ROMAN DOMINATION IN CARTESIAN PRODUCT GRAPHS AND STRONG PRODUCT GRAPHS

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A map $f : V \to \{0, 1, 2\}$ is a Roman dominating function for G if for every vertex v with $f(v) = 0$, there exists a vertex u, adjacent to v, with $f(u) = 2$. The weight of a Roman dominating function is $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on G is the Roman domination number of G. In this paper we study the Roman domination number of Cartesian product graphs and strong product graphs.

1. INTRODUCTION

The behavior of several graph parameters in product graphs has become an interesting topic of research [**10, 11**]. For instance, we emphasize the Shannon capacity of a graph [**14**], which is a certain limiting value involving the vertex independence number of strong product powers of a graph, and Hedetniemi's coloring conjecture for the categorical product [**8, 11**], which states that the chromatic number of any categorical product graph is equal to the minimum value of the chromatic numbers of its factors. Also, one of the oldest open problems on domination in graphs is related to the Cartesian product graphs. The problem was presented first by Vizing in 1963 [**16**]. Vizing's conjecture states that the domination number of any Cartesian product graph is greater than or equal to the product of the domination numbers of its factors.

Vizing's conjecture has become one of the most interesting problems on domination in graphs, and has led to other Vizing-like results for several parameters,

including some not related to standard domination. Much research has been developed in this sense and the conjecture has been proved for several families of graphs. The surveys [1, 6] contain almost all the results obtained on the conjecture. Also, these surveys contain some references to similar open problems on product graphs. Nevertheless, the conjecture remains open. One variant of domination is the concept of Roman domination introduced by COCKAYNE et al. in [3], according to some connections with historical problems of defending the Roman Empire, for instance [15]. Roman domination has been studied further by other authors, see [4, 5, 9, 18]. In this article we obtain Vizing-like results for the Roman domination number of Cartesian product graphs and strong product graphs.

We begin by establishing the principal terminology and notation which we use throughout the article. Hereafter $G = (V, E)$ denotes a finite simple graph. For two adjacent vertices u and v of G we use the notation $u \sim v$ and, in this case, we say that uv is an edge of G, *i.e.*, $uv \in E$. For a vertex v of G, $N(v) = \{u \in V : u \sim v\}$ denotes the set of neighbors that v has in G, and is called the *open neighborhood of v*. The *closed neighborhood of v* is defined as $N[v] = N(v) \cup \{v\}$. For a set $D \subseteq V$, the *open neighborhood* is $N(D) = \cup_{v \in D} N(v)$ and the *closed neighborhood* is $N[D] = N(D) \cup D$. A set D is a *dominating set* if $N[D] = V$. The *domination number $\gamma(G)$* is the minimum cardinality of a dominating set in G. We say that a set S is a $\gamma(G)$-set if it is a dominating set and $|S| = \gamma(G)$.

A map $f : V \rightarrow \{0, 1, 2\}$ is a *Roman dominating function* for a graph G if for every vertex v with $f(v) = 0$, there exists a vertex $u \in N(v)$ such that $f(u) = 2$. The *weight* of a Roman dominating function is given by $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on G is called the *Roman domination number* of G and it is denoted by $\gamma_R(G)$.

Any Roman dominating function f on a graph G induces three sets B_0, B_1, B_2, where $B_i = \{v \in V : f(v) = i\}$. Thus, we write $f = (B_0, B_1, B_2)$. It is clear that for any Roman dominating function $f = (B_0, B_1, B_2)$ on a graph $G = (V, E)$ of order n we have that $f(V) = \sum_{u \in V} f(u) = 2|B_2| + |B_1|$ and $|B_0| + |B_1| + |B_2| = n$. We say that a function $f = (B_0, B_1, B_2)$ is a $\gamma_R(G)$-function if it is a Roman dominating function and $f(V) = \gamma_R(G)$.

Several results about Roman dominating sets have been obtained recently [3, 4, 5, 9, 15, 18], and it is natural to try to relate the Roman domination number to the standard domination number. For instance, [3, 9] contain the following result, which we use as a tool in this article.

Lemma 1. [3, 9] *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Results about Roman domination in product graphs have been developed in [12, 13]. In the first one the exact value for the Roman domination number of lexicographic products of graphs was obtained, and in the second one, some particular cases of the Cartesian products of paths and cycles were studied. In this article we study the Roman domination number of Cartesian product graphs

and strong product graphs. More precisely, we study the relationships between the Roman domination number of product graphs and the domination number (Roman domination number) of the factors.

For two graphs G and H with sets of vertices $V_1 = \{v_1, \ldots, v_{n_1}\}$ and $V_2 = \{u_1, \ldots, u_{n_2}\}$, respectively, the Cartesian product of G and H is the graph $G \square H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (v_i, u_j) and (v_k, u_ℓ) are adjacent in $G \square H$ if and only if

- $v_i = v_k$ and $u_j \sim u_\ell$, or

- $v_i \sim v_k$ and $u_j = u_\ell$.

The strong product $G \boxtimes H$ of the graphs G and H is defined on the Cartesian product of the vertex sets of the factors. Two distinct vertices (v_i, u_j) and (v_k, u_ℓ) of $G \boxtimes H$ are adjacent with respect to the strong product if and only if

- $v_i = v_k$ and $u_j \sim u_\ell$, or

- $v_i \sim v_k$ and $u_j = u_\ell$, or

- $v_i \sim v_k$ and $u_j \sim u_\ell$.

So, the Cartesian product graph $G \square H$ is a subgraph of the strong product graph $G \boxtimes H$.

2. CARTESIAN PRODUCT GRAPHS

Currently there are few known results on the Roman domination number of Cartesian product graphs. As far as we know, the only results on this topic are as follows. In [**13**] some particular cases of Cartesian product of paths and cycles were studied. Also, the Roman domination number of $C_{5t} \square C_{5k}$ was studied in [**18**] and the Roman domination number of some grid graphs was studied in [**3, 4**]. Also, the following general relationship between the Roman domination number of Cartesian product graphs and the domination number of its factors was obtained in [**17**]:

$$(1) \qquad \gamma_R(G \square H) \geq \gamma(G)\gamma(H).$$

The following lemma will be helpful in obtaining the results presented here.

Lemma 2. *Let G be a graph. For any $\gamma_R(G)$-function $f = (B_0, B_1, B_2)$,*

(i) $|B_2| \leq \gamma_R(G) - \gamma(G)$.

(ii) $|B_1| \geq 2\gamma(G) - \gamma_R(G)$.

Proof. Since $B_2 \cup B_1$ is a dominating set for G and $B_1 \cap B_2 = \emptyset$, we have $\gamma(G) \leq |B_2| + |B_1|$. So, (i) is deduced as $\gamma(G) \leq 2|B_2| + |B_1| - |B_2| = \gamma_R(G) - |B_2|$, and (ii) is obtained as $2\gamma(G) \leq 2|B_2| + 2|B_1| = 2|B_2| + |B_1| + |B_1| = \gamma_R(G) + |B_1|$.

Theorem 3. *For any graphs G and H,*

(i) $\gamma_R(G\Box H) \geq \dfrac{2\gamma(G)\gamma_R(H)}{3}.$

(ii) $\gamma_R(G\Box H) \geq \dfrac{\gamma(G)\gamma_R(H) + \gamma(G\Box H)}{2}.$

Proof. Let V_1 and V_2 be the vertex sets of G and H, respectively. Let $f = (B_0, B_1, B_2)$ be a $\gamma_R(G\Box H)$-function. Let $S = \{u_1, u_2, \ldots, u_{\gamma(G)}\}$ be a dominating set for G. Let $\{A_1, A_2, \ldots, A_{\gamma(G)}\}$ be a vertex partition of G such that $u_i \in A_i$ and $A_i \subseteq N[u_i]$ (Notice that this partition always exists, and it needs not be unique). Let $\{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}$ be a vertex partition of $G\Box H$, such that $\Pi_i = A_i \times V_2$ for every $i \in \{1, \ldots, \gamma(G)\}$.

For every $i \in \{1, \ldots, \gamma(G)\}$, let $f_i : V_2 \to \{0, 1, 2\}$ be a function such that $f_i(v) = \max\{f(u, v) : u \in A_i\}$. For every $j \in \{0, 1, 2\}$, let $X_j^{(i)} = \{v \in V_2 : f_i(v) = j\}$. Now, let $Y_0^{(i)} \subseteq X_0^{(i)}$ such that for every $v \in Y_0^{(i)}$, $N(v) \cap X_2^{(i)} = \emptyset$. Hence, we have that $f_i' = (X_0^{(i)} - Y_0^{(i)}, X_1^{(i)} + Y_0^{(i)}, X_2^{(i)})$ is a Roman dominating function on H. Thus,

$$\gamma_R(H) \leq 2|X_2^{(i)}| + |X_1^{(i)}| + |Y_0^{(i)}| \leq 2|B_2 \cap \Pi_i| + |B_1 \cap \Pi_i| + |Y_0^{(i)}|.$$

Hence,

$$\gamma_R(G\Box H) = 2|B_2| + |B_1| = \sum_{i=1}^{\gamma(G)} (2|B_2 \cap \Pi_i| + |B_1 \cap \Pi_i|)$$

$$\geq \sum_{i=1}^{\gamma(G)} (\gamma_R(H) - |Y_0^{(i)}|) = \gamma(G)\gamma_R(H) - \sum_{i=1}^{\gamma(G)} |Y_0^{(i)}|.$$

So,

$$(2) \qquad \sum_{i=1}^{\gamma(G)} |Y_0^{(i)}| \geq \gamma(G)\gamma_R(H) - \gamma_R(G\Box H).$$

Now, for every $v \in V_2$, let $Z^v \in \{0, 1\}^{\gamma(G)}$ be a binary vector associated to v as follows: $Z_i^v = 1$ if $v \in Y_0^{(i)}$ and $Z_i^v = 0$ if $v \notin Y_0^{(i)}$. So, $t_v = \|Z^v\|^2$ counts the number of components of Z^v equal to one. Hence,

$$(3) \qquad \sum_{v \in V_2} t_v = \sum_{i=1}^{\gamma(G)} |Y_0^{(i)}|.$$

Notice that, if $Z_i^v = 1$ and $u \in A_i$, then the vertex (u, v) belongs to B_0. Moreover, (u, v) is not adjacent to vertices of $B_2 \cap \Pi_i$. So, since B_0 is dominated by B_2, there exists $u' \in X_v = \{x \in V_1 : (x, v) \in B_2\}$ which is adjacent to u. Hence, $S_v = (S - \{u_i \in S : Z_i^v = 1\}) \cup X_v$ is a dominating set for G.

Now, if $t_v > |X_v|$, then we have

$$|S_v| = |S| - t_v + |X_v| = \gamma(G) - t_v + |X_v|$$
$$< \gamma(G) - t_v + t_v = \gamma(G),$$

which is a contradiction. So, we have $t_v \leq |X_v|$ and we obtain

$$(4) \qquad \sum_{v \in V_2} t_v \leq \sum_{v \in V_2} |X_v| = |B_2|,$$

which leads to,

$$(5) \qquad 2 \sum_{v \in V_2} t_v \leq 2|B_2| + |B_1| = \gamma_R(G \square H).$$

Thus, by (2), (3) and (5) we deduce

$$\gamma_R(G \square H) \geq \gamma(G)\gamma_R(H) - \frac{\gamma_R(G \square H)}{2},$$

and, as a consequence, (i) follows.

Now, by Lemma 2 (i) and (4) we have

$$(6) \qquad \sum_{v \in V_2} t_v \leq |B_2| \leq \gamma_R(G \square H) - \gamma(G \square H).$$

Thus, by (2), (3) and (6) we obtain (ii). $\qquad \square$

Lemma 1 and Theorem 3 lead to the following result.

Corollary 4. *For any graphs G and H,*

(i) $\gamma_R(G \square H) \geq \dfrac{\gamma_R(G)\gamma_R(H)}{3}$.

(ii) $\gamma(G \square H) \geq \dfrac{\gamma(G)\gamma_R(H)}{3}$.

Note that if there exists a graph that satisfies the above inequalities, then Vizing's conjecture is false.

The following inequality related to Vizing's conjecture was obtained in [**2**]:

$$(7) \qquad \gamma(G \square H) \geq \frac{\gamma(G)\gamma(H)}{2}.$$

If $\gamma_R(H) > \left\lceil \dfrac{3\gamma(H)}{2} \right\rceil$, then $\gamma_R(H) \geq \left\lceil \dfrac{3\gamma(H)}{2} \right\rceil + 1$. Thus, Corollary 4 (ii) leads to a result which improves the above inequality.

REMARK 5. Let G and H be two graphs. If $\gamma_R(H) > \left\lceil \dfrac{3\gamma(H)}{2} \right\rceil$, then

$$\gamma(G \square H) \geq \frac{\gamma(G)\gamma(H)}{2} + \frac{\gamma(G)}{3}.$$

A graph H is a *Roman graph* if $\gamma_R(H) = 2\gamma(H)$. Roman graphs were introduced in [**3**] where the authors presented some classes of Roman graphs and they proposed some open problems. Theorem 3 (i) leads to the following result.

Corollary 6. *For any graph G and any Roman graph H,*

(i) $\gamma_R(G\square H) \geq \dfrac{4}{3}\gamma(G)\gamma(H)$.

(ii) $\gamma(G\square H) \geq \dfrac{2}{3}\gamma(G)\gamma(H)$.

Let $\mathfrak{F}$ be the class of all graphs with a dominating set $S = \{u_1, u_2, \ldots, u_{\gamma(G)}\}$ such that $N[u_i] \cap N[u_j] = \emptyset$, for every $i, j \in \{1, \ldots, \gamma(G)\}$, $i \neq j$. In this case the set S is called an *efficient dominating set*. Notice that $\mathfrak{F}$ is the family of all graphs having a perfect code (a subset $S \subset V$ is a perfect code in a graph G if $|N[v] \cap S| = 1$, for every $v \in V$.) Examples of graphs belonging to $\mathfrak{F}$ are the path graphs P_n, the cycle graphs C_{3k} and the cube graph $Q_3 = K_2\square K_2\square K_2$. Examples of Roman graphs belonging to $\mathfrak{F}$ are C_{3k}, P_{3k}, P_{3k+2} and Q_3. Note that $P_{3k+1} \in \mathfrak{F}$ but P_{3k+1} are not Roman paths, while C_{3k+2} are Roman cycles but $C_{3k+2} \notin \mathfrak{F}$.

A 2-packing of a graph G is a set of vertices in G that are pairwise at distance more than two. The 2-packing number $P_2(G)$ of a graph G is the size of a largest 2-packing in G. The 2-packing number is a graph invariant closely related to the domination number. In fact, it is well known that $P_2(G) \leq \gamma(G)$, cf. [**10, 11**].

Let $G \in \mathfrak{F}$. Since every efficient dominating set $S = \{u_1, u_2, \ldots, u_{\gamma(G)}\}$ is a 2-packing, we have $\gamma(G) \leq P_2(G)$. So, we conclude that if $G \in \mathfrak{F}$, then $P_2(G) = \gamma(G)$ (The converse is not true.) We recall that if $P_2(G) = \gamma(G)$, then Vizing's conjecture holds for G [**11**]. As a consequence, by Theorem 3 (ii) we deduce the following result, which improves the inequality (1) when $G \in \mathfrak{F}$.

Corollary 7. *Let G and H be two graphs. If $G \in \mathfrak{F}$, then*

$$\gamma_R(G\square H) \geq \frac{1}{2}\max\left\{\gamma(G)\left(\gamma_R(H) + \gamma(H)\right), \gamma(H)\left(\gamma_R(G) + \gamma(G)\right)\right\}.$$

Theorem 8. *Let G and H be two graphs. If $G \in \mathfrak{F}$, then*

$$\gamma_R(G\square H) \geq \gamma(G)\gamma_R(H).$$

Proof. Let V_1 and V_2 be the vertex sets of G and H, respectively. Let $S = \{u_1, \ldots, u_{\gamma(G)}\}$ be an efficient dominating set for G, *i.e.*, $\{N[u_1], \ldots, N[u_{\gamma(G)}]\}$ is a vertex partition of G and, as a consequence, $\{\Pi_1', \Pi_2', \ldots, \Pi_{\gamma(G)}'\}$ is a vertex partition of $G\square H$, where $\Pi_i' = N[u_i] \times V_2$ for every $i \in \{1, \ldots, \gamma(G)\}$.

Proceeding analogously to the proof of Theorem 3, we consider a $\gamma_R(G\square H)$-function $f = (B_0, B_1, B_2)$ and, for every $i \in \{1, \ldots, \gamma(G)\}$, we define the function $f_i : V_2 \to \{0, 1, 2\}$ as $f_i(v) = \max\{f(u, v) \; : \; u \in N[u_i]\}$. In addition, for every $j \in \{0, 1, 2\}$ we define $X_j^{(i)} = \{v \in V_2 : f_i(v) = j\}$.

Now, if $v \in X_0^{(i)}$, then for every $u \in N[u_i]$ we have that $(u, v) \in B_0$. Hence, since u_i has no neighbors in $V_1 - N[u_i]$ and B_2 dominates B_0, there exists $(u_i, v') \in$

B_2 such that v' is adjacent to v. We conclude that every $v \in X_0^{(i)}$ has a neighbor $v' \in X_2^{(i)}$ and, as a consequence, $f_i = (X_0^{(i)}, X_1^{(i)}, X_2^{(i)})$ is a Roman dominating function on H, for every $i \in \{1, \dots, \gamma(G)\}$. Therefore, the result is deduced as follows:

$$\gamma_R(G\Box H) = 2|B_2| + |B_1| = \sum_{i=1}^{\gamma(G)} \left(2|B_2 \cap \Pi_i'| + |B_1 \cap \Pi_i'|\right)$$

$$\geq \sum_{i=1}^{\gamma(G)} \left(2|X_2^{(i)}| + |X_1^{(i)}|\right) \geq \gamma(G)\gamma_R(H). \qquad \square$$

An interesting consequence of Theorem 8 is the following result.

Corollary 9. *Let G and H be two graphs. If $G \in \mathfrak{F}$ and H is a Roman graph, then*

$$\gamma_R(G\Box H) \geq 2\gamma(G)\gamma(H).$$

Theorem 10. *For any graphs G and H of order n_1 and n_2, respectively,*

$$\gamma_R(G\Box H) \leq \min\{n_1\gamma_R(H), n_2\gamma_R(G)\}.$$

Proof. Let f_1 be a $\gamma_R(G)$-function. Let $f : V_1 \times V_2 \to \{0, 1, 2\}$ be a function defined by $f(u, v) = f_1(u)$. If there exists a vertex $(x, y) \in V_1 \times V_2$ such that $f(x, y) = 0$, then $f_1(x) = 0$. Since f_1 is Roman, there exists $u \in V_1$, adjacent to x, such that $f_1(u) = 2$. Hence, we obtain that $f(u, y) = 2$ and (x, y) is adjacent to (u, y). So, f is a Roman dominating function. Therefore,

$$\gamma_R(G\Box H) \leq \sum_{(u,v)\in V_1\times V_2} f(u, v) = \sum_{v\in V_2}\sum_{u\in V_1} f_1(u) = \sum_{v\in V_2} \gamma_R(G) = n_2\gamma_R(G).$$

Analogously we obtain that $\gamma_R(G\Box H) \leq n_1\gamma_R(H)$ and the result follows. $\qquad \square$

The above inequality is tight. It is achieved, for instance, for $G = P_n$, a path graph of order n, and $H = S_{1,r}$, a star graph with $r \geq 2$ leaves. In this case we have $\gamma_R(S_{1,r}) = 2 = 2\gamma(S_{1,r})$, $\gamma(P_n) = \left\lceil \dfrac{n}{3} \right\rceil$, $\gamma_R(P_n) = \dfrac{2n+1}{3}$ if $n \equiv 1(3)$ and $\gamma_R(P_n) = 2\left\lceil \dfrac{n}{3} \right\rceil$ if $n \not\equiv 1(3)$. So, $\gamma_R(G\Box H) = 2n = n\gamma_R(H)$.

Corollary 11. *For any graphs G and H of order n_1 and n_2, respectively,*

$$\gamma_R(G\Box H) \leq 2\min\{n_1\gamma(H), n_2\gamma(G)\}.$$

Lemma 12. [3] *A graph G is Roman if and only if it has a $\gamma_R(G)$-function $f = (A_0, A_1, A_2)$ with $|A_1| = 0$.*

Theorem 13. *Let G be a graph of order n and let H be a graph.*

(i) *If G has at least one connected component of order greater than two, then*

$$\gamma_R(G\square H) \leq (n+1)\gamma_R(H) - 2\gamma(H).$$

(ii) *If G is a Roman graph, then*

$$\gamma_R(G\square H) \leq 2n\left(\gamma_R(H) - \gamma(H)\right) + 2\gamma(G)\left(2\gamma(H) - \gamma_R(H)\right).$$

Proof. Let $f_1 = (A_0, A_1, A_2)$ be a $\gamma_R(G)$-function and let $f_2 = (B_0, B_1, B_2)$ be a $\gamma_R(H)$-function. We define the map $f : V_1 \times V_2 \to \{0, 1, 2\}$ as follows.

- $f(u, v) = f_2(v)$ for every $(u, v) \notin (A_0 \cup A_2) \times B_1$.

- If $(u, v) \in A_0 \times B_1$, then $f(u, v) = 0$.

- If $(u, v) \in A_2 \times B_1$, then $f(u, v) = 2$.

Since every vertex from $A_0 \times B_1$ has a neighbor in $A_2 \times B_1$ and every vertex of $V_1 \times B_0$ has a neighbor in $V_1 \times B_2$, we have that f is a Roman dominating function on $G\square H$. Thus,

$$(8) \quad \gamma_R(G\square H) \leq n\gamma_R(H) - |A_0||B_1| + |A_2||B_1| = n\gamma_R(H) - |B_1|(|A_0| - |A_2|).$$

Since G has at least one component of order greater than two, it is satisfied that $|A_0| \geq |A_2| + 1$ and, by Lemma 2 (ii), $|B_1|(|A_0| - |A_2|) \geq 2\gamma(H) - \gamma_R(H)$. Therefore, by (8) we deduce (i).

Now, if G is a Roman graph, then by Lemma 12 there exists a $\gamma_R(G)$-function $f = (A_0, A_1, A_2)$ with $|A_1| = 0$. Thus, $|A_0| + |A_2| = n$ and, as a consequence, $|A_0| - |A_2| = n - 2\gamma(G)$. Therefore, by (8) we deduce (ii):

$$\begin{aligned}
\gamma_R(G\square H) &\leq n\gamma_R(H) - |B_1|(|A_0| - |A_2|) \\
&\leq n\gamma_R(H) - (2\gamma(H) - \gamma_R(H))(n - 2\gamma(G)) \\
&= 2n\left(\gamma_R(H) - \gamma(H)\right) + 2\gamma(G)\left(2\gamma(H) - \gamma_R(H)\right). \qquad \square
\end{aligned}$$

For any Roman graph H, Theorem 13 leads to $\gamma_R(G\square H) \leq 2n\gamma(H)$. Now, for any non-Roman graph H we have $\gamma_R(H) - 2\gamma(H) \leq -1$ and, as a consequence, Theorem 13 leads to the following result.

Corollary 14. *Let G be a graph of order n and let H be a graph. If G has at least one connected component of order greater than two and H is not Roman, then*

$$\gamma_R(G\square H) \leq n\gamma_R(H) - 1.$$

Proposition 15. [3] *If G is a connected graph of order n, then $\gamma_R(G) = \gamma(G) + 1$ if and only if there exists a vertex of G of degree $n - \gamma(G)$.*

From Proposition 15 and Theorem 13 we derive the following result.

Proposition 16. *If G is a graph of order n_1 having at least one connected component of order greater than two and H is a connected graph of order n_2 having a vertex of degree $n_2 - \gamma(H)$, then*

$$\gamma_R(G\Box H) \leq n_1(\gamma(H) + 1) - \gamma(H) + 1.$$

The above inequality is tight. For instance, if G is a path graph of order three and H is the star $K_{1,3}$ with one of its edges subdivided, then we have $\gamma(H) = 2$ and $\gamma_R(G\Box H) = 8$. So, Proposition 16 leads to the exact value of $\gamma_R(G\Box H)$.

Theorem 17. *For any graphs G and H of order n_1 and n_2, respectively,*

$$\gamma_R(G\Box H) \leq 2\gamma(G)\gamma(H) + (n_1 - \gamma(G))(n_2 - \gamma(H)).$$

Proof. Let S_1 be a $\gamma(G)$-set and let S_2 be a $\gamma(H)$-set. Let $B_2 = S_1 \times S_2$, $B_1 = (V_1 - S_1) \times (V_2 - S_2)$ and $B_0 = S_1 \times (V_2 - S_2) \cup (V_1 - S_1) \times S_2$. Since B_2 dominates B_0, the map $f : V_1 \times V_2 \to \{0, 1, 2\}$ defined by $f(u, v) = i$, for every $(u, v) \in B_i$, is a Roman dominating function on $G\Box H$. Therefore, the result is obtained as follows,

$$\gamma_R(G\Box H) \leq 2|B_2| + |B_1| = 2|S_1||S_2| + |V_1 - S_1||V_2 - S_2|$$
$$= 2\gamma(G)\gamma(H) + (n_1 - \gamma(G))(n_2 - \gamma(H)). \qquad \Box$$

We know that $\gamma_R(P_{3k+2}) = 2\gamma(P_{3k+2}) = 2(k+1)$, $\gamma_R(P_{3k+1}) = 2k+1$ and $\gamma(P_{3k+1}) = k+1$. So, Theorem 17 leads to $\gamma_R(P_{3k+1}\Box P_{3k+2}) \leq 6k^2 + 6k + 2$, while by Theorem 10 we only get $\gamma_R(P_{3k+1}\Box P_{3k+2}) \leq 6k^2 + 7k + 2$ and by Theorem 13 we only get $\gamma_R(P_{3k+2}\Box P_{3k+1}) \leq 6k^2 + 7k + 1$.

From the above results we have that the bounds on the Roman domination number and the domination number of the factor graphs lead to bounds on the Roman domination number of the Cartesian product graphs. For example, it is well known that for any graph G of order n and maximum degree Δ, $\gamma(G) \geq \dfrac{n}{\Delta + 1}$, cf. [**7**]. The following straightforward result allows us to derive several bounds on $\gamma_R(G\Box H)$.

REMARK 18. For any graph $G \in \mathfrak{F}$ of order n and minimum degree δ, $\gamma(G) \leq \dfrac{n}{\delta + 1}$. As a consequence, for any δ-regular graph $G \in \mathfrak{F}$ it follows, $\gamma(G) = \dfrac{n}{\delta + 1}$.

An example of a result derived from the above remark, Theorem 8 and Theorem 10, is the following one.

Proposition 19. *For any δ-regular graph $G \in \mathfrak{F}$ of order n,*

$$\frac{2n}{\delta + 1} \leq \gamma_R(G\Box K_2) \leq \frac{4n}{\delta + 1}.$$

3. STRONG PRODUCT GRAPHS

In this section we obtain some results on the Roman domination number of strong product graphs. We begin by recalling the following well-known result, cf. [**11**].

Theorem 20. [**11**] *For any graphs G and H,*

$$\max\{P_2(G)\gamma(H), \gamma(G)P_2(H)\} \leq \gamma(G \boxtimes H) \leq \gamma(G)\gamma(H).$$

One immediate consequence of Theorem 20 is the following result.

Corollary 21. *For any graph $G \in \mathfrak{F}$ and any graph H, $\gamma(G \boxtimes H) = \gamma(G)\gamma(H)$.*

The next result follows from Lemma 1 and Theorem 20.

Corollary 22. *For any graphs G and H,*

$$\max\{P_2(G)\gamma(H), \gamma(G)P_2(H)\} \leq \gamma_R(G \boxtimes H) \leq 2\gamma(G)\gamma(H).$$

Theorem 23. *Let $f_1 = (A_0, A_1, A_2)$ be a $\gamma_R(G)$-function and let $f_2 = (B_0, B_1, B_2)$ be a $\gamma_R(H)$-function. Then,*

$$\gamma_R(G \boxtimes H) \leq \gamma_R(G)\gamma_R(H) - 2|A_2||B_2|.$$

Proof. We define the function f on $G \boxtimes H$ as follows:

$$f(u,v) = \begin{cases} 2, & (u,v) \in (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2), \\ 1, & (u,v) \in A_1 \times B_1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the set $(A_0 \times B_0) \cup (A_0 \times B_2) \cup (A_2 \times B_0)$ is dominated by $A_2 \times B_2$, the set $A_1 \times B_0$ is dominated by $A_1 \times B_2$, and $A_0 \times B_1$ is dominated by $A_2 \times B_1$. Then we have that f is a Roman dominating function on $G \boxtimes H$.

Therefore,

$$\begin{aligned} \gamma_R(G \boxtimes H) &\leq 2|A_2||B_2| + 2|A_1||B_2| + 2|A_2||B_1| + |A_1||B_1| \\ &= 4|A_2||B_2| + 2|A_1||B_2| + 2|A_2||B_1| + |A_1||B_1| - 2|A_2||B_2| \\ &= 2|A_2|(2|B_2| + |B_1|) + |A_1|(2|B_2| + |B_1|) - 2|A_2||B_2| \\ &= (2|A_2| + |A_1|)(2|B_2| + |B_1|) - 2|A_2||B_2| \\ &= \gamma_R(G)\gamma_R(H) - 2|A_2||B_2|. \qquad \square \end{aligned}$$

Now we present some interesting consequences of Theorem 23.

Corollary 24. *For any nontrivial graphs G and H,*

$$\gamma_R(G \boxtimes H) \leq \gamma_R(G)\gamma_R(H) - 2.$$

The above inequality is achieved, for instance, if G and H are graphs of order n_1 and n_2, containing a vertex of degree $n_1 - 1$ and $n_2 - 1$, respectively. In this case, we have $\gamma_R(G \boxtimes H) \leq \gamma_R(G)\gamma_R(H) - 2 = 2 \cdot 2 - 2 = 2$.

In order to show one example where Corollary 24 leads to a better result than Corollary 22 we take a graph G such that $\gamma_R(G) = \gamma(G) + 1 > 3$ (see Proposition 15). In this case Corollary 24 leads to $\gamma_R(G \boxtimes G) \leq (\gamma(G))^2 + 2\gamma(G) - 1$, while Corollary 22 leads to $\gamma_R(G \boxtimes G) \leq 2(\gamma(G))^2$.

If $H = P_n$ or $H = C_n$, then we have that for any $\gamma_R(H)$-function $f = (B_0, B_1, B_2)$, $|B_2| = \left\lfloor \dfrac{n}{3} \right\rfloor$. Hence, Theorem 23 leads to the following result.

Corollary 25. *Let G be a nontrivial graph. If $H = P_n$ or $H = C_n$, then*

$$
\gamma_R(G \boxtimes H) \leq
\begin{cases}
\dfrac{2n+1}{3}\gamma_R(G) - 2\left\lfloor \dfrac{n}{3} \right\rfloor, & n \equiv 1(3) \\[3mm]
2\left\lceil \dfrac{n}{3} \right\rceil \gamma_R(G) - 2\left\lfloor \dfrac{n}{3} \right\rfloor, & n \not\equiv 1(3).
\end{cases}
$$

Every star graph $G = K_{1,r}$ satisfies the above inequality for $n \not\equiv 2(3)$. That way we have $\gamma_R(C_n \boxtimes K_{1,r}) = \gamma_R(P_n \boxtimes K_{1,r}) = 2\left\lceil \dfrac{n}{3} \right\rceil$. Note that $C_n \boxtimes K_{1,r}$ and $P_n \boxtimes K_{1,r}$ are Roman graphs for $n \not\equiv 2(3)$.

Theorem 26. *Let G and H be two graphs. If $G \in \mathfrak{F}$, then*

$$\gamma_R(G \boxtimes H) \geq \gamma(G)\gamma_R(H).$$

Proof. Let V_1 and V_2 be the vertex sets of G and H, respectively. Let $S = \{u_1, \ldots, u_{\gamma(G)}\}$ be an efficient dominating set for G, *i.e.*, $\{N_G[u_1], \ldots, N_G[u_{\gamma(G)}]\}$ is a vertex partition for G. Let $\{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}$ be the vertex partition of $G \boxtimes H$ defined as $\Pi_i = N_G[u_i] \times V_2$, for every $i \in \{1, \ldots, \gamma(G)\}$.

Now, let $f = (B_0, B_1, B_2)$ be a $\gamma_R(G \boxtimes H)$-function and, for every $i \in \{1, \ldots, \gamma(G)\}$, let the function $f^{(i)} : V_2 \to \{0, 1, 2\}$ defined by $f^{(i)}(v) = \max\{f(u, v) : (u, v) \in \Pi_i\}$. Let $\{B_0^{(i)}, B_1^{(i)}, B_2^{(i)}\}$ such that $B_j^{(i)} = \{v \in V_2 : f^{(i)}(v) = j\}$ with $j \in \{0, 1, 2\}$ and $i \in \{1, \ldots, \gamma(G)\}$.

If there is a vertex y of H such that $f^{(i)}(y) = 0$ and $N_H[y] \cap B_2^{(i)} = \emptyset$, then $f(u_i, y) = 0$ and (u_i, y) is not adjacent to any vertex (a, b) of $G \boxtimes H$ with $f(a, b) = 2$, a contradiction. Thus, $f^{(i)} = (B_0^{(i)}, B_1^{(i)}, B_2^{(i)})$ is a Roman dominating function on H for every $i \in \{1, \ldots, \gamma(G)\}$. As a consequence,

$$
\gamma_R(G \boxtimes H) = 2|B_2| + |B_1| = \sum_{i=1}^{\gamma(G)} (2|B_2 \cap \Pi_i| + |B_1 \cap \Pi_i|)
$$

$$
\geq \sum_{i=1}^{\gamma(G)} (2|B_2^{(i)}| + |B_1^{(i)}|) \geq \sum_{i=1}^{\gamma(G)} \gamma_R(H) = \gamma(G)\gamma_R(H).
$$

Therefore, the proof is complete. $\square$

Cockayne et al. [**3**] gave some classes of Roman graphs and they posed the following question: Can you find other classes of Roman graphs? The next result is an answer to this question.

Theorem 27. *If $G \in \mathfrak{F}$ and H is a Roman graph, then $G \boxtimes H$ is a Roman graph.*

Proof. If $G \in \mathfrak{F}$ and H is Roman, then Theorem 26 leads to $\gamma_R(G \boxtimes H) \geq 2\gamma(G)\gamma(H)$. So, by Corollary 22 we obtain $\gamma_R(G \boxtimes H) = 2\gamma(G)\gamma(H)$. Corollary 21 concludes the proof.

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