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MULTIPLE POSITIVE SOLUTIONS FOR DISCRETE p-LAPLACIAN EQUATIONS WITH POTENTIAL TERM

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We study the existence of solutions to nonlinear discrete boundary value problems with the discrete p-Laplacian, potential, and nonlinear source terms. Using variational methods, we demonstrate that there exist at least two positive solutions. The existence strongly depends on the smallest positive eigenvalue of Dirichlet eigenvalue problems and the growth conditions of the source terms.

1. INTRODUCTION

Discrete boundary value problem is one of the most important mathematical equations and has rich applications in the area such as astrophysics, gas dynamics, fluid mechanics, computer science, image processing, chemically reacting systems, and others. Study of the properties of the operators plays a key role in dealing with these problems. Recently the discrete p-Laplacian, which appears in various discrete problems, has received great attention from many researchers. For more details, see [**3, 4, 5, 6, 11, 12, 13**].

In [**1**], Agarwal, Perera and O'Regan proved the existence of multiple positive solutions to the following boundary value problem involving the discrete p-Laplacian:

$$(1.1)\qquad \begin{cases} -\mathcal{D}(\phi_p(\mathcal{D}u(k-1))) = f(k,u(k)), & k \in [1,T] := \{1,\dots,T\} \\ u(0) = u(T+1) = 0, \end{cases}$$

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where T is a fixed positive integer, $\mathcal{D}u(k) := u(k+1) - u(k)$ is the forward difference operator, $\phi_p(t) := |t|^{p-2}t$, $t \in \mathbb{R}$, $1 < p < \infty$, and the function $f \in C([1,T] \times (0,\infty); \mathbb{R})$ satisfies

$$a_0(k) \le f(k,t) \le a_1(k)t^{-\gamma}, \quad (k,t) \in [1,T] \times (0,t_0) \tag{1.2}$$

for some nontrivial functions a_0, $a_1 \ge 0$ and γ, $t_0 > 0$. Their first result is that if the function f satisfies (1.2) and

$$\limsup_{t\to\infty} \frac{f(k,t)}{t^{p-1}} < \lambda_1, \quad k \in [1,T], \tag{1.3}$$

where λ_1 is the smallest positive eigenvalue of

$$\begin{cases} -\mathcal{D}(\phi_p(\mathcal{D}u(k-1))) = \lambda\phi_p(u(k)), & k \in [1,T] \\ u(0) = u(T+1) = 0 \end{cases}$$

then (1.1) has a solution. Their second result is that if f satisfies (1.2) and

$$f(k,t_1) \le 0, \quad k \in [1,T], \tag{1.4}$$

for some $t_1 > t_0$, then (1.1) has a solution $u_1 < t_1$. If, in addition, f satisfies

$$\liminf_{t\to\infty} \frac{f(k,t)}{t^{p-1}} > \lambda_1, \quad k \in [1,T], \tag{1.5}$$

then there exists a second solution $u_1 < u_2$.

The main purpose of this paper is to generalize the graph structure and the main equation and improve the growth conditions in [**1**]. To do this, we consider a discrete boundary value problem including potential terms on a graph. Namely, we deal with the following equation on a simple and connected graph $G = G(S \cup \partial S, E)$:

$$\begin{cases} -\Delta_{p,\omega}u(x) + V(x)|u(x)|^{p-2}u(x) = f(x,u(x)), & x \in S \\ u(x) = \sigma(x) \ge 0, & x \in \partial S \end{cases} \tag{1.6}$$

where $\Delta_{p,\omega}$ is the *discrete p-Laplacian* defined by

$$\Delta_{p,\omega}u(x) := \sum_{y\in\overline{S}} |u(y)-u(x)|^{p-2}(u(y)-u(x))\omega(x,y), \ x \in S$$

and $\omega : S \cup \partial S \times S \cup \partial S \to [0,\infty)$ is a *weight function* defined by

(i) $\omega(x,y) = \omega(y,x) > 0$ if $\{x,y\} \in E$,

(ii) $\omega(x,y) = 0$ if and only if $\{x,y\} \notin E$.

We note that the operator $\mathcal{D}(\phi_p(\mathcal{D}u))$ in (1.1) is the discrete p-Laplacian $\Delta_{p,\omega}$ on a path with standard weights.

We now propose the following assumptions:

(H1) V satisfies $\lambda_{1,V} > 0$.

(H2) f satisfies

$$a_0(x) \leq f(x,t), \quad t \in (0, \tau_0(x)), \quad x \in S,$$

where τ_0 is a positive function on S and a_0 is a non-negative function satisfying $a_0(x) \neq 0$ for some $x \in S$.

(H3) f satisfies

$$\limsup_{t\to\infty} \frac{f(x,t)}{t^{p-1}} < \lambda_{1,V}, \quad x \in S.$$

(H3)′ (1.6) has a supersolution w_0.

(H4) f satisfies

$$\liminf_{t\to\infty} \frac{f(x,t)}{t^{p-1}} > \lambda_{1,V}, \quad x \in S.$$

The assumptions **(H3)**, **(H3)′**, and **(H4)** provide more improved bounds than (1.3), (1.4) and (1.5). Main results in this paper are as follows:

Theorem 1. *Let* **(H1)**, **(H2)**, *and* **(H3)** *hold. Then* (1.6) *has a positive solution* u.

Theorem 2. *If* **(H1)**, **(H2)**, *and* **(H3)′** *hold, then* (1.6) *has a positive solution* u_1 *satisfying* $u_1 < w_0$. *If, in addition,* **(H4)** *holds, then* (1.6) *has the second positive solutions* u_2 *satisfying* $u_1 < u_2$.

The rest of this paper is organized as follows: in Section 2, we present graph theoretic notions used frequently throughout this paper. We also introduce a comparison principle and the sub-supersolution method for the discrete p-Laplacian with potential terms which are proved in [**7, 8**]. In Section 3, we show the existence of a positive solution to our problem. In Section 4, we prove that there exist at least two positive solutions, and verify that one of them is strictly greater than the other. Finally, in Section 5, we give some examples for the results in Section 4.

2. PRELIMINARIES

In this section, we start with graph theoretic notions used frequently throughout this paper.

By a graph $G = G(S \cup \partial S, E)$ we mean a two finite and disjoint set S and ∂S of *vertices*, called *interior* and *boundary* respectively, with a set E of unordered pairs of distinct elements of $S \cup \partial S$ whose elements are called *edges*. As conventionally used, we denote by $x \in \overline{S}$ the facts that x is a vertex in $S \cup \partial S$.

A graph G is said to be *simple* if it has neither multiple edges nor loops, and G is said to be connected if for every pair of vertices x and y, there exists a sequence (termed a *path*) of vertices $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ such that x_{j-1} and x_j are connected by an edge (termed *adjacent*) for $j = 1, \ldots, n$.

A graph $G' = G'(T, E')$ is said to be a *subgraph* of $G = G(\overline{S}, E)$. If $T \subset \overline{S}$ and $E' \subset E$. If E' consists of all the edges from E which connect the vertices T in a graph G then G' is called an *induced subgraph*.

A *weight* on a graph G is a function $\omega : \overline{S} \times \overline{S} \to [0, \infty)$ satisfying

(i) $\omega(x, y) = \omega(y, x) > 0 \qquad \text{if } \{x, y\} \in E,$

(ii) $\omega(x, y) = 0 \qquad \text{if and only if } \{x, y\} \notin E.$

A graph associated with a weight is said to be a *weighted graph* (or *network*). In this paper, we only consider a simple and connected graph G with weight ω. We note that since a graph G is simple, it is trivial that $\omega(x, x) = 0$ for all $x \in \overline{S}$.

Throughout this paper, a function on a graph is understood as a function defined on the set of vertices of the graph. For a nonempty subset T of vertices in G, the integration of a function $u : T \to \mathbb{R}$ is defined by

$$\int_T u := \sum_{x \in T} u(x).$$

For $p > 1$, the *p-directional derivative* of a function $u : \overline{S} \to \mathbb{R}$ in the direction y is defined by

$$D_{p,\omega,y} u(x) := |u(y) - u(x)|^{p-2}(u(y) - u(x))\sqrt{\omega(x, y)}$$

for $x \in \overline{S}$. The *p-gradient* $\nabla_{p,\omega}$ of a function $u : \overline{S} \to \mathbb{R}$ is defined to be

$$\nabla_{p,\omega} u(x) := \left(D_{p,\omega,y} u(x)\right)_{y \in \overline{S}}$$

for $x \in \overline{S}$. In the case of $p = 2$, we write simply ∇_ω instead of $\nabla_{2,\omega}$.

The *discrete p-Laplacian* $\Delta_{p,\omega}$ of a function $u : \overline{S} \to \mathbb{R}$ is defined by

$$\Delta_{p,\omega} u(x) := \sum_{y \in \overline{S}} |u(y) - u(x)|^{p-2}(u(y) - u(x))\omega(x, y), \ \ x \in S.$$

We note that for any pair of functions $u : \overline{S} \to \mathbb{R}$ and $v : \overline{S} \to \mathbb{R}$, we have

$$2 \int_{\overline{S}} v(-\Delta_{p,\omega} u) = \int_{\overline{S}} \nabla_\omega v \cdot \nabla_{p,\omega} u. \tag{2.7}$$

where $\mathbb{A} \cdot \mathbb{B} := \sum_{i=1}^{n} a_i b_i$ for $\mathbb{A} = (a_1, \ldots, a_n)$ and $\mathbb{B} = (b_1, \ldots, b_n)$. We remark here that other authors define the p-Laplacian as generalizations of the combinatorial graph Laplacian which then has opposite sign, see e.g. [**2**]. In this paper we follow the notions in [**1**].

In this paper, we define a set $\mathcal{A}_\sigma$ for a function $\sigma : \partial S \to [0, \infty)$ as follows:

$$\mathcal{A}_\sigma := \{u : \overline{S} \to \mathbb{R} \mid u(z) = \sigma(z), z \in \partial S\}.$$

In particular, in the case of $\sigma \equiv 0$, we write $\mathcal{A}_0$.

For a function $V : S \to \mathbb{R}$, Dirichlet eigenvalue problem for the discrete p-Laplacian with potential term is defined as follows:

$$(2.8) \qquad \begin{cases} -\Delta_{p,\omega}\phi(x) + V(x)|\phi(x)|^{p-2}\phi(x) = \lambda|\phi(x)|^{p-2}\phi(x), & x \in S \\ \phi(z) = 0, & z \in \partial S. \end{cases}$$

This problem has the first eigenvalue which is given by

$$(2.9) \qquad \lambda_{1,V} := \inf_{\substack{\phi \not\equiv 0 \\ \phi \in \mathcal{A}_0}} \frac{\frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi \cdot \nabla_{p,\omega}\phi + \int_S V|\phi|^p}{\int_S |\phi|^p}$$

and there exists a positive eigenfunction $\phi_1 \in \mathcal{A}_0$ corresponding to $\lambda_{1,V}$ satisfying

$$\int_S |\phi_1|^p = 1.$$

We note that the first eigenvalue $\lambda_{1,V}$ can be considered as a functional with respect to V and it has the following properties:

(i) $\lambda_{1,V}$ is continuous on $\mathcal{A}_0$,
(ii) the multiplicity of $\lambda_{1,V}$ is one,
(iii) $\lambda_{1,V}$ is isolated.

Particularly, if we put $V \equiv 0$ and $\partial S \neq \emptyset$, then $\lambda_{1,V} > 0$ (for more details, see [**9**]).

We include here a (strong) comparison principle and the method of sub-supersolutions for the discrete p-Laplacian for future use which are proved in [**7, 8**].

Theorem 2.1 (Comparison principle)**.** *For a function $V : S \to \mathbb{R}$, suppose that $\lambda_{1,V} > 0$ and $u_i : \overline{S} \to \mathbb{R}$, $i = 1, 2$ satisfies that*

$$(2.10) \qquad \begin{cases} -\Delta_{p,\omega}u_2(x) + V(x)|u_2(x)|^{p-2}u_2(x) \geq \\ \qquad\qquad -\Delta_{p,\omega}u_1(x) + V(x)|u_1(x)|^{p-2}u_1(x), & x \in S \\ u_2(x) \geq u_1(x), & x \in \partial S. \end{cases}$$

If we assume in addition that

$$\begin{cases} -\Delta_{p,\omega}u_2(x) + V(x)|u_2(x)|^{p-2}u_2(x) \geq 0, & x \in S \\ u_2(x) \geq 0, & x \in \partial S \end{cases}$$

then $u_2(x) \geq u_1(x)$ for all $x \in S$. Moreover, the equalities hold in (2.10) *if and only if $u_2 \equiv u_1$.*

Theorem 2.2 (Sub-supersolution method)**.** *For a function $f \in C(S \times \mathbb{R}; \mathbb{R})$, suppose that $\underline{u}$ and $\overline{u}$ in $\mathcal{A}_\sigma$ are subsolution and supersolution with $\underline{u} \leq \overline{u}$ to the equation*

$$(2.11) \qquad \begin{cases} -\Delta_{p,\omega}u(x) = f(x, u(x)), & x \in S \\ u(x) = \sigma(x), & x \in \partial S. \end{cases}$$

If a given function f satisfies that there exists $\lambda > 0$ such that $f(\cdot, t) + \lambda|t|^{p-2}t$ is nondecreasing in S, then there exists a solution u of (2.11) *such that $\underline{u} \leq u \leq \overline{u}$.*

We note that for a non-zero function $a_0 : S \to [0, \infty)$, the equation

$$\begin{cases} -\Delta_{p,\omega} u(x) + V(x)|u(x)|^{p-2}u(x) = a_0(x), & x \in S \\ u(x) = \sigma(x) \geq 0, & x \in \partial S \end{cases} \tag{2.12}$$

has a unique solution u_0 if the first eigenvalue $\lambda_{1,V}$ is bigger than zero. Moreover, the condition $a_0 \not\equiv 0$ implies that the solution u_0 is strictly positive on S (for details, see [**8**]).

3. A POSITIVE SOLUTION

For a function $g \in C(S \times \mathbb{R}; \mathbb{R})$, we consider a functional E_g defined by

$$E_g[u] := \frac{1}{2p} \int_S \nabla_{p,\omega} u \cdot \nabla_\omega u + \frac{1}{p} \int_S V|u|^p - \int_S \mathcal{G}_u, \quad u \in \mathcal{A}_\sigma$$

where $\mathcal{G}_u : S \to \mathbb{R}$ is defined by

$$\mathcal{G}_u(x) := \int_0^{u(x)} g(x,t)\, \mathrm{d}t.$$

We note that since the functional E_g is differentiable, a critical point of E_g is a solution to (1.6).

Lemma 3.3. *Suppose that a function $g \in C(S \times \mathbb{R}; \mathbb{R})$ satisfies*

$$\limsup_{|t| \to \infty} \frac{g(x,t)}{|t|^{p-2}t} < \lambda_{1,V}, \quad x \in S. \tag{3.13}$$

Then there exists a solution to

$$\begin{cases} -\Delta_{p,\omega} u(x) + V(x)|u(x)|^{p-2}u(x) = g(x, u(x)), & x \in S \\ u(x) = \sigma(x) \geq 0, & x \in \partial S. \end{cases}$$

Proof. It follows from (3.13) that there exists $M_0 < \lambda_{1,V}$ such that

$$\int_0^t g(x,s)\, \mathrm{d}s \leq \frac{M_0}{p}|t|^p + C, \quad x \in S,\ t \in \mathbb{R} \tag{3.14}$$

for some constant C. Since $\lambda_{1,V}$ is continuous with respect to V, there exists a sufficiently small value $\epsilon > 0$ such that $M_0 < (1-\epsilon)\lambda_{1,\frac{V}{1-\epsilon}}$. We now define two functions u_t and $\tilde{u}$ by

$$u_t(x) := \begin{cases} tu(x), & x \in S \\ \sigma(x), & x \in \partial S \end{cases}$$

and

$$\tilde{u}(x) := \begin{cases} u(x), & x \in S \\ 0, & x \in \partial S \end{cases}$$

for $t > 0$ and $u \in \mathcal{A}_\sigma$ satisfying $\int\limits_S |u|^p = 1$. Then for each t and $u \in \mathcal{A}_\sigma$ with $\int\limits_S |u|^p = 1$, we have

$$\begin{aligned} E_g[u_t] \geq \frac{t^p}{2p} \sum_{x,y\in S} |u(y) - u(x)|^p \omega(x,y) + \frac{1}{p} \sum_{\substack{x\in S \\ y\in\partial S}} |\sigma(y) - tu(x)|^p \omega(x,y) \\ + \frac{t^p}{p} \sum_{x\in S} V(x)|u(x)|^p + \frac{1}{2p} \sum_{x,y\in\partial S} |\sigma(y) - \sigma(x)|^p \omega(x,y) \\ - \frac{M_0 t^p}{p} \sum_{x\in S} |u(x)|^p - C'. \end{aligned}$$

We note that for $|\alpha|^p > \gamma > 0$, $\beta \geq 0$, there exists $K > 0$ such that

$$|\alpha t - \beta|^p \geq \gamma t^p - K\beta, \quad t > 0. \tag{3.15}$$

It follows from (3.15) that there exists $K > 0$ such that

$$|\sigma(y) - tu(x)|^p = |u(x)t - \sigma(y)|^p \geq (1-\epsilon)|u(x)|^p t^p - K\sigma(y)$$

for all $x \in S$ and $y \in \partial S$. Thus we have

$$\begin{aligned} &\sum_{\substack{x\in S \\ y\in\partial S}} |\sigma(y) - tu(x)|^p \omega(x,y) \geq t^p(1-\epsilon) \sum_{\substack{x\in S \\ y\in\partial S}} |u(x)|^p \omega(x,y) - K \sum_{\substack{x\in S \\ y\in\partial S}} \sigma(y)\omega(x,y) \\ &= t^p(1-\epsilon) \sum_{\substack{x\in S \\ y\in\partial S}} |\tilde{u}(y) - \tilde{u}(x)|^p \omega(x,y) - K \sum_{\substack{x\in S \\ y\in\partial S}} \sigma(y)\omega(x,y). \end{aligned}$$

Therefore the functional E_g satisfies that

$$\begin{aligned} &E_g[u_t] \geq \frac{(1-\epsilon)t^p}{2p} \sum_{x,y\in\overline{S}} |\tilde{u}(y) - \tilde{u}(x)|^p \omega(x,y) + \frac{(1-\epsilon)t^p}{p} \sum_{x\in S} \frac{V(x)}{(1-\epsilon)} |\tilde{u}(x)|^p \\ &+ \frac{1}{2p} \sum_{x,y\in\partial S} |\sigma(y) - \sigma(x)|^p \omega(x,y) - K \sum_{\substack{x\in S \\ y\in\partial S}} \sigma(y)\omega(x,y) - \frac{M_0 t^p}{p} \sum_{x\in S} |u(x)|^p - C'. \end{aligned}$$

By the definition of $\lambda_{1,\frac{V}{1-\epsilon}}$, it holds that

$$\frac{1}{2} \sum_{x,y\in S} |\tilde{u}(y) - \tilde{u}(x)|^p \omega(x,y) + \sum_{x\in S} \frac{V(x)}{(1-\epsilon)} |\tilde{u}(x)|^p \geq \lambda_{1,\frac{V}{1-\epsilon}} \sum_{x\in S} |\tilde{u}(x)|^p.$$

Thus it follows from $\int_S |u|^p = 1$ that

$$\begin{aligned} E_g[u_t] \geq & \frac{1}{p}\left((1-\epsilon)\lambda_{1,\frac{V}{1-\epsilon}} - M_0\right) t^p + \frac{1}{2p}\sum_{x,y\in\partial S} |\sigma(y)-\sigma(x)|^p\omega(x,y) \\ & - K\sum_{\substack{x\in S\\ y\in\partial S}} \sigma(y)\omega(x,y) - C' \\ \to & \infty, \quad t\to\infty. \end{aligned}$$

Thus E_g has a global minimizer which is a solution to (1.6).

Remark 3.4. In [**1**] the special case of $V \equiv 0$ and $\sigma \equiv 0$ was shown.

Example 1. Let a graph G be given by a path, V be nonnegative and non-zero on S and $\sigma \equiv 0$. Then by the definition of the first eigenvalue, it holds that $\lambda_{1,V} > \lambda_{1,0}$. We note that since G is a path, $\lambda_{1,0} = \lambda_1$ and $-\Delta_{p,\omega} u = -\mathcal{D}(\phi_p(\mathcal{D}u))$. We, in addition, assume that $V(x_0) = 0$ for some $x_0 \in S$ and a function g satisfies

$$\lambda_{1,0} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{|t|^{p-2}t} < \lambda_{1,V}, \quad x\in S.$$

We now putting $h(x,t) := g(x,t) - V(x)|t|^{p-2}t$, then h satisfies

$$\limsup_{|t|\to\infty} \frac{h(x_0,t)}{|t|^{p-2}t} \geq \lambda_{1,0}$$

and it holds that

$$\begin{cases} -\Delta_{p,\omega} u(x) = h(x,u(x)), & x\in S\\ u(x) = 0, & x\in\partial S. \end{cases} \tag{3.16}$$

In this case, the function h dose not satisfy the hypothesis in [**1**, Lemma 2.4] but by Lemma 3.3, the equation has a solution.

We now prove the first main result in this paper.

Theorem 3.5. *Suppose* **(H1)**, **(H2)**, *and* **(H3)** *hold. Then* (1.6) *has a positive solution u satisfying*

$$\epsilon_0^{\frac{1}{p-1}} u_0(x) < u(x), \quad x\in S \tag{3.17}$$

where u_0 is a strictly positive solution to (2.12) *and*

$$\epsilon_0 := \min\left\{\min_{x\in S}\left(\frac{\tau_0(x)}{u_0(x)}\right)^{p-1}, 1\right\}.$$

Proof. We first define a function $\underline{u}$ by $\underline{u}(x) := \epsilon_0^{\frac{1}{p-1}} u_0(x)$ for all $x\in\overline{S}$. Then it is clear that $\underline{u}(x) \leq \tau_0(x)$, $x\in S$. We now consider a function $\underline{f} : S\to\mathbb{R}$ defined by

$$\underline{f}(x,t) := \begin{cases} f(x,\underline{u}(x)), & t\leq \underline{u}(x)\\ f(x,t), & t > \underline{u}(x) \end{cases}$$

for $x \in S$. Then the function $\underline{f}$ satisfies

$$\limsup_{|t|\to\infty} \frac{\underline{f}(x,t)}{|t|^{p-2}t} < \lambda_{1,V}, \quad x \in S.$$

Hence by Lemma 3.3, there exists a global minimizer u_1 of $E_{\underline{f}}$. It is a solution to the equation

$$\begin{cases} -\Delta_{p,\omega} u(x) + V(x)|u(x)|^{p-2}u(x) = \underline{f}(x, u(x)), & x \in S \\ u(x) = \sigma(x), & x \in \partial S. \end{cases}$$

We now show that $u_1 > \underline{u}$ on S. Define a set $T := \{x \in S \mid u_1(x) \le \underline{u}(x)\}$ satisfying that an induced subgraph $G(T, E')$ of $\overline{S}(V, E)$ is connected and that $u_1(x) > \underline{u}(x)$, $x \in \partial T := \{x \in \overline{S} \setminus T \mid x \sim y \text{ for some } y \in T\}$. Since $u_1(x) \le \underline{u}(x)$ for $x \in T$, by the definition of $\underline{f}$, we have

$$-\Delta_{p,\omega} u_1(x) + V(x)|u_1(x)|^{p-2}u_1(x) = \underline{f}(x, u_1(x)) = f(x, \underline{u}(x)), \quad x \in T.$$

Since $f(x, \underline{u}(x)) \ge a_0(x) \ge \epsilon_0 a_0(x)$, $x \in T$, it follows that

$$-\Delta_{p,\omega} u_1(x) + V(x)|u_1(x)|^{p-2}u_1(x) \ge -\Delta_{p,\omega} \underline{u}(x) + V(x)|\underline{u}(x)|^{p-2}\underline{u}(x)$$

for all $x \in T$. Since $u_1(x) > \underline{u}(x)$ for all $x \in \partial T$, by the comparison principle, the set T is empty. Hence $u_1 > \underline{u}$ on S.

Example 2. Let a graph G be a path, $\sigma \equiv 0$ and $V \equiv -c$ where c is a constant satisfying $0 < c < \lambda_{1,0} (= \lambda_1)$. From the definition of $\lambda_{1,V}$, it follows that

$$\lambda_{1,V} \ge \lambda_{1,0} + \sum_{x\in S} V(x)|\phi_1|^p$$

where ϕ_1 is the positive eigenfunction corresponding to $\lambda_{1,V}$ satisfying $\int_S |\phi_1|^p = 1$. Hence $\lambda_{1,V} > \lambda_{1,0} (> 0)$. We now take a function f satisfying

$$\lambda_{1,0} \le \limsup_{t\to\infty} \frac{f(x,t)}{t^{p-1}} < \lambda_{1,V}, \quad x \in S.$$

Then it follows from Theorem 3.5 that there exists a positive solution to (1.6). On the other hand, it follows from the assumptions of V and f that

$$\limsup_{t\to\infty} \frac{h(x,t)}{t^{p-1}} \ge \lambda_{1,0} + c, \quad x \in S$$

where $h(x,t) := f(x,t) - V(x)t^{p-1}$. Hence in this example, h does not satisfy the hypotheses in [**1**, Theorem 1.1] but (3.16) has a positive solution.

4. TWO POSITIVE SOLUTIONS

In this section, we prove that (1.6) has at least two positive solutions if **(H1)**, **(H2)**, **(H3)**$'$, and **(H4)** hold. From now on, we assume that the function w_0 in **(H3)**$'$ is not a solution to (1.6) and the definitions of ϵ_0 and u_0 are the same as ones in Theorem 3.5.

Theorem 4.6. *If we assume* **(H1)**, **(H2)**, *and* **(H3)**$'$ *hold, then* (1.6) *has a positive solution* u *satisfying*

$$\epsilon_0^{\frac{1}{p-1}} u_0(x) < u(x) < w_0(x), \quad x \in S.$$

Proof. For a function $\underline{u}$ defined in the proof of Theorem 3.5, we define a function $\widetilde{f} : S \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(x,t) := \begin{cases} f(x, \underline{u}(x)), & t \leq \underline{u}(x) \\ f(x,t), & \underline{u}(x) < t < w_0(x) \\ f(x, w_0(x)), & w_0(x) \leq t. \end{cases}$$

Then it is clear that

$$\limsup_{|t|\to\infty} \frac{\widetilde{f}(x,t)}{|t|^{p-2}t} < \lambda_{1,V}.$$

Hence by Lemma 3.3, there exists a global minimizer u_1 of $E_{\widetilde{f}}$. Since $E_{\widetilde{f}}$ is differentiable, the function u_1 is a solution to

$$\begin{cases} -\Delta_{p,\omega} u(x) + V(x)|u(x)|^{p-2}u(x) = \widetilde{f}(x, u(x)), & x \in S \\ u(x) = \sigma(x), & x \in \partial S. \end{cases}$$

By using the argument in Theorem 3.5, we can show $\underline{u} < u_1$. Finally, we show that $u_1(x) < w_0(x)$, $x \in S$ by contradiction. Define a set $T := \{x \in S \mid u_1(x) \geq w_0(x)\}$ satisfying that an induced subgraph $G(T, E')$ of $\overline{S}(V,E)$ is connected and that $u_1(x) < w_0(x)$, $x \in \partial T := \{x \in \overline{S} \setminus T \mid x \sim y \text{ for some } y \in T\}$. In the case of $T = S$, by the definition of $\widetilde{f}$, we have

$$\begin{aligned} -\Delta_{p,\omega} u_1(x) + V(x)|u_1(x)|^{p-2}u_1(x) &= \widetilde{f}(x, u_1(x)) = f(x, w_0(x)) \\ &\leq -\Delta_{p,\omega} w_0(x) + V(x)|w_0(x)|^{p-2}w_0(x) \end{aligned}$$

for all $x \in S$ and $u_1(x) = w_0(x)$ for all $x \in \partial S$. Hence by the comparison principle, $u_1 \equiv w_0$ which is a contradiction. Moreover, in the case of $T \neq S$, there exists $x_0 \in S \cap \partial T$ such that $u_1(x_0) < w_0(x_0)$. Hence by the comparison principle, $u_1(x) < w_0(x)$, $x \in T$, which is also a contradiction. Thus we get the desired result. □

We now discuss the existence of the second solution to (1.6). To prove this, we first present a condition of $g \in C(S \times \mathbb{R}; \mathbb{R})$ which implies that the functional E_g satisfies the *Palais-Smale condition* (simply, (PS) condition):

(PS) Suppose that Ω is a real Banach space. A functional $E \in \mathcal{C}^1(\Omega;\mathbb{R})$ satisfies the Palais-Smale condition if for any sequence $(u_n) \subset \Omega$ satisfying
(a) $E[u_n]$ is bounded and
(b) $E'[u_n] \to 0$ as $n \to \infty$,
the sequence (u_n) has a convergent subsequence. A sequence satisfying (a) and (b) is called a (PS) sequence for E.

Lemma 4.7. *Let* **(H1)** *hold and a function* $g \in C(S \times \mathbb{R};\mathbb{R})$ *satisfies*

$$\limsup_{t\to-\infty} \frac{g(x,t)}{|t|^{p-2}t} < \lambda_{1,V} < \liminf_{t\to\infty} \frac{g(x,t)}{|t|^{p-2}t}$$

for all $x \in S$. *Then the functional* E_g *satisfies the* (PS) *condition.*

Proof. Let $\{u_n \in \mathcal{A}_\sigma\}$ be a (PS) sequence for E_g. Suppose that $\|u_n^-\|_p \to \infty$ where $u_n^-(x) := \max\{-u_n(x), 0\}$, $x \in \overline{S}$. We note that $u_n^-(x) = 0$ for all $x \in \partial S$, namely, $u_n^- \in \mathcal{A}_0$. Define a function $\epsilon_n : S \to \mathbb{R}$ by

$$\epsilon_n(x) := -\Delta_{p,\omega} u_n(x) + V(x)|u_n(x)|^{p-2}u_n(x) - g(x, u_n(x)), \quad x \in S.$$

Then we have $\epsilon_n \to 0$ by the definition of a (PS) sequence. Since g satisfies

$$\limsup_{t\to-\infty} \frac{g(x,t)}{|t|^{p-2}t} < \lambda_{1,V}, \quad x \in S,$$

there exists a real value $M < \lambda_{1,V}$ such that

$$g(x,t) > M|t|^{p-2}t + C, \quad t > 0 \tag{4.18}$$

for some constant C. It follows from the definition of ϵ_n and (4.18) that

$$\begin{aligned} 0 &= \int_S \left[-\Delta_{p,\omega} u_n(x) + V(x)|u_n(x)|^{p-2}u_n(x) - g(x,u_n(x)) - \epsilon_n(x)\right] u_n^-(x) \\ &< -(\lambda_{1,V} - M)\|u_n^-\|_p - \int_S (C + \epsilon_n) u_n^- \\ &\to -\infty, \quad n \to \infty, \end{aligned}$$

an obvious contradiction. Hence $\{u_n^-\}$ is bounded. We now suppose that $\|u_n\|_p \to \infty$. Define a function $v_n \in \mathcal{A}_\sigma$ by

$$v_n(x) := \frac{u_n(x)}{\|u_n\|_p}, \quad x \in S.$$

Then there exists a function $v_0 \in \mathcal{A}_\sigma$ such that $v_n \to v_0$. Moreover, since $\{u_n^-\}$ is bounded and $\|u_n\|_p \to \infty$, $v_0 \geq 0$ and $v_0 \not\equiv 0$. It follows from the assumption

$$\lambda_{1,V} < \liminf_{t\to\infty} \frac{g(x,t)}{|t|^{p-2}t}, \quad x \in S$$

that for sufficiently small $\epsilon > 0$,

$$g(x,t) > (\lambda_{1,V} + \epsilon)|t|^{p-2}t + C, \quad t > 0,\ x \in S$$

for some constant C. Hence we have that

$$-\Delta_{p,\omega} v_0(x) + V(x)|v_0(x)|^{p-2}v_0(x) \geq (\lambda_{1,V} + \epsilon)|v_0(x)|^{p-2}v_0(x)$$

for all $x \in S$. Therefore the function v_0 is a supersolution to

$$\text{(4.19)} \quad \begin{cases} -\Delta_{p,\omega} u(x) + V(x)|u(x)|^{p-2}u(x) = (\lambda_{1,V} + \epsilon)|u(x)|^{p-2}u(x), & x \in S, \\ u(x) = \sigma(x), & x \in \partial S. \end{cases}$$

Moreover, since the function v_0 satisfies

$$\begin{cases} -\Delta_{p,\omega} v_0(x) + V(x)|v_0(x)|^{p-2}v_0(x) \geq 0, & x \in S, \\ v_0(x) \geq 0, & x \in \partial S \end{cases}$$

and $v_0(x) > 0$ for some $x \in S$. It follows from the comparison principle that $v_0(x) > 0$ for all $x \in S$. Hence the positive eigenfunction ϕ_1, corresponding to $\lambda_{1,V}$, is a subsolution to (4.19). Moreover, by the comparison principle, $v_0(x) > \phi_1(x)$ for all $x \in S$. Thus by the sub-super solution method, for $\epsilon > 0$, the equation (4.19) has a solution which implies $\lambda_{1,V}$ is not isolated, which contradicts. Hence $\{u_n\}$ is bounded. Thus E_f satisfies the (PS) condition. □

We proved that there exists a positive solution to (1.6) in Theorem 4.6. The next result shows the existence of another positive solution.

Theorem 4.8. *Suppose that the hypotheses in Theorem* 4.6 *hold. If, in addition,* **(H4)** *holds, then* (1.6) *has at least two positive solutions* u_1 *and* u_2 *satisfying* $u_1 < u_2$.

Proof. It follows from Theorem 4.6 that there exists a positive solution $u_1 < w_0$. Now, we define a function $\widetilde{f_1} \in C(S \times \mathbb{R}; \mathbb{R})$ by

$$\widetilde{f_1}(x,t) := \begin{cases} f(x, u_1(x)), & t \leq u_1(x) \\ f(x,t), & u_1(x) < t < w_0(x) \\ f(x, w_0(x)), & t \geq w_0(x) \end{cases}$$

for $x \in S$. Then it is clear that the function f_1 satisfies

$$\limsup_{|t|\to\infty} \frac{\widetilde{f_1}(x,t)}{|t|^{p-2}t} < \lambda_{1,V}.$$

Thus by Lemma 3.3, the functional $E_{\widetilde{f_1}}$ has a global minimizer v_0. Using the similar argument in the proof of Theorem 4.6, we have $u_1 \leq v_0 < w_0$ on S. We define a function $f_1 \in C(S \times \mathbb{R}; \mathbb{R})$ by

$$f_1(x,t) := \begin{cases} f(x, u_1(x)), & t \leq u_1(x) \\ f(x,t), & u_1(x) < t. \end{cases}$$

Then the function f_1 satisfies

$$\limsup_{t\to-\infty} \frac{f_1(x,t)}{|t|^{p-2}t} < \lambda_{1,V} < \liminf_{t\to\infty} \frac{f_1(x,t)}{|t|^{p-2}t}.$$

Thus E_{f_1} satisfies the (PS) condition. Moreover, It follows from **(H4)** that there exists $M > \lambda_{1,V}$ such that

$$E_{f_1}[t\phi_1] \le \frac{\lambda_{1,V} - M}{p} t^p - Ct \to -\infty \text{ as } t \to \infty$$

for some constant C. Since $E_{\widetilde{f_1}}[u] = E_{f_1}[u]$ for all $u_1 \le u < w_0$, the function v_0 is a local minimizer of E_{f_1}. Hence by Mountain Pass Theorem, there exists a critical point u_2 of E_{f_1}. Using the comparison principle, we have $u_1 < u_2$.

5. EXAMPLES

In this section, we give some corollaries and examples for results obtained in Section 4.

Corollary 5.9. *Suppose that* **(H1)** *and* **(H2)** *hold and that there exists* $t_1 > \max_{z\in\partial S}\sigma(z)$ *such that* $f(x,t_1) \le 0$, $x \in S$. *If the function* V *satisfies*

$$V(x) \ge -\left(1 - \frac{\max_{y\in\partial S}\sigma(y) + \delta}{t_1}\right)^{p-1} \sum_{y\in\partial S} \omega(x,y), \quad x \in S \tag{5.20}$$

where $\delta \in (0, t_1 - \max_{z\in\partial S}\sigma(z))$ *then* (1.6) *has a solution* u *such that*

$$\epsilon_0^{\frac{1}{p-1}} u_0(x) < u(x) < t_1, \quad x \in S.$$

Proof. We put a function $\tau_1(x) = t_1$ for all $x \in S$ and $\tau_1(x) = \sigma(x) + \delta$ for all $x \in \partial S$. Then since $f(x,t_1) \le 0$ for all $x \in S$, we have

$$\begin{aligned} -\Delta_{p,\omega}\tau_1 + V(x)|\tau_1|^{p-2}\tau_1 &\ge (t_1 - (\max_{y\in\partial S}\sigma(y) + \delta))^{p-1} \sum_{y\in\partial S}\omega(x,y) + V(x)t_1^{p-1} \\ &\ge f(x,t_1) \end{aligned}$$

for all $x \in S$. Since $\tau_1(x) > \sigma(x)$ for all $x \in \partial S$, the function τ_1 is a supersolution to (1.6). Hence by Theorem 4.6, we have the desired result.

Example 3. We start this example with a set $\partial^\circ S$ defined by

$$\partial^\circ S := \{x \in S \mid \omega(x,y) > 0 \text{ for some y} \in \partial S\}.$$

Let a graph G be a path, $\sigma \equiv 0$ and

$$V(x) := \begin{cases} 0, & x \in S \setminus \partial^\circ S, \\ -\alpha, & x \in \partial^\circ S \end{cases}$$

where $-\alpha$ is a negative value greater than the right hand side of (5.20) and $-\lambda_1$. Then it is easily proved that $\lambda_{1,V} > 0$. We now take a function f satisfying

(i) $f(x,t) = 0$ for all t and $x \in S \setminus \partial^\circ S$;

(ii) $f(x,0) > 0$ for all $x \in \partial^\circ S$;

(iii) $f(x,t) > -\alpha t^{p-1}$ for all $t > 0$ and $x \in \partial^\circ S$;

(iv) $0 > f(x,t_1)$ for all $x \in \partial^\circ S$.

Then there exists a positive solution to (1.6) by Corollary 5.9.

Now taking $h(x,t) := f(x,t) - V(x)|t|^{p-2}t$, we have the equation

$$\begin{cases} -\Delta_{p,\omega} u(x) = h(x,u(x)), & x \in S \\ u(x) = 0, & x \in \partial S. \end{cases} \tag{5.21}$$

Then $h(x,t) > 0$, $x \in \partial^\circ S$, $t > 0$. Hence the function h does not satisfy the condition (1.4) in [**1**].

In addition, we show one more example in respect of a positive function f. To construct the function f, we introduce the eigenvalue problem without Dirichlet boundary condition:

$$-\Delta_{p,\omega}\psi(x) + \overline{V}(x)|\psi(x)|^{p-2}\psi(x) = \mu|\psi(x)|^{p-2}\psi(x), \quad x \in \overline{S}$$

for a function $\overline{V} : \overline{S} \to \mathbb{R}$. We note that this eigenvalue problem has the first eigenvalue $\mu_{1,\overline{V}}$ which is given by

$$\mu_{1,\overline{V}} := \inf_{\phi \not\equiv 0} \frac{\frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi \cdot \nabla_{p,\omega}\phi + \int_{\overline{S}} \overline{V}|\phi|^p}{\int_{\overline{S}} |\phi|^p} \tag{5.22}$$

and there exists a positive eigenfunction ψ_1 corresponding to $\mu_{1,\overline{V}}$. We note that if $\overline{V} \equiv 0$, then $\mu_{1,\overline{V}} = 0$ and $\psi_1(x) = \psi_1(y)$, $x, y \in \overline{S}$. Moreover, the multiplicity of $\mu_{1,\overline{V}}$ is one and $\mu_{1,\overline{V}}$ is isolated (for more details, see [**10**]).

We now show the next corollary with a definition of a function $\overline{V} : \overline{S} \to \mathbb{R}$ as follows: for a given $V : S \to \mathbb{R}$,

$$\overline{V}(x) := \begin{cases} V(x), & x \in S, \\ k, & x \in \partial S \end{cases}$$

where k is a constant.

Corollary 5.10. *Let* **(H1)** *and* **(H2)** *hold. Suppose that* $\mu_{1,\overline{V}}$, ψ_1, *and* f *satisfy*

$$\psi_1(x) > \max_{x \in \partial S} \sigma(x), \quad x \in \overline{S} \tag{5.23}$$

and

$$f(x, \psi_1(x)) \le \mu_{1,\overline{V}} \psi_1^{p-1}(x), \quad x \in S. \tag{5.24}$$

Then (1.6) *has a solution* u *such that*

$$\epsilon_0^{\frac{1}{p-1}} u_0(x) < u(x) < \psi_1(x), \quad x \in S.$$

Proof. It follows from (5.23), (5.24), and the definition of $\mu_{1,\overline{V}}$ that the eigenfunction ψ_1 is a supersolution but a solution to (1.6). Hence by Theorem 4.6, this corollary is proved.

Example 4. Suppose that a graph G is a path, V is positive and non-constant and $\sigma \equiv 0$. Then by definitions of $\lambda_{1,V}$, we have $\lambda_{1,V} > 0$. In addition, we take $k > \max_{x\in S} V(x)$, then $\mu_{1,\overline{V}} > 0$. Since V is non-constant, it follows from the definition of $\mu_{1,\overline{V}}$ that

$$\mu_{1,\overline{V}} > V(x_0) = \min_{x\in S} V(x).$$

We now take a function f satisfying (5.24) and $f(x,t) > 0$ for $x \in S$ and $t > 0$. Then by Corollary 5.10, there exists a positive solution to (1.6). Finally, since ψ_1 also become a supersolution to (1.6), there exists the second positive solution u_2 by Theorem 4.8 if **(H4)** holds.

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