

ON SOLUTIONS OF Q-DIFFERENCE RICCATI EQUATIONS WITH RATIONAL COEFFICIENTS

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We consider q-difference Riccati equations in complex plane. We prove that their transcendental meromorphic solutions are hypertranscendental, we investigate the value distribution of their meromorphic solutions and we consider the existence and forms of rational solutions of q-difference Riccati equations.

1. INTRODUCTION

We assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory [20]. Let $f(z)$ be a meromorphic function in the complex plane, $q \in \mathbb{C} \setminus \{0, 1\}$. The first order q-difference operator [1, p.488] is defined by

$$\Delta_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}.$$

In 2002, GUNDERSEN et al [7] showed that the order of growth of the solution of the Schröder equation

$$(1.1) \quad f(qz) = R(z, f(z))$$

is equal to $\log_{|q|}(\deg_f R)$ ($|q| > 1$). They also obtained that if the q-difference equation (1.1) has a meromorphic solution of order zero, then the equation (1.1) reduces to a q-difference equation with $\deg_f R = 1$ where

$$\deg_f R = \max\{p, q\} \quad \text{for} \quad R(z, f(z)) = \frac{\sum_{j=0}^p a_j(z) f(z)^j}{\sum_{j=0}^q b_j(z) f(z)^j}.$$

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This result is a q-difference analogue of the classical Malmquist's theorem [11, P.193]. The Schröder equation is closely connected to complex dynamics, see [3].

Several authors [4, 5, 8, 19] have recently investigated a difference Riccati equation

$$f(z+c) = \frac{a(z) + b(z)f(z)}{c(z) + d(z)f(z)},$$

where $c \in \mathbb{C} \setminus \{0\}$, and $a(z), b(z), c(z), d(z)$ are meromorphic, such that $a(z)d(z) - b(z)c(z) \neq 0$.

Similarly, we call equation (1.1) with $\deg_f R = 1$,

$$(1.2) \quad f(qz) = \frac{A(z) + f(z)}{1 - (q-1)zf(z)}$$

the q-difference Riccati equation, where $(q-1)zA(z) + 1 \neq 0$.

In Section 2, we prove that the transcendental meromorphic solutions of (1.2) are hypertranscendental, and investigate the value distribution of its meromorphic solutions. In Section 3, we investigate the value distribution of meromorphic solutions of more generalized q-difference Riccati equations, and we consider the existence and forms of rational solutions of equations (1.2).

2. HYPERTRANSCENDENCY AND VALUE DISTRIBUTION OF TRANSCENDENTAL SOLUTIONS OF (1.2)

A function $\varphi(z)$ is called differentially algebraic [16], if there exists a differential polynomial $P(z, \varphi, \varphi', \dots, \varphi^{(n)})$ such that

$$P(z, \varphi, \varphi', \dots, \varphi^{(n)}) = 0,$$

where the coefficients of P are rational functions. An analytic function which is not differentially algebraic is called hypertranscendental (or transcendentally transcendental). WITTICH [18] investigated hypertranscendency of entire solutions of the functional equation

$$(2.1) \quad y(qz) = a(z)y(z) + b(z),$$

where $a(z)$ and $b(z)$ are polynomials, q is a nonzero constant satisfying $|q| \neq 1$. ISHIZAKI [9] generalized Wittich's results, and obtained the following theorem.

Theorem 2.1 [9]. *Suppose that $a(z)$ and $b(z)$ are rational functions in (2.1). Then all transcendental meromorphic solutions of the equation (2.1) are hypertranscendental.*

RITT [14] proved that meromorphic solutions of the Schröder equation $f(qz) = R(f(z))$, where $R(z)$ is a rational function in z , are hypertranscendental, except for certain cases where they are given in terms of exponential, trigonometric, or

elliptic functions. As RUBEL posed in [15, 17], there is an open problem on hypertranscendency for the equation $f(\lambda z) = R(z, f(z))$, where λ is a complex constant and $R(z, f(z))$ is a rational function in z and f . In this section, we prove that all transcendental meromorphic solutions of the q -difference equation (1.2) are hypertranscendental.

REMARK 2.2. In the second part of [7], GUNDERSEN et al considered the q -difference equation

$$(2.2) \quad f(qz) = \frac{a(z) + b(z)f(z)}{c(z) + d(z)f(z)},$$

where $a(z), b(z), c(z), d(z)$ are meromorphic, $a(z)d(z) - b(z)c(z) \neq 0$, and $q \neq 0, |q| \neq 1$. Suppose that (2.2) has a meromorphic solution $f(z)$. Then $T(r, f)$ is the same order quantity as $\Psi(r) \log r$, i.e. $\lim_{r \rightarrow \infty} \frac{T(r, f)}{\Psi(r) \log r}$ is a nonzero finite number. In other words

$$T(r, f) = O(\Psi(r) \log r),$$

where $\Psi(r) = \max\{T(r, a), T(r, b), T(r, c), T(r, d)\}$.

Therefore, if $A(z)$ is a rational function, then the transcendental solution $f(z)$ of (1.2) satisfies $T(r, f) = O((\log r)^2)$.

Lemma 2.3 [2]. *Let $f(z)$ be a non-constant zero-order meromorphic solution of*

$$P(z, f) = 0,$$

where $P(z, f)$ is a q -difference polynomial in $f(z)$. If $P(z, \alpha) \neq 0$ for a meromorphic function $\alpha(z)$ satisfying $\lim_{r \rightarrow \infty} \frac{T(r, \alpha)}{T(r, f)} = 0$, then

$$m\left(r, \frac{1}{f - \alpha}\right) = S(r, f)$$

on a set of logarithmic density 1.

Lemma 2.3 is a q -difference variant of the Mohon'ko lemma for differential equation [13].

Theorem 2.4. *Let $q \in \mathbb{C} \setminus \{0\}$ such that $|q| \neq 1$. Suppose that $A(z)$ is a rational function.*

(1) *Suppose that (1.2) has a rational solution $a(z)$. Then all transcendental meromorphic solutions of the equation (1.2) are hypertranscendental.*

(2) *Suppose that $A(z)$ is a nonconstant rational function which is not a polynomial of degree one or three. If $f(z)$ is a transcendental meromorphic solution of (1.2), then $f(z)$ has no deficient value and has infinitely many fixed points.*

Proof. (1) Assume that (1.2) has a transcendental meromorphic solution $f(z)$. We define $k(z)$ from

$$(2.3) \quad f(z) = a(z) + \frac{1}{\alpha(z)k(z)}, \quad \text{with} \quad \alpha(z) = \frac{(q-1)q^{-1}za(q^{-1}z) - 1}{(q-1)q^{-1}zA(q^{-1}z) + 1}.$$

We note that $T(r, k) = T(r, f) + O(\log r)$. From (1.2), we have a q-difference equation for $k(z)$

$$(2.4) \quad k(qz) = B(z)k(z) + (q - 1)z, \quad \text{with} \quad B(z) = ((q - 1)za(z) - 1)\alpha(z).$$

By the assumption that $a(z)$ and $A(z)$ are rational functions, $B(z)$ is a rational function. So applying Theorem 2.1, we know that $k(z)$ is hypertranscendental. By (2.3), we know that $f(z)$ is hypertranscendental.

(2) Suppose that $f(z)$ is a transcendental solution of (1.2). By Remark 2.2, the order of growth of $f(z)$ is zero. Set $y(z) = \frac{1}{f(z)}$. Then $T(r, y) = T(r, f) + O(1)$.

Substituting $f(z) = \frac{1}{y(z)}$ into (1.2), we obtain

$$P_1(z, y) := A(z)y(qz)y(z) + y(qz) - y(z) + (q - 1)z = 0.$$

Since $P_1(z, 0) = (q - 1)z \neq 0$, by Lemma 2.3, we see that

$$m\left(r, \frac{1}{y}\right) = S(r, y),$$

on a set E of logarithmic density 1. Thus

$$N\left(r, \frac{1}{y}\right) = T(r, y) + S(r, y)$$

on the set E . By the above equation and $y(z) = \frac{1}{f(z)}$, we obtain

$$N(r, f) = N\left(r, \frac{1}{y}\right) = T(r, y) + S(r, y) = T(r, f) + S(r, f)$$

on E . So

$$\delta(\infty, f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{N(r, f)}{T(r, f)} \leq 1 - \overline{\lim}_{r \rightarrow \infty, r \in E} \frac{N(r, f)}{T(r, f)} = 1 - 1 = 0.$$

Thus, $\delta(\infty, f) = 0$, that is ∞ is not a deficient value of $f(z)$.

Now we prove that no finite values d are deficient values of $f(z)$. By (1.2), we have

$$P_2(z, f) := (q - 1)zf(qz)f(z) - f(qz) + f(z) + A(z) = 0.$$

Since $A(z)$ is not a polynomial of degree one, $P_2(z, d) = A(z) + (q - 1)d^2z \neq 0$. Using a similar method as above, we obtain the result.

Set $y(z) = f(z) - z$. Then $y(z)$ is a zero order transcendental meromorphic function, and

$$T(r, y) = T(r, f) + O(\log r).$$

Substituting $f(z) = y(z) + z$ into (1.2), we obtain

$$P_3(z, y) := (q - 1)z(y(qz) + qz)(y(z) + z) - y(qz) + y(z) + A(z) - (q - 1)z = 0.$$

Since $A(z)$ is not a polynomial of degree three, $P_3(z, 0) = (q-1)(qz^2-1)z + A(z) \neq 0$. By Lemma 2.3, we see that

$$N\left(r, \frac{1}{f-z}\right) = N\left(r, \frac{1}{y}\right) = T(r, y) + S(r, y) = T(r, f) + S(r, f)$$

on a set of logarithmic density 1. So, $f(z)$ has infinitely many fixed points.

REMARK 2.5. By (2.3) and (2.4), we can find a transcendental meromorphic solution of (1.2) through its rational solution, and we investigate the existence of rational solutions of (1.2) in section 4.

3. THE ZEROS OF DIFFERENCES OF SOLUTIONS OF Q-DIFFERENCE RICCATI EQUATIONS

Recently, FLETCHER et al. [6] investigated the zeros of $f(qz) - f(z)$ and $\frac{f(qz) - f(z)}{f(z)}$, and obtained the following theorem.

Theorem 3.1 [6]. *Let $q \in \mathbb{C}$ with $|q| > 1$. Let f be a transcendental meromorphic function in the plane with*

$$(3.1) \quad L(f) = \liminf_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} = 0.$$

Then at least one of $f(qz) - f(z)$ and $\frac{f(qz) - f(z)}{f(z)}$ has infinitely many zeros.

For the q -difference Riccati equations

$$(3.2) \quad f(qz) = \frac{a(z)f(z) + b(z)}{f(z) + c(z)},$$

where $a(z), b(z), c(z)$ are rational functions, by Remark 2.2, we know that a transcendental meromorphic solution $f(z)$ of (3.2) satisfies $T(r, f) = O((\log r)^2)$, and may not satisfy (3.1), but we prove that both $f(qz) - f(z)$ and $\frac{f(qz) - f(z)}{f(z)}$ have infinitely many zeros.

Theorem 3.2. *Suppose that $a(z), b(z), c(z)$ are rational functions and $q \in \mathbb{C}$ with $|q| \neq 0, 1$. If $f(z)$ is a transcendental meromorphic solution of (3.2), then:*

(i) *If $b(z)$ is nonconstant rational function, then $f(z)$ has infinitely many zeros and poles. Furthermore, if $\forall d \in \mathbb{C}$, it holds that $d^2 + d(c(z) - a(z)) - b(z) \neq 0$, then $f(z)$ has no deficient value.*

(ii) *If $a(z) \equiv c(z)$ is a nonzero rational function, and $b(z) \equiv (s(z))^2$, where $s(z) \neq \pm a(z)$ is a nonconstant rational function, then $f(qz) - f(z)$ and $\frac{f(qz) - f(z)}{f(z)}$ have infinitely many zeros.*

Proof. (i) Suppose that $f(z)$ is a transcendental solution of (3.2), by Remark 2.2, the order of growth of $f(z)$ is zero. We obtain that

$$f(qz)f(z) = -c(z)f(qz) + a(z)f(z) + b(z).$$

By the q-difference Clunie Lemma [2, Theorem 2.1] and the above equation, we see that

$$m(r, f) = o(T(r, f))$$

on E . Hence we have that $N(r, f) = T(r, f) + S(r, f)$ on E . On the other hand, we set

$$(3.3) \quad P(z, f) = f(qz)f(z) + c(z)f(qz) - a(z)f(z) - b(z).$$

Since $P(z, 0) = -b(z) \not\equiv 0$, by Lemma 2.3, we have that $m\left(r, \frac{1}{f}\right) = S(r, f)$ on E . Hence $N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f)$ on E . Thus, $\delta(\infty, f) = \delta(0, f) = 0$.

Furthermore, if for an arbitrary constant d , we have that $P(z, d) = d^2 + d(c(z) - a(z)) - b(z) \not\equiv 0$, by Lemma 2.3, we have that $m\left(r, \frac{1}{f-d}\right) = S(r, f)$ on E . Further $N\left(r, \frac{1}{f-d}\right) = T(r, f) + S(r, f)$ on E . Hence $f(z)$ has no deficient values.

(ii) By the equation (3.2), we obtain that

$$(3.4) \quad f(qz) - f(z) = \frac{a(z)f(z) + s^2(z)}{f(z) + a(z)} - f(z) = -\frac{(f(z) - s(z))(f(z) + s(z))}{f(z) + a(z)}.$$

By (3.3), we obtain that

$$(3.5) \quad P(z, s(z)) = s(qz)s(z) + a(z)(s(qz) - s(z)) - (s(z))^2,$$

$$(3.6) \quad P(z, -s(z)) = s(qz)s(z) - a(z)(s(qz) - s(z)) - (s(z))^2.$$

If $P(z, s(z)) \equiv 0$ and $P(z, -s(z)) \equiv 0$, by (3.5) and (3.6), we see that

$$a(z)(s(qz) - s(z)) \equiv 0.$$

By the assumption that $s(z)$ is nonconstant rational function and $|q| \neq 0, 1$, we know that $s(qz) - s(z) \not\equiv 0$. So $a(z) \equiv 0$, which contradicts with the assumption that $a(z) \not\equiv 0$. Hence $P(z, s(z)) \not\equiv 0$ or $P(z, -s(z)) \not\equiv 0$. Without loss of generality, we assume that $P(z, s(z)) \not\equiv 0$. By Lemma 2.3, we obtain that

$$m\left(r, \frac{1}{f(z) - s(z)}\right) = o(T(r, f))$$

on E . Hence

$$(3.7) \quad N\left(r, \frac{1}{f(z) - s(z)}\right) = T(r, f) + o(T(r, f))$$

on E .

Next, we claim that the zeros of $f(z) - s(z)$ are all the zeros of $f(qz) - f(z)$, except at most finite many zeros. In fact, by (3.4), if z_0 is a common zero of

$f(z) - s(z)$ and $f(z) + a(z)$, then $s(z_0) + a(z_0) = 0$. Since both $s(z)$ and $a(z)$ are rational functions and $s(z) \not\equiv -a(z)$, the number of solutions of the equation $s(z) + a(z) = 0$ is finite. Thus the number of the common zeros of $f(z) - s(z)$ and $f(z) + a(z)$ is finite. Since the poles of $f(z) - s(z)$ and $f(z) + s(z)$ are all the poles of $f(z)$, except at most finite many poles, then the number of points such as z_1 is finite, where z_1 is both a zero of $f(z) - s(z)$ and a pole of $f(z) + s(z)$. So

$$N\left(r, \frac{f(z) + a(z)}{(f(z) - s(z))(f(z) + s(z))}\right) = N\left(r, \frac{1}{f(z) - s(z)}\right) + O(\log r).$$

By the above equality and (3.4), we obtain that

$$\begin{aligned} N\left(r, \frac{1}{f(qz) - f(z)}\right) &= N\left(r, \frac{f(z) + a(z)}{(f(z) - s(z))(f(z) + s(z))}\right) \\ &= N\left(r, \frac{1}{f(z) - s(z)}\right) + O(\log r). \end{aligned}$$

From this and (3.7), we have that

$$N\left(r, \frac{1}{f(qz) - f(z)}\right) = T(r, f) + o(T(r, f))$$

on E . Hence $f(qz) - f(z)$ has infinitely many zeros.

By (3.4), we have that

$$(3.8) \quad \frac{f(qz) - f(z)}{f(z)} = -\frac{(f(z) - s(z))(f(z) + s(z))}{(f(z) + a) f(z)}.$$

Using the analogous method, we obtain that

$$N\left(r, \frac{1}{\frac{f(qz) - f(z)}{f(z)}}\right) = T(r, f) + o(T(r, f))$$

on E . Hence $\frac{f(qz) - f(z)}{f(z)}$ has infinitely many zeros.

4. THE EXISTENCE AND FORMS OF RATIONAL SOLUTIONS OF Q-DIFFERENCE RICCATI EQUATIONS

CHEN and SHON [4] consider existence and forms of rational solutions of difference Riccati equation. In this section, we do the same for q-difference Riccati equation (1.2).

Theorem 4.1. *Let $q \in \mathbb{C} \setminus \{0\}$ such that $|q| \neq 1$ and $A(z) = \frac{P(z)}{Q(z)} = \frac{c_m z^m + \cdots + c_0}{d_n z^n + \cdots + d_0}$ an irreducible rational function, where $c_m (\neq 0), \dots, c_0, d_n (\neq 0), \dots, d_0$ are constants.*

(1) If $m \geq n$ and $m - n$ is an even number, then the equation (1.2) has no rational solutions.

(2) If (1.2) has a rational solution $f(z) = \frac{R(z)}{S(z)} = \frac{a_r z^r + \dots + a_0}{b_s z^s + \dots + b_0}$, where $a_r (\neq 0), \dots, a_0, b_s (\neq 0), \dots, b_0$ are constants, then:

(i) If $m > n$ and $m - n$ is an odd number, then $r - s = \frac{m - n - 1}{2}$.

(ii) If $n - m \geq 2$, then

$$r = s - 1, a_r = -b_s \quad \text{or} \quad s - r = n - m, d_n a_r = (d_n a_r + c_m b_s) q^{s-r}.$$

(iii) If $n - m = 1$, then $r = s - 1$ and $(q - 1)d_n a_r (a_r + b_s) + q c_m b_s^2 = 0$.

Examples 4.1 and 4.2 show that there exist rational solutions satisfying Theorem 4.1 (2)(i). Examples 4.3 and 4.4 show that there exist two types of rational solutions satisfying Theorem 4.1 (2)(ii). Example 4.5 shows that there exist rational solutions satisfying Theorem 4.1(2)(iii).

EXAMPLE 4.1. For $q = \frac{1}{2}$, $A(z) = \frac{\frac{1}{2}z^3 - \frac{3}{2}z^2 - z}{(z + 1)(z + 2)}$, equation (1.2) has a rational solution $f(z) = \frac{z - 1}{z + 1}$, where $r = s, m = 3, n = 2$.

EXAMPLE 4.2. Let $q = 2$, $A(z) = \frac{-4z^5 + 2z^3 - 3z^2}{(z - 1)(2z - 1)}$. The equation (1.2) has a rational solution $f(z) = \frac{z^2}{z - 1}$, where $r = 2, s = 1, m = 5, n = 2$.

EXAMPLE 4.3. Let $q = \frac{1}{2}$, $A(z) = \frac{4z}{(z^2 + 4)(z^2 + 1)}$. The equation (1.2) has a rational solution $f(z) = \frac{-z + 1}{z^2 + 1}$, where $r = 1, s = 2, a_r = -1, b_s = 1$.

EXAMPLE 4.4. Let $q = \frac{1}{2}$, $A(z) = \frac{3z^3 - z^2 - 4z + 2}{z^5}$. Then the equation (1.2) has a rational solution

$$f(z) = \frac{z - 1}{z^3}.$$

EXAMPLE 4.5. Let $q = \frac{1}{2}$. The q-difference Riccati equation

$$f(qz) = \frac{\frac{2z^3 - 6z^2}{(z^2 + 4)(z^2 + 1)} + f(z)}{1 - (q - 1)zf(z)}$$

has a rational solution

$$f(z) = \frac{z - 1}{z^2 + 1}.$$

The following lemma will be used in the proof of Theorem 4.1.

Lemma 4.2. Let $R(z)$ and $S(z)$ be polynomials with $\deg R(z) = r$ and $\deg S(z) = s$ respectively. Let q be a complex constant such that $|q| \notin \{0, 1\}$.

(1) If $r \neq s$, then $\deg (R(z)S(qz) - R(qz)S(z)) = s + r$.

(2) If $r = s$, then $\deg(R(z)S(qz) - R(qz)S(z)) \leq s + r - 1$.

Proof. Suppose that

$$R(z) = a_r z^r + a_{r-1} z^{r-1} + \cdots + a_0, \quad S(z) = b_s z^s + b_{s-1} z^{s-1} + \cdots + b_0,$$

where $a_r (\neq 0), \dots, a_0, b_s (\neq 0), \dots, b_0$, are constants. Since

$$\begin{aligned} R(z)S(qz) - R(qz)S(z) &= a_r b_s (q^s - q^r) z^{r+s} \\ &\quad + (a_r b_{s-1} (q^{s-1} - q^r) + a_{r-1} b_s (q^s - q^{r-1})) z^{r+s-1} + \cdots, \end{aligned}$$

and $|q| \in \mathbb{R} \setminus \{0, 1\}$, if $r \neq s$, then $a_r b_s (q^s - q^r) \neq 0$, so

$$\deg(R(z)S(qz) - R(qz)S(z)) = s + r.$$

If $r = s$, then

$$R(z)S(qz) - R(qz)S(z) = (a_r b_{s-1} - a_{r-1} b_s) (1 - q) q^s z^{r+s-1} + \cdots,$$

so $\deg(R(z)S(qz) - R(qz)S(z)) \leq s + r - 1$.

Proof of Theorem 4.1. Suppose that $f(z) = \frac{R(z)}{S(z)}$ is a rational solution of (1.2), where $R(z)$ and $S(z)$ are polynomials with $\deg R(z) = r$ and $\deg S(z) = s$. Equation (1.2) can be written as

$$(4.1) \quad P(z)S(qz)S(z) + Q(z)(R(z)S(qz) - R(qz)S(z)) + (q-1)zR(qz)R(z)Q(z) = 0.$$

(1) Suppose that $m \geq n$ and $m - n$ is an even number. For $q \in \mathbb{C} \setminus \{0, 1\}$, we have

$$\deg(Q(z)(R(z)S(qz) - R(qz)S(z))) \leq n + s + r.$$

If $r \leq s - 1$, then

$$n + s + r \leq n + 2s - 1 < m + 2s = \deg(P(z)S(qz)S(z)),$$

and

$$\begin{aligned} \deg((q-1)zR(qz)R(z)Q(z)) &= 1 + 2r + n \leq n + 2s - 1 \\ &< m + 2s = \deg(P(z)S(qz)S(z)). \end{aligned}$$

Thus, in the left side of (4.1), there is only one term of the highest degree, which is a contradiction.

If $r \geq s$, then

$$\begin{aligned} \deg(Q(z)(R(z)S(qz) - R(qz)S(z))) &\leq n + r + s \leq n + 2r \\ &< n + 2r + 1 = \deg((q-1)zR(qz)R(z)Q(z)). \end{aligned}$$

Thus (4.1) implies

$$\deg((q-1)zR(qz)R(z)Q(z)) = n + 2r + 1 = m + 2s = \deg(P(z)S(qz)S(z)).$$

It gives $r - s = \frac{m - n - 1}{2}$. It contradicts with the fact that $m - n$ is an even number. So, the equation (1.2) has no rational solution.

(2) For other cases, suppose that (1.2) has a rational solution $f(z) = \frac{R(z)}{S(z)} = \frac{a_r z^r + \dots + a_0}{b_s z^s + \dots + b_0}$, where $a_r (\neq 0), \dots, a_0, b_s (\neq 0), \dots, b_0$ are constants, we consider three subcases:

(i) Suppose that $m > n$ and $m - n$ is an odd number. If $r \leq s - 1$, then we see that (4.1) is a contradiction by using the same method as in (1). If $r \geq s$, then

$$n + r + s \leq n + 2r < n + 2r + 1,$$

(4.1) implies

$$\deg((q - 1)zR(qz)R(z)Q(z)) = n + 2r + 1 = m + 2s = \deg(P(z)S(qz)S(z)).$$

It gives $r - s = \frac{m - n - 1}{2}$.

(ii) Suppose that $n - m \geq 2$. If $r \geq s$, then

$$\begin{aligned} \deg(Q(z)(R(z)S(qz) - R(qz)S(z))) &\leq n + r + s \leq n + 2r \\ &< n + 2r + 1 = \deg((q - 1)zR(qz)R(z)Q(z)). \end{aligned}$$

Since $m < n$,

$$\begin{aligned} \deg(P(z)S(qz)S(z)) &= m + 2s \leq n + 2r \\ &< n + 2r + 1 = \deg((q - 1)zR(qz)R(z)Q(z)), \end{aligned}$$

which contradicts (4.1).

If $r = s - 1$, then combining $m + 1 < n$, we obtain that

$$\begin{aligned} \deg(P(z)S(qz)S(z)) &= m + 2s = m + 2r + 2 \\ &< n + 2r + 1 = \deg((q - 1)zR(qz)R(z)Q(z)). \end{aligned}$$

Since $|q| \neq 0, 1$, by Lemma 4.2 and $r = s - 1$, we have that

$$\deg(Q(z)(R(z)S(qz) - R(qz)S(z))) = n + r + s = n + 2r + 1.$$

Since

$$(4.2) \quad R(z) = a_r z^r + \dots + a_0, S(z) = b_s z^s + \dots + b_0,$$

we see that

$$(4.3) \quad R(qz) = a_r q^r z^r + \dots + a_0, S(qz) = b_s q^s z^s + \dots + b_0.$$

Substituting (4.2), (4.3) and $P(z) = c_m z^m + \dots + c_0, Q(z) = d_n z^n + \dots + d_0$ into (4.1), we obtain that

$$(q - 1)a_r d_n (a_r + b_s) q^r z^{n+2r+1} + T_{n+2r}(z) = 0,$$

where $T_{n+2r}(z)$ is a polynomial such that $\deg T_{n+2r} \leq n+2r$. Since $q \neq 0, 1, a_r d_n \neq 0$, by the above equation, it follows that

$$a_r = -b_s.$$

If $r < s - 1$, then by Lemma 4.2 and $|q| \neq 0, 1$, we obtain

$$\begin{aligned} \deg((q-1)zR(qz)R(z)Q(z)) &= n+2r+1 \\ &< n+r+s = \deg(Q(z)(R(z)S(qz) - R(qz)S(z))). \end{aligned}$$

Thus, (4.1) implies that

$\deg(Q(z)(R(z)S(qz) - R(qz)S(z))) = n+r+s = m+2s = \deg(P(z)S(qz)S(z))$,
so $s-r = n-m$. Substituting (4.2), (4.3) and $P(z) = c_m z^m + \dots + c_0, Q(z) = d_n z^n + \dots + d_0$ into (4.1), we obtain that

$$(d_n(a_r b_s q^{s-r} - a_r b_s) + c_m b_s^2 q^{s-r})q^r z^{n+r+s} + T_{n+r+s-1}(z) = 0,$$

where $T_{n+r+s-1}(z)$ is a polynomial such that $\deg T_{n+r+s-1} \leq n+r+s-1$. Since $q \neq 0, 1$, by the above equation, we see that

$$d_n a_r = (d_n a_r + c_m b_s)q^{s-r}.$$

(iii) Suppose that $n-m=1$. If $r \geq s$, using the same method as in (ii), we see that (4.1) is a contradiction. If $r < s-1$, then by Lemma 4.2 and $|q| \neq 0, 1$, we obtain

$$\begin{aligned} \deg((q-1)zR(qz)R(z)Q(z)) &= n+2r+1 \\ &< n+r+s = \deg(Q(z)(R(z)S(qz) - R(qz)S(z))). \end{aligned}$$

Thus, (4.1) implies

$$\deg(Q(z)(R(z)S(qz) - R(qz)S(z))) = n+r+s = m+2s = \deg(P(z)S(qz)S(z)),$$

so $s-r = n-m$. It contradicts with $n-m=1$.

If $r = s-1$, by Lemma 4.2,

$$\begin{aligned} \deg(P(z)S(qz)S(z)) &= m+2s = n+2r+1 = \deg((q-1)zR(qz)R(z)Q(z)) \\ &= n+r+s = \deg(Q(z)(R(z)S(qz) - R(qz)S(z))). \end{aligned}$$

Substituting (4.2), (4.3) and $P(z) = c_m z^m + \dots + c_0, Q(z) = d_n z^n + \dots + d_0$ into (4.1) again, we obtain that

$$((q-1)d_n a_r (a_r + b_s) + q c_m b_s^2)q^r z^{m+2s} + T_{m+2s-1}(z) = 0,$$

where $T_{m+2s-1}(z)$ is a polynomial such that $\deg T_{m+2s-1} \leq m+2s-1$. Since $q \neq 0, 1$, by the above equation, we see that

$$(q-1)d_n a_r (a_r + b_s) + q c_m b_s^2 = 0.$$

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