2}{(1-xt)(1-x^2t)} + \frac{xyt}{1-xt} D(xt) + \frac{qx^3y^2t^2}{(1-xt)(1-x^2t)} (F(1) - F(x^2t)). \end{aligned} \tag{4}$$

By (3) and (4) we obtain

$$\begin{aligned} I(t)-D(t) &= \frac{x^2y^2t}{(1-x)(1-x^2t)}+\frac{xyt}{1-xt}I(1)+\frac{qx^2y^2t}{(1-xt)(1-x)}F(x) \\ &\quad - \frac{x^3y^2t^2}{(1-xt)(1-x^2t)}-\frac{qx^3y^2t^2}{(1-xt)(1-x^2t)}F(1)-\frac{xyt}{1-xt}(I(xt)+D(xt)), \end{aligned}$$

which, by (2), implies

$$\begin{aligned} I(t)-D(t) &= \frac{x^2y^2t}{(1-x)(1-x^2t)}+\frac{xyt}{1-xt}I(1)+\frac{qx^2y^2t}{(1-xt)(1-x)}F(x) \\ &\quad - \frac{qx^3y^2t^2}{(1-xt)(1-x^2t)}F(1)-\frac{xyt}{1-xt}F(xt). \end{aligned} \tag{5}$$

By summing (3) and (4) and using (2), we obtain

$$\begin{aligned} F(t)-\frac{xyt}{1-xt} &= I(t)+D(t) \\ &= \frac{x^2y^2t}{(1-x)(1-x^2t)}+\frac{xyt}{1-xt}I(1)+\frac{qx^3y^2t^2}{(1-xt)(1-x^2t)}F(1) \\ &\quad + \frac{qx^2y^2t}{(1-xt)(1-x)}F(x)-\frac{2qx^3y^2t^2}{(1-xt)(1-x^2t)}F(x^2t), \\ &\quad + \frac{x^3y^2t^2}{(1-xt)(1-x^2t)}+\frac{xyt}{1-xt}(D(xt)-I(xt)). \end{aligned}$$

Thus by (5), we obtain

$$\begin{aligned} F(t)-\frac{xyt}{1-xt} &= \frac{x^2y^2t}{(1-x)(1-x^2t)}+\frac{xyt}{1-xt}I(1)+\frac{qx^3y^2t^2}{(1-xt)(1-x^2t)}F(1) \\ &\quad + \frac{qx^2y^2t}{(1-xt)(1-x)}F(x)-\frac{2qx^3y^2t^2}{(1-xt)(1-x^2t)}F(x^2t), \\ &\quad + \frac{x^3y^2t^2}{(1-xt)(1-x^2t)}-\frac{x^4y^3t^2}{(1-x)(1-xt)(1-x^3t)}-\frac{x^3y^2t^2}{(1-xt)(1-x^2t)}I(1) \\ &\quad - \frac{qx^4y^3t^2}{(1-x)(1-xt)(1-x^2t)}F(x)+\frac{qx^6y^3t^3}{(1-xt)(1-x^2t)(1-x^3t)}F(1) \\ &\quad + \frac{x^3y^2t^2}{(1-xt)(1-x^2t)}F(x^2t), \end{aligned}$$

which implies that the generating function $F(t)=F(t;x,y,q)$ satisfies

$$\begin{aligned} F(t) &= \frac{(x^4(1-y)t-x^3(1+y^2)t+x(y-1)+1)xyt}{(1-x)(1-xt)(1-x^3t)}+\frac{(1-x^2(y+1)t)xyt}{(1-xt)(1-x^2t)}I(1) \\ &\quad + \frac{q(x^3t(y-1)+1)x^3y^2t^2}{(1-xt)(1-x^2t)(1-x^3t)}F(1)+\frac{q(1-x^2(y+1)t)x^2y^2t}{(1-xt)(1-x^2t)(1-x)}F(x) \end{aligned} \tag{6}$$

$$+\frac{(1-2q)x^3y^2t^2}{(1-xt)(1-x^2t)}F(x^2t).$$

Now, let us find an expression for $I(1)$ in terms of $F(1)$ and $F(x)$. By setting $t=1$ in (5) and (2), then summing the resulting two equations, we obtain

$$I(1)=\frac{xy(xy-1+x^2)}{(2-2x-xy)(1-x^2)}+\frac{(1-x)(1-x^2)-qx^3y^2}{(1-x^2)(2-2x-xy)}F(1) + \frac{xy(qxy-1+x)}{(1-x)(2-2x-xy)}F(x). \tag{7}$$

By substituting (7) in (6) we get that the generating function $F(t)=F(t;x,y,q)$ satisfies

$$F(t)=M_0(t)+M_1(t)F(1)+M_2(t)F(x)+M_3(t)F(x^2t), \tag{8}$$

where

$$\begin{aligned}
M_0(t)&=\frac{xyt}{\rho}\big(2(1-x)(1-x^2)(1-x^2t)(1-x^3t)\\
&\quad-x^3y^2(1+t(1-2x^2-x^3)-x^2t^2(2-x-2x^2))+x^5y^3t(1-xt)\big),\\
M_1(t)&=\frac{xyt}{\rho}\big((1-x)(1-x^2)(1-x^2t)(1-x^3t)\\
&\quad-x^2yt(1-x)(1-x^2)(1-2q)(1-x^3t)\\
&\quad-x^3y^2q(1+t-2x^2t+2x^4t^2+x^3t^2-x^3t-2x^2t^2)+x^5y^3tq(1-xt)\big),\\
M_2(t)&=-\frac{(1-2q)(1-x^2yt-x^2t)x^2y^2t}{(2-2x-xy)(1-xt)(1-x^2t)},\\
M_3(t)&=\frac{(1-2q)x^3y^2t^2}{(1-xt)(1-x^2t)},
\end{aligned}$$

and $\rho=(1-x^2)(2-2x-xy)(1-xt)(1-x^2t)(1-x^3t)$.

Thus, by iterating (8) and using the fact that $F(x^jt)\to 0$ when $j\to\infty$, we obtain

$$F(t)=\sum_{j\geq 0}M_0(x^{2j}t)\prod_{i=0}^{j-1}M_3(x^{2i}t) + F(1)\sum_{j\geq 0}M_1(x^{2j}t)\prod_{i=0}^{j-1}M_3(x^{2i}t)+F(x)\sum_{j\geq 0}M_2(x^{2j}t)\prod_{i=0}^{j-1}M_3(x^{2i}t). \tag{9}$$

By setting $t=1$ and $t=x$ and solving the resulting equations, we derive the following result.

Theorem 1. *Let* $(a;b)_n = \prod_{j=0}^{n-1}(1-ab^j)$. *Define for* $s=0,1,2$

$$A_s(t) = \sum_{j\geq 0} M_s(x^{2j}t)\prod_{i=0}^{j-1} M_3(x^{2i}t) = \sum_{j\geq 0}\frac{M_s(x^{2j}t)(1-2q)^j x^{2j^2+j}y^{2j}t^{2j}}{(xt;x)_{2j}}.$$

Then

$$F(x,y,q) = \frac{A_0(1)(1-A_2(x)) + A_0(x)A_2(1)}{(1-A_1(1))(1-A_2(x)) - A_1(x)A_2(1)}.$$

Our final goal is to write the statement of the above theorem in a simple form. In order to achieve this, we need the following definition and lemmas. Recall that $P(t) = \prod_{j\geq 0}(1-tx^j) = (t;x)_\infty$ and $H_{a,b}(t) = aP(t)+bP(-t)$, for any constants a,b.

Lemma 1. *Let* $q' = \sqrt{1-2q}$. *We have*

$$\sum_{j\geq 0}\frac{x^{2ja}q'^{2j}x^{2j^2-j}y^{2j}}{(x;x)_{2j}} = H_{1/2,1/2}(x^a yq')$$

and

$$\sum_{j\geq 0}\frac{x^{(2j+1)a}q'^{2j+1}x^{2j^2+j}y^{2j+1}}{(x;x)_{2j+1}} = -H_{1/2,-1/2}(x^a yq').$$

Proof. We show the first identity. The second also holds using very similar arguments. From the fact that $\sum_{j\geq 0} z^j\frac{u^{\binom{j}{2}}}{(u;u)_j} = (-z;u)_\infty$ and the definition of $H_{a,b}(x)$, we have

$$\sum_{j\geq 0}\frac{x^{2ja}q'^{2j}x^{2j^2-j}y^{2j}}{(x;x)_{2j}} = \sum_{j\geq 0}\frac{q'^{2j}x^{2j^2+(2a-1)j}y^{2j}}{(x;x)_{2j}} = \sum_{j\geq 0}\frac{(x^a yq')^{2j}x^{\binom{2j}{2}}}{(x;x)_{2j}}$$
$$= H_{1/2,1/2}(x^a yq'),$$

as claimed.

Lemma 2. *Let* $q' = \sqrt{1-2q}$. *Define* $H'(x,y) = \frac{1}{2}H_{1,-1}(yq'/x) + \frac{q'y}{x(1-x)}$ *and* $H''(x,y) = \frac{1}{2}H_{1,1}\left(\frac{yq'}{x}\right) - 1$. *We have*

$$A_2(1) = \frac{-xy}{2-2x-xy} + \frac{xy}{2(2-2x-xy)}H_{1+q',1-q'}(xyq'),$$
$$A_2(x) = \frac{1-x-xy}{2-2x-xy} - \frac{1-x}{2q'(2-2x-xy)}H_{1+q',q'-1}(xyq'),$$

$$A_1(1) = \frac{-x((1-x^2)(1-x)-qx^3y^2)}{q'^3y^2(2-2x-xy)(1-x^2)}H'(x,y)$$
$$+\frac{(1-x^2)^2+(1-x^2)(1-x)q'^2y-xq(1+x)(x^2+x-1)y^2-qx^3y^3}{q'^3y^2(2-2x-xy)(1-x^2)}H'(x,xy)$$
$$-\frac{(1+x)(1-x)^2+(1-x^2)(1-x)q'y-q(x+2x^2-2)y^2-xqy^3}{q'^3y^2(2-2x-xy)(1-x^2)}H'(x,x^2y),$$
$$A_1(x) = \frac{(1-x^2)(1-x)-qx^3y^2}{q'^4y^3(2-2x-xy)(1+x)}\left(H''(x,y)-\frac{y^2}{q'^2x(1-x)(1-x^2)}\right)$$
$$-\frac{(1-x)^2(1+x+q'y)-xq(x^2+x-1)y^2-\dfrac{qx^3y^3}{1+x}}{q'^4xy^3(2-2x-xy)}$$
$$\times\left(H''(x,xy)-\frac{q'^2y^2x}{(1-x)(1-x^2)}\right)$$
$$+\frac{(1-x)^2(1+q'y)-\dfrac{q(x+2x^2-2)y^2}{1+x}-\dfrac{qxy^3}{1+x}}{q'^4xy^3(2-2x-xy)}\left(H''(x,x^2y)-\frac{q'^2y^2x^3}{(1-x)(1-x^2)}\right),$$
$$A_0(1) = -\frac{x(2-2x-2x^2+2x^3-x^3y^2)}{q'^3y^2(2-2x-xy)(1-x^2)}H'(x,y)$$
$$+\frac{2(1-x^2)^2-x(x+1)(x^2+x-1)y^2-x^3y^3}{q'^3y^2(2-2x-xy)(1-x^2)}H'(x,xy)$$
$$+\frac{-2(1+x)(1-x)^2+(x+2x^2-2)y^2+xy^3}{q'^3y^2(2-2x-xy)(1-x^2)}H'(x,x^2y),$$
$$A_0(x) = \frac{2-2x-2x^2+2x^3-x^3y^2}{q'^4y^3(2-2x-xy)(1+x)}\left(H''(x,y)-\frac{q'^2y^2}{x(1-x)(1-x^2)}\right)$$
$$+\frac{-2(1-x^2)^2+x(x+1)(x^2+x-1)y^2+x^3y^3}{q'^4xy^3(2-2x-xy)(1+x)}\left(H''(x,xy)-\frac{q'^2xy^2}{(1-x)(1-x^2)}\right)$$
$$+\frac{2(1+x)(1-x)^2-(x+2x^2-2)y^2-xy^3}{q'^4xy^3(2-2x-xy)(1+x)}\left(H''(x,x^2y)-\frac{q'^2x^3y^2}{(1-x)(1-x^2)}\right).$$

Proof. Here, we give only the proof of the identities for $A_2(1)$ and $A_2(x)$. All the others can be obtained using similar arguments. Theorem 1 gives

$$A_2(t) = \sum_{j\geq 0}\frac{M_2(x^{2j}t)(yq't)^{2j}x^{2j^2+j}}{(xt;x)_{2j}}.$$

Since $M_2(t) = -\frac{(1-2q)(1-x^2yt-x^2t)x^2y^2t}{(2-2x-xy)(1-xt)(1-x^2t)}$, this implies

$$A_2(t) = -\frac{q'^2y^2}{2-2x-xy}\sum_{j\geq 0}\frac{(1-x^{2j+2}t(1+y))(yq')^{2j}x^{2j^2+3j+2}t^{2j+1}}{(xt;x)_{2j+2}}. \tag{10}$$

Therefore, by Lemma 1, we obtain

$$\begin{aligned}A_2(1) &= \frac{1+y}{2-2x-xy}\sum_{j\geq 0}\frac{(yq')^{2j+2}x^{2j^2+5j+4}}{(x;x)_{2j+2}} - \frac{1}{2-2x-xy}\sum_{j\geq 0}\frac{(yq')^{2j+2}x^{2j^2+3j+2}}{(x;x)_{2j+2}} \\ &= \frac{x(1+y)}{2-2x-xy}\sum_{j\geq 1}\frac{(xyq')^{2j}x^{\binom{2j}{2}}}{(x;x)_{2j}} - \frac{x}{2-2x-xy}\sum_{j\geq 1}\frac{(yq')^{2j}x^{\binom{2j}{2}}}{(x;x)_{2j}}.\end{aligned}$$

Thus, $A_2(1)$ can be written in term of $P(x)$ as follows:

$$\begin{aligned}A_2(1) &= \frac{x}{2(2-2x-xy)}((1+y)(P(xyq')+P(-xyq')-2)-(P(yq') \\ &\quad +P(-yq')-2)) \\ &= \frac{-xy}{2-2x-xy}+\frac{x}{2(2-2x-xy)}(y(1+q')P(xyq')+y(1-q')P(-xyq')) \\ &= \frac{-xy}{2-2x-xy}+\frac{xy}{2(2-2x-xy)}H_{1+q',1-q'}(xyq'),\end{aligned}$$

as required.

Now, let us prove the formula for $A_2(x)$. By (10) and Lemma 1, we obtain

$$\begin{aligned}A_2(x) &= -\frac{1}{2-2x-xy}\sum_{j\geq 0}\frac{(1-x^{2j+3}(1+y))(yq')^{2j+2}x^{2j^2+5j+3}}{(x^2;x)_{2j+2}} \\ &= -\frac{1-x}{q'y(2-2x-xy)}\sum_{j\geq 1}\frac{(1-x^{2j+1}(1+y))(yq')^{2j+1}x^{\binom{2j+1}{2}}}{(x;x)_{2j+1}}.\end{aligned}$$

Thus, $A_2(x)$ can be written in term of $P(x)$ as follows:

$$\begin{aligned}A_2(x) &= \left(P(-yq')-P(yq')-(1+y)P(-xyq')+(1+y)P(xyq')+\frac{2q'y(x+xy-1)}{1-x}\right) \\ &\quad\times\frac{-(1-x)}{2q'y(2-2x-xy)} \\ &= -\frac{(1-x)\left((1+q')P(xyq')+(q'-1)P(-xyq')+\frac{2q'(x+xy-1)}{1-x}\right)}{2q'(2-2x-xy)} \\ &= \frac{1-x-xy}{2-2x-xy}-\frac{1-x}{2q'(2-2x-xy)}H_{1+q',q'-1}(xyq'),\end{aligned}$$

as required. □

Now we are ready to present our main result.

Theorem 2. *Let $q'=\sqrt{1-2q}$, then the generating function $F(x,y,q)$ is given by*

$$\frac{A}{q'^3H_{(1+q')^2,-(1-q')^2}(xyq')-qA},$$

where

$$A = 4(1-q)(1+P(xyq')P(-xyq')) - \left(\frac{q'^2xy}{1-x} + 2 + 2q'^2\right) H_{1+q',1-q'}(xyq')$$
$$- \frac{q'xy}{1-x} H_{1+q',-1+q'}(xyq').$$

Proof. Theorem 1 yields

$$F(x,y,q) = \frac{F_1}{F_2} \equiv \frac{A_0(1)(1-A_2(x)) + A_0(x)A_2(1)}{(1-A_1(1))(1-A_2(x)) - A_1(x)A_2(1)}.$$

So, using Lemma 2, we may write F_1 and F_2 as

$$F_1 = e_{11}P(q'y)P(-q'y) + e_{10}P(q'y) + e_{01}P(-q'y) + e_{00}$$

and

$$F_2 = -\,qe_{11}P(-q'y)P(q'y) + \left(-qe_{10} + \frac{(1+q')^2(1-x)}{2q'(1-yq')(2-2x-xy)}\right) P(q'y)$$
$$+ \left(-qe_{01} - \frac{(1-q')^2(1-x)}{2q'(1+yq')(2-2x-xy)}\right) P(-yq') - qe_{00},$$

where

$$e_{11} = \frac{2(x-1)(q-1)}{q'^4(1-y^2q'^2)(2-2x-xy)},$$
$$e_{10} = \frac{2(1+q'^2)(1+q')(x-1) + q'(q'+1)(2(x-1)q'^2 - xq' + x - 2)y - xq'^2(1+q')^2y^2}{2q'^4(1-y^2q'^2)(2-2x-xy)},$$
$$e_{01} = \frac{2(1+q'^2)(1-q')(x-1) + q'(q'-1)(2(x-1)q'^2 + xq' + x - 2)y - xq'^2(1-q')^2y^2}{2q'^4(1-y^2q'^2)(2-2x-xy)},$$
$$e_{00} = \frac{2(q-1)(x-1)}{q'^4(2-2x-xy)}.$$

After several algebraic operations, one may write the generating functions F_2 and F_1 as

$$F_2 = \frac{(1+q')^2(1-x)}{2q'(1-yq')(2-2x-xy)} P(q'y) - \frac{(1-q')^2(1-x)}{2q'(1+yq')(2-2x-xy)} P(-yq') - qF_1$$
$$= \frac{(1+q')^2(1-x)}{2q'(2-2x-xy)} P(xyq') - \frac{(1-q')^2(1-x)}{2q'(2-2x-xy)} P(-xyq') - qF_1$$
$$= \frac{1-x}{2q'(2-2x-xy)} H_{(1+q')^2,-(1-q')^2}(xyq') - qF_1$$

and

$$F_1 = e_{11}P(q'y)P(-q'y) + e_{10}P(q'y) + e_{01}P(-q'y) + e_{00}$$

$$= \frac{1-x}{2q'^4(2-2x-xy)}A.$$

Hence, the generating function $F(x,y,q)$ is given by

$$F(x,y,q) = \frac{F_1}{F_2} = \frac{\frac{1-x}{2q'^4(2-2x-xy)}A}{\frac{1-x}{2q'(2-2x-xy)}H_{(1+q')^2,-(1-q')^2}(xyq') - q\frac{1-x}{2q'^4(2-2x-xy)}A}$$
$$= \frac{A}{q'^3 H_{(1+q')^2,-(1-q')^2}(xyq') - qA},$$

which completes the proof.

EXAMPLE 1. Putting $y = q = 1$ in Theorem 2 gives

$$F(x,1,1) = \frac{\frac{x}{1-x}H_{1+i,1-i}(ix) - \frac{ix}{1-x}H_{1+i,-1+i}(ix)}{-iH_{2i,2i}(ix) - \frac{x}{1-x}H_{1+i,1-i}(ix) + \frac{ix}{1-x}H_{1+i,-1+i}(ix)},$$

where $i^2 = -1$. Thus, by definition of the generating function $H_{a,b}(x)$ and $P(x)$, we obtain

$$F(x,1,1) = \frac{\frac{ix}{1-x}H_{1,-1}(i) + \frac{x}{1-x}H_{1,1}(ix)}{2H_{1,1}(i) - \frac{ix}{1-x}H_{1,-1}(i) - \frac{x}{1-x}H_{1,1}(i)} = \frac{1}{\frac{1-x}{x}\cdot\frac{2H_{1,1}(ix)}{iH_{1,-1}(i)+H_{1,1}(i)} - 1}$$
$$= \frac{1}{\frac{1-x}{x}\cdot\frac{2H_{1,1}(i)}{(1+i)P(i)+(1-i)P(-i)} - 1}$$
$$= \frac{1}{\frac{1-x}{x}\cdot\frac{2H_{1,1}(ix)}{(1+i)(1-i)P(ix)+(1-i)(1+i)P(-ix)} - 1}$$
$$= \frac{1}{\frac{1-x}{x}\cdot\frac{2H_{1,1}(ix)}{2P(ix)+2P(-ix)} - 1} = \frac{1}{\frac{1-x}{x} - 1} = \frac{x}{1-2x},$$

which agrees with the fact that the generating function for the number of non-empty compositions of n is $\frac{x}{1-2x}$.

EXAMPLE 2. For $q = 0$, Theorem 2 leads to the generating function $F(x,1,0)$ being given by

$$\frac{1+P(x)P(-x)}{P(x)} - 2 - \frac{x}{1-x} = p(x) + r(x) - 1,$$

where $p(x)$ is the generating function for integer partitions of n and $r(x)$ is the generating function for integer partitions of n with distinct parts. This fact can be explained as follows: since we have a composition with exactly one partition block (recall that q counts the number of blocks minus 1), the composition is either non-decreasing or decreasing.

Notice that if the composition has one block then it is either a non-decreasing or a decreasing partition. Thus, the generating function for the number of compositions of n with exactly one partition block is $p(x)+r(x)-1$.

Now we find the generating function for the total number of partition blocks minus one in compositions of n.

Corollary 1. *The generating function* $\frac{\partial}{\partial q}F(x,1,q)\Big|_{q=1}$ *is given by*

$$\frac{2-6x+5x^2}{(2x-1)^2}-\frac{2(x-1)^2}{(2x-1)^2}\frac{1}{\dfrac{1}{\prod_{j\geq 1}(1-ix^j)+i}+\dfrac{1}{\prod_{j\geq 1}(1+ix^j)-i}},$$

where $i^2=-1$.

Proof. After several algebraic operations, Theorem 2 yields

$$\frac{\partial}{\partial q}F(x,1,q)\Big|_{q=1}$$
$$=\frac{2(x-1)^2\left(iH_{1,1}(ix)-1-\prod_{j\geq 1}(1+x^{2j})\right)+(2-6x+5x^2)H_{1,1}(ix)}{(2x-1)^2H_{1,1}(ix)},$$

which leads to our result.

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School of Mathematics,
University of the Witwatersrand,
Private Bag 3, Wits 2050, Johannesburg
South Africa
E-mail: aubrey.blecher@wits.ac.za

The John Knopfmacher Centre
for Applicable Analysis and Number Theory,
School of Mathematics,
University of the Witwatersrand,
Private Bag 3, Wits 2050, Johannesburg
South Africa
E-mail: charlotte.brennan@wits.ac.za

Department of Mathematics,
University of Haifa, 31905 Haifa
Israel
E-mail: tmansour@univ.haifa.ac.il

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